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HIGH FREQUENCY DIFFRACTION BY A HARD CIRCULAR DISC.

BY

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Introduction

The high frequency scattering coefficients for the soft and hard circular discs have been calculated by Jones (1), (2) using an integral equation approach. The author used Jones' method for the soft disc, with some minor changes, to find the far field off the axis of symmetry for a normally incident plane wave (7) and extended the method to calculate the scattering coefficient for the plane wave at oblique incident (8) In this work the author demonstrated the fundamental nature of certain functions which occurred naturally after a contour deformation had produced extensive cancellations. Subsequently, Williams (9) has shown that these functions may he obtained more directly.

The purpose of the present paper is to calculate in detail the leading terms in a high frequency asymptotic expansion of the far scattered field off the axis of symmetry due to a normally incident plane harmonic scalar wave impinging on a hard circular disc. To do this use is made of techniques developed for work on the soft disc.

1. The Integral Equation

If the problem is normalised and represented in a cylindrical polar co-ordinate system with the hard disc occupying the region $0 \leq r \leq 1, z = 0$ then Jones (2) has shown that $f(w)$, the discontinuity in the field across the disc, satisfies the integral equation

$$\int_0^1 f(w)\left\{\frac{e^{-L\propto(w-u)}}{w-u} - \frac{e^{-L\propto(w+u)}}{w+u}\right\}dw = G(v) \tag{1}$$

where

$$G(u) = -2(2\pi\propto)^{\frac{1}{2}}\int_0^u \frac{\cos\{\propto(u^2-x^2)^{\frac{1}{2}}\}}{(U^2-x^2)^{\frac{1}{2}}}\int_0^x \frac{I_{-\frac{1}{2}}\{\propto(x^2-r^2)^{\frac{1}{2}}\}}{\left(x^2-r^2\right)^{\frac{1}{4}}}\left(\frac{\partial u_0(r,z)}{\partial z}\right)\Big/_{z=0} rdrdx \tag{2}$$

The parameter α is the product of the wave number of the incident field and the disc radius, and is large. The axisymmetric incident field is represented by $u_o(r,z)$ and is assumed to be harmonic and of small amplitude. Time dependence $e^{i\omega t}$ is understood and omitted throughout.

A new unknown is defined by

$$F(u) = \int_0^1 f(w)\,\frac{e - i\propto(w-u)}{w-u}\,dw, \tag{3}$$

Using the edge condition $f(l) = 0$ and inverting , Jones has shown that

$$f(W) = \frac{1}{4}\left(\frac{1-w}{w}\right)^{\frac{1}{2}} e^{i\propto w}\int_0^1\left(\frac{U}{1-u}\right)^{\frac{1}{2}}\frac{F(u)}{w-u}e^{-l\propto u}\,du \tag{4}$$

and hence that

$$F(U) = G(u) - \frac{1}{4}\left(\frac{1+U}{U}\right)^{\frac{1}{2}} e^{-L\propto u}\int_0^1\left(\frac{t}{1-t}\right)^{\frac{1}{2}}\frac{F(t)}{t+u}e^{-L\propto t}\,dt. \tag{5}$$

In considering the analogous equation for the soft disc the author redefined the unknown function and after some contour deformations produced a more amenable known function depending upon a fundamental function. Williams (9) subsequently obtained this fundamental function by a more direct method. In either case the required field could be expressed using Jones' iteration scheme.

A similar analysis in the present case shows that

$$F(u) = \sum_{k=0}^{\infty} \int_0^{\infty} \overline{\Psi}_k(w) \frac{e - L \propto (w - u)}{w - u} dw \tag{6}$$

The iteration scheme is

$$\Psi_{k+1}(u) = -\int_1^{\infty} \left(\frac{w}{w-1}\right)^{\frac{1}{2}} \frac{e - 1 \propto w}{w+u} \Psi_k(w)dw \tag{7}$$

$$\Psi_{k+1}(w) = \frac{1 \propto}{4\pi} \int_0^{\infty} \left(\frac{1+u}{u}\right)^{\frac{1}{2}} \Psi_{k+1}(u) M_2(u, w) e^{-L \propto u} du \tag{8}$$

The function $M_2(u,w)$ is defined by (Jones (2))

$$M_2(u, w) = \frac{4u}{u^2 - w^2} \left\{ w H_0^{(2)} (\propto u) J_1(\propto w) - u H^{(2)} (\propto u) J_0(\propto w) \right\}. \tag{9}$$

