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# SMOOTH POLYNOMIAL INTERPOLATION TO BOUNDARY DATA ON TRIANGLES

by

R. E. Barnhill and J. A. Gregory

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A B S T R A C T

Boolean sum interpolation theory is used to derive a polynomial interpolant which interpolates a function $F \in C^N(\partial T)$, and its derivatives of order N and less, on the boundary 3T of a triangle T. A triangle with one curved side is also considered.

## 1. INTRODUCTION

Boolean sum Interpolation theory was first used on triangles by Barnhill, Birkhoff, and Gordon [1] to derive rational functions interpolating the boundary data. The general theory of Boolean sum interpolation is briefly discussed in this paper and a polynomial Boolean sum interpolant is presented, which, for any positive integer N, interpolates a function $F \in C\ (\partial T)$, and its derivatives of order N and less, on the boundary $\partial T$ of a triangle T. The case N=0 corresponds to an interpolant constructed by other means by Nielson [6]. The interpolant requires that certain derivatives of F be compatible at the vertices of T, but these conditions can be removed by adding suitable rational Terms. The theory is generalized for a triangle with one curved side.

The interpolant can be used to define a piecewise function which is $C^N(\Omega)$ over a triangular subdivision of a polygonal region $\Omega$. This has applications to computer aided geometric design and finite element analysis. Finite dimensional piecewise $C^N\ (\Omega)$ interpolants can be derived by taking the boundary data to be functions interpolating discrete data along the sides. Alternatively the blending function can be incorporated with finite elements so as to match exactly a given boundary function on $\Omega$, see, for example, Marshall and Mitchell [5].

## 2. BOOLEAN SUM INTERPOLATION THEORY

This section considers conditions which are sufficient for the application of Boolean sum interpolation theory. These conditions motivate the formulation of the projectors considered in Section 3.

The interpolation of the function F is first discussed and this is then generalized to the interpolation of the function F and its derivatives.

<u>THEOREM 2.1</u>,

Let $\Gamma_1$ and $\Gamma_2$ be two subsets of R and let F be a function defined on $\Gamma_1 \cup \Gamma_2$ . Let $P_1$ and $P_2$ be two interpolation projectors such that $P_iF = F$ on $\Gamma_i$ and $P_iF$ is defined on $\Gamma_1 \cup \Gamma_2$ , $i = 1,2$ .

Then the Boolean sum function

$$(P_1 \oplus P_2)F = (P_1 + P_2 - P_1P_2)F \tag{2.1}$$

(i) interpolates F on $\Gamma_1$

(ii) interpolates F on $\Gamma_2 - \Gamma_1$ if $P_1F$ on $\Gamma_2 - \Gamma_1$ is a linear combination of function evaluations on $\Gamma_2$.

<u>Proof</u>, (i) Since $I - P_1$ is null on $\Gamma_1$ , where I is the identity operator, it follows that

$$F - (P_1 \oplus P_2)F \equiv (P_1 + P_2 - P_1P_2)F$$

is zero on $\Gamma_1$ .

(ii) Also, since $(I - P_2)F = 0$ on $\Gamma_2$ ,

$$F - (P_1 \oplus P_2)F \equiv (I-P_2)F - P_1(I-P_2)F$$

is zero on $\Gamma_2 - \Gamma_1$ if $P_1$ on $\Gamma_2 - \Gamma_1$ . is a linear combination of function evaluations on $\Gamma_2$ . Q.E.D.

In practice, $P_iF$ usually involves F only on $\Gamma_i$ . The hypothesis of Theorem 2.1 then becomes that $P_iF$ on $\Gamma_2 - \Gamma_1$ is a linear combination of function evaluations on $\Gamma_1 \cap \Gamma_2$ , where it is a necessary condition that $\Gamma_1 \cap \Gamma_2$ is not null.