Using (4) and (6) f(w) can be written

$$f(w) = \sum_{k=0}^{\infty} \int_k (w) \tag{10}$$

and

$$f_k(w) = \Psi_k(w) + \frac{1}{\pi}\left(\frac{1-w}{w}\right)^{\frac{1}{2}} e^{i \propto w} \int_1^{\infty} \left(\frac{t}{t-1}\right)^{\frac{1}{2}} \frac{e^{-L \propto t}}{w-t} \Psi_k(t)dt$$

$$= \Psi_k(w) + \frac{1}{\pi}\left(\frac{1-w}{w}\right)^{\frac{1}{2}} \Psi_{k+1}(-w) e^{i \propto w}. \tag{11}$$

The function fundamental to the problem is ψ_o (w) which depends upon the particular incident field being considered. Once the incidental field has been specified then ψ_o (w) is determined by the methods of (7) or more directly by William's method (9). In either case if the incident field is a plane wave from the negative *z* direction

$$\Psi_0(w) = -2 . \qquad (12)$$

2. The Far Scattered Field

The scattered field is given by

$$4_s(\underset{\sim}{R}) = \frac{1}{4\pi}\int_s f(r_1)\frac{\partial}{\partial z_1}\left[\frac{e - L \propto |\underset{\sim}{R} - \underset{\sim}{R}|}{|\underset{\sim}{R} - \underset{\sim}{R}|}\right]_{Z_1=0} ds \qquad (13)$$

where S is the unit circle, centre the origin, in the plane $z = 0$ and $\underset{\sim}{R}_1$ is a point on S. For points $\underset{\sim}{R}$ far from the disc it is more convenient to use spherical polar co-ordinates (R,θ,ϕ). Since the incident field is assumed axisymmetric the far field will be independent of ϕ. The point on the disc $\underset{\sim}{R}_1$ is expressed in co-ordinates $(r_1,\theta_1,\phi_1,)$ Where $r_1 = w$, $0 \leq w \leq 1$ and $\theta_T = \frac{\pi}{2}$. In these co-ordinates

$$\frac{\partial}{\partial z_1}\left[\frac{e^{-i\propto|\underset{\sim}{R}-\underset{\sim}{R}|}}{|\underset{\sim}{R}-\underset{\sim}{R},|}\right]_{z_1 0} = \frac{\partial}{w\,\partial\theta_1}\left[\frac{e^{-i\propto|\underset{\sim}{R}-\underset{\sim}{R}|}}{|\underset{\sim}{R}-\underset{\sim}{R}|}\right]_{\theta_1=\frac{\pi}{2}}$$

$$\sim \frac{e^{-i\propto R}}{R} i \propto \cos\theta\, e^{i\propto w \sin\theta \cos(\phi-\phi)} + O(\frac{1}{R})$$

The far scattered field may therefore he written

$$U_s(\underset{\sim}{R}) \sim \frac{1}{4\pi}\frac{e - L \propto R}{R} i \propto \cos\theta \int_0^1 w\, f(w) \int_0^{2\pi} e^{i\propto w \sin\theta\cos(\phi-\phi_1)} d\phi, dw + O\left(\frac{1}{R^2}\right)$$

$$\sim \frac{1}{2}\frac{e^{-L\propto R}}{R} L \propto \cos\theta\, F(\theta) + O\left(\frac{1}{R^2}\right) \qquad (14)$$

where

$$F(\theta(= \int_0^1 w\, f(w) \int_0 (\propto w \sin\theta)\, dw. \qquad (15)$$

Using (10) and (11) in this last expression and an analysis similar to that used in (7) shows that

$$F(\theta(= \int_0^1 w\Psi_0(w)\int_0(\propto w \sin)\, dw - \frac{1}{\pi}\int_1^{1+iw} w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}\Psi_1(-w)e^{i\propto w}\int_0 (\propto w \sin\theta)dw$$

$$+ \sum_{k=1}^{\infty}\left\{-\int_1^{\infty} w\Psi_k(w)\int_0(\propto w \sin\theta)dw - \frac{1}{\pi}\int_1^{1+i\propto} w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}\Psi_{k+1}(-w)e^{i\propto w}\int_0(\propto w \sin\theta)dw\right\} \quad (16)$$

Call the first pair of terms the zero order contribution to $F(\theta)$ and denote them by $F^{(o)}(\theta)$. Similarly call each subsequent pair of terms for each k the k^{th} order contribution and denote them by $F^{(k)}(\theta)$.

Equations (14) and (16) determine the far scattered field though detailed evaluation of the integrals is required.

3. An Asymptotic Form of $\Psi_k(w)$

Equation (16) contains the Functions $\Psi_k(w)$ and $\Psi_{k+1}(-w)$

Using (7) $\Psi_{k+1}(-w)$ can be expressed as an integral involving $\Psi_k(w)$. Hence in the cases where $\Psi_k(w)$ occurs $|w| \geq 1$ and so an asymptotic form can be used.

In order to develop a suitable asymptotic expansion equations (8) and (9) are employed. Using the fact that M_2 (v,w) has poles at $v = \pm w$ we have

$$\Psi_k(w) = -\frac{i\propto}{4\pi}\int_0^\infty \left(\frac{I+u}{u}\right)^{\frac{1}{2}} \Psi_k(u)\, M_2(u,w) e^{-L\propto u} du + \frac{L}{\pi}\left(\frac{1+w}{w}\right)^{\frac{1}{2}} \Psi_k(w) e^{-i\propto w}$$

Deforming the contour onto the negative imaginary axis and changing v to -iv gives

$$\Psi_k(w) = \frac{2\propto}{\pi^2} e^{i\frac{\pi}{4}} \int_0^\infty \left(\frac{1-iu}{u}\right)^{\frac{1}{2}} \Psi_k(-iu) \frac{u}{u^2+w^2}\left\{ w\int_1(\propto w) K_0(\propto u) - u\int_0(\propto w) k(\propto u)\right\} e^{-\propto u} du$$
$$+ \frac{i}{\pi}\left(\frac{1+W}{w}\right)^{\frac{1}{2}} \Psi_k(w) e^{-i\propto w}$$

The integrand is expanded for small v assuming $|w| \geq 1$ and use is made of the result

$$\int_0^\infty t^{\mu-1} K_\mu(t) e^{-t} dt = \frac{(\mu - v - i)!\ (\mu + V - l)!}{2^\mu \left(\mu - \frac{1}{2}\right)!} \pi^{\frac{1}{2}}$$

Finally, for $|w| \geq 1$

$$\Psi_k(w) \sim \frac{L}{\pi}\left(\frac{1+w}{w}\right)^{\frac{1}{2}} \Psi_k(w) e^{-i\propto w} + \frac{2\propto}{\pi^{\frac{1}{2}}} e^{i\frac{\pi}{4}} \sum_{m=o}^\infty \frac{\frac{1}{2}!(-i)^m}{m!(\frac{1}{2}-m)!} \sum_{n=0}^\infty \frac{(-i)^n}{n!} \Psi_k^{(n)}(o)$$

$$X \sum_{p=o}^\infty \frac{\left(m+n+2p+\frac{1}{2}\right)!(-1)^p}{(2\propto)^{m+n+2p\frac{1}{2}}} \left\{ \frac{\left(m+n+2p+\frac{1}{2}\right)!}{(m+n+2p+1)!} \frac{J_1(\propto w)}{w^{2p+1}} - \frac{\left(m+n+2p+\frac{5}{2}\right)!}{2\propto(m+n+2p+2)!} \frac{J_o(\propto w)}{w^{2p+2}} \right\} \qquad (17)$$

4. The Zero Order Contribution to $F(\theta)$

The zero order contribution is given from equation (16) by

$$C^{(o)}(\theta) = \int_0^1 w\Psi_0(w)J_0(\propto w\sin\theta)\,dw - \frac{1}{\pi}\int_i^{i+i\infty} w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}\Psi_1(-w)e^{i\propto w}\int_c(\propto w\,\sin\theta)\,dw$$

For the plane wave at normal incidence

$$\Psi_0(w) = -2$$

and so

$$F^{(o)}(\theta) = -2\int_0^1 w\;J_0(\propto w\sin\theta)\,dw$$

$$-\frac{2}{\pi}\int_1^{1-i\infty}\left(\frac{t}{t-1}\right)^{\frac{1}{2}}e^{-i\propto t}\int_i^{1+i\propto}\frac{w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}{t-w}J_0(\propto w\sin\theta)e^{i\propto w}\,dw\,dt.$$