<u>REMARK.</u>. If the dual hypothesis holds for $(P_2 \oplus P_1)F$, that is, $P_2F$ on $\Gamma_1 - \Gamma_2$ is a linear combination of function evaluations on $\Gamma_1$, then

$$(P_1 \oplus P_2)F = (P_2 \oplus P_1)F \quad \text{on} \quad \Gamma_1 \cup \Gamma_2 \quad \text{and hence}$$

$$P_1P_2F - P_2P_1F \text{ on } \Gamma_1 \cup \Gamma_2 .$$

We thus have sufficient conditions that the projectors safisfy the definition of weak commutativity of Gordon and Wixom [4].

The generalization of Theorem 2.1 to the interpolation of function and derivatives on $\Gamma_1 \cup \Gamma_2$ is the following:

THEOREM 2.2.

Let $P_1$ and $P_2$ be two interpolation projectors such that $D^\alpha P_iF = D^\alpha F$ on $\Gamma_i$ and $D^\alpha P_iF$ is defined on $\Gamma_1 \cup \Gamma_2$, $i = 1,2$, for all $|\alpha| \le N$, where

$$\alpha = (\alpha_1, \ldots\ldots, \alpha_n) and D^\alpha = \frac{\partial |\alpha|}{\partial x_1{}^\alpha 1 \ldots \partial x_n{}^\alpha n}.$$

Then

(i) $D^\alpha(P_1 \oplus P_2)F = D^\alpha F$ on $\Gamma_1$ for all $|\alpha| \le N$,

(ii) $D^\alpha(P_1 \oplus P_2)F = D^\alpha F$ on $\Gamma_2 - \Gamma_1$ for all $|a| \le N$ if $D^\alpha P_1F$ on $\Gamma_2 - \Gamma_1$ is a linear combination of function and derivative evaluations on $\Gamma_2$ which are interpolated by $P_2F$.

<u>Proof</u>, The proof is an extension of the proof of Theorem 2.1. The only complication is on $\Gamma_2 - \Gamma_1$ where

$$D^{\alpha} F - D^{\alpha} (P_1 \oplus P_2)F \equiv D^{\alpha} (I - P_2)F - D^{\alpha} P_1 (I - P_2)F$$

is zero for all $|\alpha| \leq N$ if and only if $D^{\alpha} P_1 (I - P_2)F = 0$. A sufficient condition for this to hold is that $D^{\alpha} P_1$ on $\Gamma_2 - \Gamma_1$ is a linear combination of function and derivative evaluations on $\Gamma_2$ . For $|\alpha| \geq 1$ some of these derivatives may be of order greater than N and thus we require that these be interpolated by $P_2$. Q.E.D.

NOTE. Since $P_2 F$ interpolates $D^{\alpha} F$ on $\Gamma_2$ for all $|\alpha| \leq N$ , then, assuming its existence, $\partial^{\beta} / \partial s^{\beta} (D^{a}F)$ is also interpolated on $\Gamma_2$ , where $\partial/\partial s$ is any derivative along the set $\Gamma_2$. such derivatives, assuming any necessary compatibility to allow change of order of differentiation, frequently include those required by Theorem 2.2.

EXAMPLE OF RATIONAL INTERPOLATION ON TRIANGLES. Consider the standard triangle T with vertices at $V_1 = (0,1)$, $V_2 = (1,0)$, and $V_3 = (0,0)$, where the side opposite the vertex V, is denoted by $E_k$ . Rational Hermite projectors on T are defined by

$$T_1 F = \sum_{i \leq N} \psi_i \left[\frac{x}{1-y}\right] (1-y)^i \, Fi,0(0, y) + \sum_{i \leq n} \psi_i \left[\frac{x}{1-y}\right] (1-y)^i \, F_{i,0} (1-y, y), \tag{2.2}$$

$$T_2 F = \sum_{i \leq N} \psi_i \left[\frac{y}{1-x}\right] (1-x)^i \, F_{0,i} (x,0) + \sum_{i \leq n} \psi_i \left[\frac{y}{1-x}\right] (1-x)^i \, F_{0,i} (x,1-x), \tag{2.3}$$