The integrals are similar to those encounted when considering the soft disc problem and are evaluated in a similar way. Again the first integral cancels completely with part of the second integral. Finally

$$F^{(0)}\,(\theta\theta \sim \left(\frac{2}{i\pi}\right)^{\frac{1}{2}}\frac{e^{i\frac{\pi}{2}}}{(\propto\sin\theta i^{\frac{1}{2}}}\left\{[(1+\sin\theta i^{-\frac{1}{2}}e^{i\propto\sin\theta} - i(1-\sin\theta i^{-\frac{1}{2}}e^{-i\propto\sin\theta}\right]$$

$$+\frac{i}{8\propto\sin\theta}\left([(1+\sin\theta i^{-\frac{1}{2}}e^{i\propto\sin\theta} + i(i-\sin\theta i^{-\frac{1}{2}}e^{-i\propto\sin\theta}\right]$$

$$\left. + 2[(1+\sin\theta i^{\frac{1}{2}}e^{i\propto\sin\theta}\,i(1-\sin\theta i^{\frac{1}{2}}e^{-i\propto\sin\theta}]\right)$$

$$\frac{1}{16(\propto\sin\theta i^{2}} = ([1+\sin\theta i^{\frac{1}{2}}e^{i\propto\sin\theta} - 1(1-\sin\theta i^{-\frac{1}{2}}e^{-i\propto\sin\theta}\;]$$

$$-\frac{17}{8}[(1+\sin\theta i^{-\frac{1}{2}}e i\propto\sin\theta - i(1-\sin\theta i - \frac{1}{2}e^{-i\propto\sin\theta}]$$

$$+\frac{7}{2}[(1+\sin\theta i^{\frac{1}{2}}e^{i\propto\sin\theta} - i(1-\sin\theta i^{\frac{1}{2}}e^{-i\propto\sin\theta}]$$

$$-\frac{1}{2}[(1+\sin\theta i^{\frac{1}{2}}e^{i\propto\sin\theta} - i(1-\sin\theta i^{\frac{1}{2}}e - i\propto\sin\theta in$$

$$+ O\left(\propto - \frac{2}{2}\right). \qquad (18)$$

5. The k^{th} Order Contribution to

from equation (16) this contribution is given by

$$F^{(k)}(\theta) = -\int_1^{\infty} w\Psi_k(w) \int_o (\alpha w \sin\theta)\, dw - \frac{1}{\pi}\int_1^{1+i\infty} w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}\Psi_{k+1}(-w)e^{i\propto w}\int_c (\propto w \sin\theta)\, dw.$$

The asumptotic form of $\Psi_k(w)$ 'developed earlier may be

employed in the first integral. Evaluation of the resulting integrals then yields

$$\int_1^{\infty} w\Psi_k(w)\int_0 (\propto w \sin\theta)dw$$

$$\sim \frac{e - i(\propto + \frac{\pi}{r})}{(\sin\theta s^{\frac{1}{2}}(\pi\propto)^{\frac{1}{2}}} \Psi_k(1)\left(\frac{e^{i\propto\sin\theta}}{1-\sin\theta} + i\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\right)$$

$$-\frac{i}{\propto}(\Psi_k'(l) + \frac{1}{4}\psi_k(l)\left(\frac{e^{i\propto\sin\theta}}{(1-\sin\theta)^2} + i\frac{e^{-i\propto\sin\theta}}{(1+\sin\theta)^2}\right)$$

$$+\frac{\Psi_k(l)}{8\sin\theta}\left(\frac{e^{i\propto\sin\theta}}{1-\sin\theta} - i\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\ ]\right)$$

$$+\frac{\Psi_k(O)e - i\frac{\pi}{4}}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}(\mathrm{Sin}\,\theta)^{\frac{1}{2}}\propto^{\frac{5}{2}}}\left(\frac{e^{i\propto(l+\sin\theta)} - e^{-i\propto(1+\sin\theta)}}{l+\sin\theta} + i\frac{e^{-i\propto(l-\sin\theta)} + e^{--i\propto(l-\sin\theta)}}{1-\sin\theta}\right)$$

$$+O\left(\frac{\Psi_k}{\propto^{\frac{7}{2}}}\right) \tag{19}$$

Asymptotic evaluation of the second integral is more involved and requires some labour. The result is

$$\frac{i}{\pi}\int_1^{1+i\propto} w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}\psi_{k+1}(-w)\ J_0\ (\propto w \sin\theta)dw$$

$$\sim -\frac{e - i(\propto + \frac{\pi}{4})}{(\sin\theta)^{\frac{1}{2}}(\pi\propto)^{\frac{1}{2}}}\left\{\psi_k(l)\left(\frac{e^{i\propto\sin\theta}}{1-\sin\theta}\left(1-\left(\frac{1+\sin\theta}{2}\right)^{-\frac{1}{2}}\right) + i\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\left(1-\left(\frac{1-\sin\theta}{2}\right)^{-\frac{1}{2}}\right)\right)\right.$$