$$T_3 F = \sum_{i \leq N} \psi_i \left[\frac{x}{x+y}\right] (x+y)^i \left( \left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i F \right) (0, x+y) \tag{2.4}$$

$$+ \sum_{I \leq N} \psi_i \left[\frac{x}{x+y}\right] (x+y)^i \left( \left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i F \right) (x+y,0)$$

where the $\psi_i(t)$ and $\psi_i(t) = (-1)^i \psi_i(1-t)$ are the cardinal basis functions for Hermite two paint Taylor interpolation on the interval [0,1]. Boolean sum interpolation using these projectors was first considered by Barnhill, Birkhoff, and Gordon [1]. Application of Theorem 2,2 gives the following theorem.

THEOREM 2.3. The Boolean sum functions

$(T_i \oplus T_i)F - (T. + T. - T_i T_i)F, \quad i \neq j; i,j\ 1,2,3,$

interpolate $F \in C^N(\partial T)$ and its derivatives of order N and less on $\partial T$, provided that F satisfies the compatibility conditions

$$\left[\frac{\partial^{m+n} F}{\partial s_i^m \, \partial s_j^n}\right](V_k) = \left[\frac{\partial^{n+m} F}{\partial s_j^n \, \partial s_i^m}\right](V_k), \quad m, n < N; m + n \geq N, \tag{2.5}$$

where V, is the vertex with adjacent sides E. and $E_i$, and $\partial/\partial s\ell$ denotes differentiation along the side $E\ell$.

Proof. By affine transformation and symmetry it is sufficient to consider the case $(T_1 \oplus T_2)F$. $T_1F$ and $T_2F$ interpolate on $\Gamma_1 = E_2 \cup E_3$ and $\Gamma_3 = E_1 \cup E_3$ respectively. With reference to the hypotheses of Theorem 2.2, $D\alpha P_1F$, $|\alpha[ \leq N$, on $\Gamma_2 - \Gamma_1. = E_{-2}$ involves linear combinations of

$$\left[\frac{\partial^{n+m} F}{\partial y^n \, \partial x^m}\right](0,0) \quad \text{and} \quad \left(-\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]^n \frac{\partial^m F}{\partial x^m}\right)(1,0), \quad 0 \leq m, n \leq N. \tag{2.6}$$

The latter derivative is interpolated by $P_2F$, since tangential derivatives along the side are automatically interpolated. Also $P_2F$ interpolates $F_{0,n}(x,0)$ and hence interpolates

$$\left[\frac{\partial^{m+n} F}{\partial x^m \partial y^n}\right](0,0), \quad 0 \le m, n \le N.$$

Thus the hypotheses of Theorem 2.2 are satisfied if $F \in C^N(\partial T)$ and satisfies the compatibility conditions (2.5) at the vertex $V_3 = (0,0)$. Q.E.D.

PRECISION. The precision set is the set of polynomials for which the interpolant is exact and is important in that it indicates the order of accuracy of the interpolant. The precision set of the Boolean sum operator $P_1 \oplus P_2$ is at least that of $P_2$ since

$$I - (P_1 \oplus P_2) \equiv I - P_2 - P_1(I - P_2)$$

and $I - P_2$ is null on the precision set of $P_2$. Thus the Boolean sum operator $P_1 \oplus P_2$ has at least the interpolation properties of the projector $P_1$ and the precision set of the projector $P_2$.