$$-\frac{i}{\propto}\frac{1}{2}\psi_k(l) + \frac{1}{4}\psi_k(l)\frac{e^{i\propto\sin\theta}}{(l-\sin\theta)^2}\left(2-\left(\frac{1+\sin\theta}{2}\right)^{-\frac{1}{2}} - \left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right)$$

$$+i\frac{e^{-i\propto\sin\theta}}{(l+\sin\theta)^2}\left(2-\left(\frac{1-\sin\theta}{2}\right)^{-\frac{1}{2}} - \left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\right)$$

$$\left.+\frac{\psi_k(l)}{8\sin\theta}\frac{e^{i\propto\sin\theta}}{1-\sin\theta}\left(1-\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right) - i\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\left(1-\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\right)\Bigg)\Bigg)\Bigg]\Bigg]\right\}$$

$$+\frac{i\psi_k(o)}{8\pi\propto}\frac{ei(\propto + \frac{\pi}{4}}{(2\pi\pi\sin\ )^{\frac{1}{2}}\propto^{\frac{1}{2}}}\left(\frac{e^{i\propto\sin\theta}}{1+\sin\theta} + i\frac{e^{-i\propto\sin\theta}}{1-\sin\theta}\right) - \frac{ei\ \propto}{(2\sin\theta 2^{\frac{1}{2}}\propto^2}\left(\frac{e^{i\propto\sin\theta}}{1+\sin\theta i^{\frac{1}{2}}} + i\frac{e^{-i\propto\sin\theta}}{(1-\sin\theta i^{\frac{1}{2}}}\right)$$

$$-\frac{e - i(\propto + \frac{\Pi}{4}}{(2\sin\theta 2^{\frac{1}{2}}\propto^{\frac{1}{2}}}\left(\frac{e^{i\propto\sin\theta}}{1-\sin\theta}\left(1-\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right) + i\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\left(1-\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right)\right)\Bigg\} + O\left(\frac{\psi_k}{\propto^{\frac{1}{2}}}\right). \tag{20}$$

When equations (19) and (20) are combined to give $F^{(k)}(\theta)$ the first cancels completely with part of the second. The resulting asymptotic development is given by

$$F^{(k)}(\theta) \sim -\frac{e^{-i(\propto + \frac{\pi}{4})}}{(\sin \theta si^{\frac{1}{2}}(\pi \propto)^{\frac{1}{2}}} \{\psi_k(l)\left(\left(\frac{1+\sin\theta}{2}\right)^{-\frac{1}{2}} \frac{e^{i\propto \sin\theta}}{1-\sin\theta} + i\left(\frac{1-\sin\theta}{2}\right)^{-\frac{1}{2}} \frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\right)$$

$$-\frac{1}{2\propto}\left[\left(\psi_k(l) + \frac{1}{4}\psi_k(l)\right)\left(\left\{\left(\frac{1+\sin\theta}{2}\right)^{-\frac{1}{2}} + \left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{ei \propto \sin\theta}{(1-\sin\theta i^{2}}\right.\right.$$

$$\left. + i\left\{\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}} + \left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{-i\propto\sin\theta}}{(1+\sin\theta)^2}\right)$$

$$\left.\left. + \frac{\Psi_k(1)}{4\sin\theta}\left(\left(\frac{1+\sin\theta}{2}\right)^{-\frac{1}{2}}\frac{e^{i\propto\sin\theta}}{1-\sin\theta} - i\left(\frac{1-\sin\theta}{2}\right)^{-\frac{1}{2}}\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\right)\right]\right\}$$

$$+ \frac{i\psi_k(o)ei\ \propto}{2^{\frac{1}{2}}\pi(\sin\theta)^{\frac{1}{2}}\propto^3}\left(\frac{e^{i\propto\sin\theta}}{(1+\sin\theta)^{\frac{1}{2}}} + i\frac{e^{-i\propto\sin\theta}}{(1-\sin\theta)^{\frac{1}{2}}}\right)$$

$$-\frac{\psi_k(O)e^{-i(\propto - \frac{\pi}{4})}}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}(\sin\theta)^{\frac{1}{2}}\propto^{\frac{5}{2}}}\left(\left(\frac{1-\sin\theta}{2}\right)^{-\frac{1}{2}}\frac{ei\propto\sin\theta}{l-\sin\theta} + i\left(\frac{1-\sin\theta}{2}\right)^{-\frac{1}{2}}\frac{e^{-i\propto\sin\theta}}{l+\sin\theta}\right)$$

$$+ O\left(\frac{\psi_k}{\propto^{\frac{2}{3}}}\right) \qquad (21)$$

6. Asymptotic Forms of the

In order to find explicit forms of the various contributions to F (θ) using the results of the previous section it is necessary to obtain expressions for the ψ_k involved

for each k, Such expressions may be found from the iteration scheme (7) and the particular form of $\Psi_0(w)$ given in (12).