## 3. POLYNOMIAL INTERPOLATION ON TRIANGLES

By affine invariance it is sufficient to consider the standard triangle $T$ defined above. Projectors $P_1$ and $P_2$ are considered, which satisfy the conditions of Theorem 2,2 and which respectively interpolate $F \in C^N((\partial T)$ and its derivatives of order $N$ and less, on the hypotenuse $\Gamma_1 = E_3$ and on the x and y axes $\Gamma_2 = E_1 \cup E_2$. These projectors involve suitable combinations of the Taylor projectors which interpolate $F$ and its derivatives on the sides of the triangle $T$ along parallels to the $x$ and $y$ axes. Explicitly the Taylor projectors are defined by

$$T_x^2 F = \sum_{i\leq N} x^{(i)} F_{i,0}(0, y),$$

$$T_x^3 F = \sum_{i\leq N} (x + y - 1)^{(i)} F_{i,0}(1 - y, y), \qquad (3.1)$$

$$T_y^1 F = \sum_{j\leq N} y^{(j)} F_{0,j}(x,0) \quad ,$$

$$T_y^3 F = \sum_{j\leq N} (x + y - 1)^{(j)} F_{0,j}(x, 1 - x) ,$$

where $x^{(i)} = x^i/i$; and $T_x^2$ denotes the Taylor projector across the side $E_2$ along the line through $(x, y)$ parallel to the $x$ axis etc.

Let

$$P_2 F = (T_x^2 \oplus T_y^1)F$$

$$= \sum_{i\leq N} x^{(i)} F_{i,0}(0,y) + \sum_{j\leq N} y^{(j)} F_{0,j}(x,0) \qquad (3.2)$$

$$- \sum_{i,\, j\leq N} x^{(i)}\, y^{(j)} \left( \frac{\partial^{\,i+j} F}{\partial x^i \partial y^j} \right) (0,0)$$

Then it is easily shown that for $F \in C^N(\partial T)$ the conditions of Theorem 2,2 are satisfied for the Boolean sum of the projectors on $E_2$ and $T_y^1$ on $E_1$ if

$$\left( \frac{\partial^{\,m+n} F}{\partial x^m \partial y^n} \right) (0,0) = \left[ \frac{\partial^{\,n+m} F}{\partial y^n \partial x^m} \right] (0,0), \quad m, n \leq N;\ m + n > N \qquad (3.3)$$

(in which case the Taylor projectors are commutative). Thus for $F$ satisfying the compatibility condition (3.3), $P_2 F$ interpolates $F$ and its derivatives of order $N$ and less on $\Gamma_2 = E_1 \cup E_2$ . The precision set of $P_2$ is the union of those of the two Taylor projectors $T_x^2$ and $T_y^1$, namely

$$x^i y^j, \begin{cases} 0 \le i \le N & \text{for all } j, \\ 0 \le j \le N & \text{for all } i. \end{cases} \tag{3.4}$$

A projector $P_1$ is required which interpolates F and its derivatives on $\Gamma_1 = E_3$ and which satisfies the conditions of Theorem 2.2, namely that on $\Gamma_2$ is a linear combination of function and derivative evaluations on $\Gamma_2$ which are interpolated by $P_2F$. This is accomplished by taking a suitable combination of the two hypotenuse Taylor projectors.

'<u>LINEAR *CASE*</u>. (Nielsen's interpolate Let

$$P_1F = y\ F(x, l\text{-}x) + x\,F(l\text{-}y,y) \ , \tag{3.5}$$

then $P_1F$ interpolates F *on* $\Gamma_1 = \{x+y = 1\}$. Also, on $x = 0$, $P_1F = y\ F(0,l)$ and, on $y = 0$, $P_1F = x\ F(1,0)$. Thus $P_1F$ on $\Gamma_2$ is a linear combination of function evaluations on $\Gamma_2$, and these are interpolated by

$$P_2F = F(0,y) + F(x,0) - F(0,0) \tag{3.6}$$

The conditions of Theorem 2.1 are thus satisfied and

$$\begin{aligned}(P_1 \oplus P_2)F &= y\ F(x,l\text{-}x) + x\ F(l\text{-}y.y) + F(x,0) + F(0,y) - F(0,0) \\ &\quad -y\ \{F(0,l\text{-}x) + F(x,0) - F(0,0)\} - x\{F(0,y) + F(l\text{-}y,0) - F(0,0)\}\end{aligned} \tag{3.7}$$

interpolates F on the boundary $\partial T$ of the triangle T. This is a Boolean sum derivation of Nielson's polynomial interpolant.