Since

$$\psi_{k+1}(u) = -\int_1^\infty \left(\frac{w}{w-1}\right)^{\frac{1}{2}} \frac{e^{-i\propto w}}{w+u} \psi_k(w) \; dw$$

and

$$\psi_0(w) = -2$$

then

$$\psi, (u) \sim 2\left(\frac{\pi}{\propto}\right)^{\frac{1}{2}} e^{i(\propto + \frac{\pi}{4})} \left\{ \frac{1}{1+u} + \frac{i}{2\propto}\left(\frac{1}{(1+u)^2} - \frac{1}{2(1+u)}\right) - \frac{3}{4\propto^2}\left(\frac{i}{(1+u)^3} - \frac{1}{2(1+u)^2} - \frac{1}{8(1+u)}\right)\right\} + O\left(\propto^{-\frac{2}{3}}\right) . \quad (22)$$

The explicit forms of the required ψ_1 are therefore

$$\psi_1(1) \sim \left(\frac{\pi}{\propto}\right)^{\frac{1}{2}} e^{-i(\propto + \frac{\pi}{4})} \left\{1 + \frac{3}{32\propto^2}\right\} + O\left(\propto^{-\frac{1}{2}}\right)$$

$$\psi_1(1) \sim -\frac{1}{2}\left(\frac{\pi}{\propto}\right)^{\frac{1}{2}} e^{-i(\propto + \frac{\pi}{4})} \left\{1 + \frac{i}{4\propto} - \frac{3}{32\propto^2}\right\} + O\left(\propto^{-\frac{1}{2}}\right),$$

$$\psi_1(O) \sim 2\left(\frac{\pi}{\propto}\right)^{\frac{1}{2}} e^{-i(\propto + \frac{\pi}{4})} \left\{1 + \frac{i}{4\propto} - \frac{9}{32\propto^2}\right\} + O\left(\propto^{-\frac{1}{2}}\right),$$

$$\psi_1(O) \sim -2\left(\frac{\pi}{\propto}\right)^{\frac{1}{2}} e^{-i(\propto + \frac{\pi}{4})} \left\{1 + \frac{3i}{4\propto} - \frac{45}{32\propto^2}\right\} + O\left(\propto^{-\frac{1}{2}}\right).$$

In order to calculate the asymptotic forms of ψ_2 , ψ_3 etc. a more general formulation is required. In the integral for $\psi_{k+1}(u)$, (7) and above, the asymptotic form of $\psi_k(w)$ from (17) may be employed. Evaluation of the integrals then shows that

$$\psi_{k+1}(u) \sim -\frac{e^{i(2\propto-\frac{\pi}{4})}}{(\pi\propto)^{\frac{1}{2}}}\frac{\psi_k(1)}{1+u}-\frac{i}{8\propto}\frac{\psi_k(u)}{u}\left(\frac{1}{(1+U)^{\frac{1}{2}}}-1\right)$$
$$+\frac{e^{-i(2\propto+\frac{\pi}{4})}}{4\pi^{\frac{1}{2}}\propto^{\frac{1}{2}}}\left(\frac{\psi_k(l)}{(1+u)^2}-\frac{\psi_k(1)+\frac{1}{4}\psi_k(1)}{1+u}+\frac{\psi_k(0)}{2^{\frac{1}{2}}(1+u)}\right)$$
$$+\quad O(\frac{\psi_k}{\propto^2}) \tag{23}$$

This, together with the expansions for ψ_1, above, enable the ψ_2 to be calculated.

$$\psi_2(1) \sim -\frac{e^{-3i\propto}}{2\propto}+\frac{\pi^{\frac{1}{2}}e^{-i(\propto-\frac{\pi}{4})}}{4\propto^{\frac{1}{2}}}\left(1-\frac{1}{2^{\frac{1}{2}}}\right)-\frac{ie^{-3i\propto}}{8\propto^2}\left(\frac{3}{4}+\frac{1}{2^{\frac{1}{2}}}\right)+O\left(\propto^{-\frac{5}{2}}\right),$$

$$\psi_2(1) \sim \frac{e-3i\propto}{4\propto}+\frac{\pi^{\frac{1}{2}}e^{-i(\propto-\frac{\pi}{4})}}{16\propto^{\frac{1}{2}}}\left(\frac{5}{2^{\frac{1}{2}}}-4\right)+\frac{ie^{-3i\propto}}{16\propto^2}\left(\frac{5}{4}+\frac{1}{2^{\frac{1}{2}}}\right)+O\left(\propto^{-\frac{5}{2}}\right),$$

$$\psi_2(O) \sim -\frac{e-3i\propto}{\propto}+\frac{\pi^{\frac{1}{2}}e^{-i(\propto-\frac{\pi}{4})}}{8\propto^{\frac{1}{2}}}-\frac{ie^{-3i\propto}}{4\propto^2}\left(\frac{5}{4}+\frac{1}{2^{\frac{1}{2}}}\right)+O\left(\propto^{-\frac{5}{2}}\right).$$