If we let

$\widetilde{F}(x,0) = (1\text{-}x)\ F(0,0) + x\ F(1,0)$, $\qquad \widetilde{F}(0,y) = (l\text{-}y)\ F(0,0) + y\ F(0,1)$,

and $\widetilde{F}(x,1\text{-}x) = F(x,1\text{-}x)$, then

$$(P_i \oplus P_2)\,\widetilde{F} = y\ F(x,1\text{-}x) + x\ F(1\text{-}y,y) + (1\text{-}x\text{-}y)\ F(0,0) \tag{3.8}$$

is an interpolation function which is linear on two sides of the triangle, whilst matching the function F on the other side. This interpolant could be incorporated with piecewise linear finite elements on a triangulated polygon so as to satisfy given boundary conditions exactly.

GENERAL CASE. Let

$$\begin{aligned} &\sum_{i,\, j \leq N} \alpha_{i,j}(x, y) \left( \frac{\partial^i}{\partial y^i} \left[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right]^j F \right)(0,1) \\ &+ \sum_{i,\, j \leq N} \beta_{i,j}(x, y) \left( \frac{\partial^i}{\partial x^i} \left[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right]^j F \right)(1,0) \qquad (3.9) \\ &+ \sum_{i,\, j \leq N} \gamma_{i,j}(x, y) \left( \frac{\partial^{i+j} F}{\partial x^i \, \partial y^j} \right)(0,0) \quad , \end{aligned}$$

where $\alpha_{i,j}$, $\beta_{i,j}$, and $\gamma_{i,j}$ are the appropriate cardinal functions, be the polynomial interpolant over the $3(N + 1)$ dimensional set of polynomials which are of degree $2N + 1$ along parallels to the three sides of T. The case $N = 1$ is the tricubic polynomial interpolant of Birkhoff [3] and, for general N, the existence of this interpolant is implied by Lemma 4.1 of Barnhill and Mansfield [2], Then

$$\alpha_{0,0}(x,y) + \beta_{0,0}(x,y) + \gamma_{0,0}(x,y) = 1$$

and

$$\left(D^{\alpha} \alpha_{0,0}\right)\left(E_1\right) = \left(D^{\alpha} \beta_{0,0}\right)\left(E_2\right) = \left(D^{\alpha} \gamma_{0,0}\right)\left(E_3\right) = 0 \quad \text{for all } |\alpha| \leq N,$$

where $\left(D^{\alpha} \alpha_{0,0}\right)\left(E_1\right)$ represents $D^{\alpha} \alpha_{0,0}(x, y)$, evaluated on the side $E_1$ etc. Hence

$$\left(\alpha_{0,0} + \beta_{0,0}\right)\left(E_3\right) = 1 \tag{3.10}$$

and

$$\left(D^{\alpha}\left[\alpha_{0,0} + \beta_{0,0}\right]\right)\left(E_3\right) = 0, 1 \le |\alpha| \le N. \tag{3.11}$$

Thus

$$\begin{aligned} P_1 F &= \alpha_{0,0}(x, y)\ T_y^3 F + \beta_{0,0}(x, y)\ T_x^3 F \\ &= \alpha_{0,0}(x, y) \sum_{j \le N} (x + y - 1)^{(j)}\ F_{0,j}(x, 1, -x) \\ &= \beta_{0,0}(x, y) \sum_{j \le N} (x + y - 1)^{(j)}\ F_{i,0}(1 - y, y) \end{aligned} \tag{3.12}$$

is a projector which interpolates $F \in C^N(\partial T)$ and its derivatives of order N and less on $\Gamma_1 - E_3$. Also, for all, $|\alpha| \le N$, $D^{\alpha}\ P_1 F$ on $y = 0$ involves the derivatives

$$\left(\left[-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]^n \frac{\partial^m F}{\partial x^m}\right)(1,0), \qquad 0 \le m, n \le N.$$