In turn these results are used to find ψ_3

$$\psi_3(1) \sim \frac{e^{i(5\propto-\frac{\pi}{4})}}{4\pi^{\frac{1}{2}}\propto^{\frac{1}{2}}}-\frac{ie^{-3i\propto}}{8\propto^2}-\left(1-\frac{1}{2^{\frac{1}{2}}}\right)+O\left(\propto^{-\frac{3}{2}}\right)$$

Similarly ψ_4 is given by

$$\psi_k(l) \sim -\frac{ie-7i\propto}{8\pi\propto^2}+O\left(\propto^{-\frac{5}{2}}\right)$$

These are sufficient to find the first to fourth order contributions to $F(\theta)$.

7. The First to Fourth Order Contributions to F(θ).

Equation (21) gives the general k^{th} order contribution to F(θ) in terms of ψ_k. Using the results of the previous section explicit forms of the first to fourth order contributions can be found.

These are

$$
\begin{aligned}
F^{(1)}(\theta) \sim{}& \frac{ie^{-2i\propto}}{\pi(\sin\theta)^{\frac{1}{2}}\propto^{2}}\Bigg\{\Bigg(\Big(\frac{1+\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{i\propto\sin\theta}}{1-\sin\theta}+i\Big(\frac{1-\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\Bigg)\\
&+\frac{i}{8\propto}\Bigg[\Bigg(\Bigg\{\Big(\frac{1+\sin\theta}{2}\Big)^{-\frac{1}{2}}+\Big(\frac{1+\sin\theta}{2}\Big)^{\frac{1}{2}}\Bigg\}\frac{e^{i\propto\sin\theta}}{(1-\sin\theta)^{2}}+i\Bigg\{\Big(\frac{1-\sin\theta}{2}\Big)^{-\frac{1}{2}}+\Big(\frac{1-\sin\theta}{2}\Big)^{\frac{1}{2}}\Bigg\}\frac{e^{-i\propto\sin\theta}}{(1+\sin\theta)^{2}}\Bigg)\\
&-\frac{1}{\sin\theta}\Bigg(\Big(\frac{1+\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{i\propto\sin\theta}}{1-\sin\theta}-i\Big(\frac{1-\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\Bigg)\Bigg]\Bigg\}\\
&-\frac{e^{-2i\propto}}{2^{\frac{5}{2}}\pi(\sin\theta)^{\frac{1}{2}}\propto^{3}}\Bigg(\Big(\frac{1+\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{i\propto\sin\theta}}{1-\sin\theta}+i\Big(\frac{1-\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\Bigg)\\
&+\frac{e^{i\frac{\pi}{4}}}{2^{\frac{5}{2}}\pi^{\frac{1}{2}}(\sin\theta)^{\frac{1}{2}}\propto^{\frac{1}{2}}}\Bigg(\frac{e^{i\propto\sin\theta}}{(1+\sin\theta)^{\frac{1}{2}}}+i\frac{e^{-i\propto\sin\theta}}{(1-\sin\theta)^{\frac{1}{2}}}\Bigg)\\
&+O(\propto^{-4})
\end{aligned}
$$

24.