$P_2 F$ defined by equation (3.2) interpolates these values provided **that** $F \in C^N(\partial T)$ satisfies the compatibility condition

$$\left(\left[-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]^n \frac{\partial^m F}{\partial x^m}\right)(1,0) = \left(\frac{\partial^m}{\partial x^m}\left[-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]^n F\right)(1,0), \quad m, n \le N;\ m + n > N. \tag{3.13}$$

Similarly on $x = 0$ we require that

$$\left(\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]^m \frac{\partial^n F}{\partial y^n}\right)(0,1) = \left(\frac{\partial^n}{\partial y^n}\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^m F\right)(0,1), \quad m, n \le N;\ m + n > N. \tag{3.14}$$

The conditions of Theorem (2.2) are then satisfied and with (3.3) we have:

THEOREM 3.1. Let $F \in C^N(\partial T)$ and satisfy the compatibility conditions

$$\left( \frac{\partial^{m+n} F}{\partial s_i^{\,m}\, \partial s_j^{\,n}} \right)(V_k) = \left( \frac{\partial^{n+m} F}{\partial s_j^{\,n}\, \partial s_i^{\,m}} \right)(V_k), \quad m,\ n \le N; \quad m + n > N, \tag{3.5}$$

at each vertex $V_k$ with adjacent sides $E_i$ and $E_j$, where $\partial/\partial s_\ell$ denotes differentiation along the side $E_\ell$. Then the polynomial Boolean sum function

$$(P_1 \oplus P_2)F = (P_1 + P_2 - P_1 P_2)F,$$

where P.. is defined by (3.12) and $P_2$ is defined by (3.2), interpolates F and its derivatives of order N and less on the boundary $\partial T$ of the triangle T.

The precision set of the interpolant is that of the projector $P_2$, see (3.4).

<u>EXAMPLES</u>, (i) For N = 0,

$\alpha_{0,0}(x,y) = y$ and $\beta_{0,0}(x,y) = x$,

giving the linear case (3.7).

(ii) For N = 1,

$\alpha_{0,0}(x,y)$ - $y^2[3-2y + 6x\ (1-x-y)]$ and $ß_{0,0}(x,y) = x^2[3-2x + 6y\ (1-x-y)]$.

This case is discussed further in Section 4.

## 4. <u>REMOVAL OF COMPATIBILITY CONDITIONS</u>.

The compatibility conditions (3.15) of Theorem 3.1 can be removed by adding suitable rational terms to the Boolean sum interpolant $(P_1 \oplus P_2)$ F. We consider the rational Hermite projectors on the standard triangle T defined by equations (2.2) - (2.4).

Firstly, since $T_3$ interpolates $F$ on $E_1 \cup E_2$, the projector $P_2$, defined by (3.2), can be modified to

$$\tilde{P}_2F = P_2F + T_3(F - P_2F) \tag{4.1}$$

where $T_3(F - P_2F)$ is a rational compatibility correction term. We consider now the modified Boolean sum interpolant

$$(P_1 \oplus \tilde{P}_2)F = (P_1 \oplus P_2)F + (I - P_1)\, T_3(F - P_2F)$$

where $P_1$ is defined by (3.12), This interpolant requires the compatibility conditions (3.15) at the vertices $V_1 = (0,1)$ and $V_2 = (1,0)$. Then $F - (P_1 \oplus \tilde{P}_2)F$ has compatible derivatives at the vertex $V_3 = (0,0)$ and can thus be interpolated by either of the rational Boolean sum operators $T_1 \oplus T_2$ or $T_2 \oplus T_1$.

Thus

$$(P_1 \notin \tilde{P}_2)F + (T_1 \oplus T_2)\; [F - (P_1 \oplus P_2)F] \tag{4.2}$$

Interpolates $F \in C^N(\partial T)$ and its derivatives of order $N$ and less on $\partial T$, where $(T_1 \oplus T_2)[F - (P_1 \oplus P_2)F]$ is another rational compatibility correction term. The rational terms are zero if the compatibility conditions (3.15) hold.