$$
\begin{aligned}
F^{(2)}(\theta)\sim{}&\frac{e^{-i(4\propto+\frac{\pi}{4})}}{2\pi^{\frac{1}{2}}(\sin\theta)^{\frac{1}{2}}\propto^{\frac{5}{2}}}\Bigg(\Big(\frac{1+\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{i\propto\sin\theta}}{1-\sin\theta}+i\Big(\frac{1-\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\Bigg)\\
&\sim\frac{e^{-2i\propto}}{4\pi(\sin\theta)^{\frac{1}{2}}\propto^{3}}\Bigg(1-\frac{1}{2^{\frac{1}{2}}}\Bigg)\Bigg(\Big(\frac{1+\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{i\propto\sin\theta}}{1-\sin\theta}+i\Big(\frac{1-\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\Bigg)\\
&+\frac{e^{-i(4\propto-\frac{\pi}{4})}}{8\pi^{\frac{1}{2}}(\sin\theta)^{\frac{1}{2}}\propto^{\frac{7}{2}}}\Bigg\{\Bigg(\frac{3}{4}+2^{\frac{1}{2}}\Bigg)\Bigg(\Big(\frac{1+\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{i\propto\sin\theta}}{1-\sin\theta}+i\Big(\frac{1-\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\Bigg)\\
&+\frac{1}{2}\Bigg[\Bigg\{\Big(\frac{1+\sin\theta}{2}\Big)^{-\frac{1}{2}}+\Big(\frac{1+\sin\theta}{2}\Big)^{\frac{1}{2}}\Bigg\}\frac{e^{i\propto\sin\theta}}{(1-\sin\theta)^{2}}+i\Bigg\{\Big(\frac{1-\sin\theta}{2}\Big)^{-\frac{1}{2}}+\Big(\frac{1-\sin\theta}{2}\Big)^{\frac{1}{2}}\Bigg\}\frac{e^{-i\propto\sin\theta}}{(1+\sin\theta)^{2}}\Bigg\}\\
&-\frac{1}{\sin\theta}\Bigg(\Big(\frac{1+\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{i\propto\sin\theta}}{1-\sin\theta}-i\Big(\frac{1-\sin\theta}{2}\Big)^{-\frac{1}{2}}\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\Bigg\}\Bigg)\\
&+O(\propto^{-4})
\end{aligned}
$$

(25)

$$F^{(3)}(\theta) \sim -\frac{e^{-6i\propto}}{4\pi^2(\sin\theta)^{\frac{1}{2}}\propto^3}\left(\left(\frac{1+\sin\theta}{2}\right)^{-\frac{1}{2}}\frac{e^{i\propto\sin\theta}}{1-\sin\theta}+i\left(\frac{1-\sin\theta}{2}\right)^{-\frac{1}{2}}\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\right)$$

$$+\frac{e^{-i(4\propto-\frac{\pi}{4})}}{8\pi^{\frac{3}{2}}(\sin\theta)^{\frac{1}{2}}\propto^{\frac{7}{2}}}\left(1-\frac{1}{2^{\frac{1}{2}}}\right)\left(\left(\frac{1+\sin\theta}{2}\right)^{-\frac{1}{2}}\frac{e^{i\propto\sin\theta}}{1-\sin\theta}+i\left(\frac{1-\sin\theta}{2}\right)^{-\frac{1}{2}}\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\right)$$

$$+O(\propto^{-4}), \qquad (26)$$

$$F^{(4)}(\theta) \sim \frac{e^{-i(8\propto-\frac{\pi}{4})}}{8\pi^{\frac{5}{2}}(\sin\theta)^{\frac{1}{2}}\propto^{\frac{1}{2}}}\left(\left(\frac{1+\sin\theta}{2}\right)^{-\frac{1}{2}}\frac{e^{i\propto\sin\theta}}{1-\sin\theta}+i\left(\frac{1-\sin\theta}{2}\right)^{-\frac{1}{2}}\frac{e^{-i\propto\sin\theta}}{1+\sin\theta}\right)$$

$$+O(\propto^{-4}) \qquad (27)$$

In addition

$$F^{(5)}(\theta) \sim 0(\alpha^{-4}).$$

8. An Explicit Form of the Far Scattered Field.

It has been shown in equation (14) that the far scattered field may be written in the form

$$U_s(\underset{\sim}{R}) \sim \frac{1}{2}\,\frac{e^{-i\alpha R}}{R}\, i\alpha \cos\theta\, F(\theta) + O\left(\frac{1}{R^2}\right)$$

where

$$F(\theta) = \sum_{k=0}^{\infty} F^{(k)}(\theta)$$

Equations (18), (24), (25) and (26) show that

$$F(\theta) = F^{(0)}(\theta) + F^{(1)}(\theta) + F^{(2)}(\theta) + F^{(3)}(\theta) + F^{(4)}(\theta) + O(\alpha^{-4})$$

An explicit form of the far scattered field is therefore available upto and including the term of $0(\alpha^{-3})$. As in the case of the soft disc this expression for the scattered field is subject to the restrictions that $R \gg 1$ and $\theta \neq 0$ or $\frac{\pi}{2}$. This represents the far field off the axis of symmetry and away from the plane of the disc.

The leading term of the zero and of the first order contributions have been found using Keller's Ray Theory (3-6) The other terms found in this work are believed to be new.

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