<u>EXAMPLE</u>. For $N = 1$, the average of (3.2) with the dual expression for $(T_y^1 \oplus T_x^2)F$ gives the symmetric projector

$$P_2F = F(0,y) + x\, F_{1,0}(0,y) + F(x,0) + y\, F_{0,1}(x,0) \tag{4.3}$$

$$- F(0,0) - yF_{0,1}(0,0) - xF_{1,0}(0,0) - \frac{xy}{2}\left\{\left[\frac{\partial^2 F}{\partial x \partial y}\right](0,0) + \left[\frac{\partial^2 F}{\partial y \partial x}\right](0,0)\right\}.$$

Then

$$T_3(F \sim P_2F) = \frac{xy(x-y)}{2(x+y)}\left\{\left[\frac{\partial^2 F}{\partial x \partial y}\right](0,0) - \left[\frac{\partial^2 F}{\partial y \partial x}\right](0,0)\right\} \qquad (4.4)$$

and the projector

$$\tilde{P}_2 F = P_2F + T_3(F - P_2F) \qquad (4.5)$$

interpolates $F \in C^N(\partial T)$ on $\Gamma_2 = E_1 \cup E_2$ . Now

$$P_1 F = y^2[3\text{-}2y\ 4\text{-}\ 6x(l\text{-}x\text{-}y)]\,[F(x,l\text{-}x) + (x+y\text{-}1)\ F_{0\ 1}(x,l\text{-}x)]$$

$$+ x^2[3\text{-}2x + 6y(l\text{-}x\text{-}y)3[F(l\text{-}y,y) + (x+y\text{-}1)\ F_{1,\ 0}(l\text{-}y,y)] \qquad (4.\ 6)$$

and the Boolean sum $(P_1 \oplus \tilde{P}_2)F = (P_1 + P_2 - P_1\tilde{P}_2)F$ can be determined from equations (4. 3) - (4. 7) where

$$P_1\tilde{P}_2F = y^2[3-2y+6x(1-x-y)][(\tilde{P}_2F)(x,1-x) + (x+y-1)\left[\frac{\partial \tilde{P}_2 F}{\partial y}\right](x,1\text{ - }x)]$$

$$+ x2[3-2x+6y(1-x-y)][(\tilde{P}_2F)(1-y,y) + (x+y-1)\left[\frac{\partial \tilde{P}_2 F}{\partial x}\right](1-y,y)]\,. \qquad (4.7)$$

It can then be shown that

$$(T_1 \oplus T_2)[F - (P_1 \oplus \tilde{P}_2)F] \qquad (4.\ 8)$$

$$= \frac{(x+y-1)^2 x^2 y\ (3-2x)}{x-1}\left\{\left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]\frac{\partial F}{\partial x}\right)(1,0) - \left(\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]F\right)(1,0)\right\}$$

$$= \frac{(x+y-1)^2 xy^2\ (3-2y)}{y-1}\left\{\left(\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]\frac{\partial F}{\partial y}\right)(0,1) - \left(\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]F\right)(0,1)\right\}$$

giving the compatibly corrected interpolant

$$(P_1 \oplus \tilde{P}_2)F + (T_1 \oplus T_2)[E - (P_1 \oplus \tilde{P}_2)F]\ .$$

TRIANGLE WITH ONE CURVED SIDE

By affine transformation, it is sufficient to consider the triangle with vertices at $V_1 = (0,1)$, $V_2 = (1,0)$ and $V_3 = (0,0)$ and two straight edges along the coordinate axes. We assume that the third side $E_3$ opposite the vertex $V_3$ is defined by the one-to-one function

$$y = f(x) \quad \text{or} \quad x = g(y),$$

where $g$ is the inverse function of $f$. The Taylor projectors on $E_3$ are now

$$T_x^3 F = \sum_{i \leq N} [x - g(y)]^{(i)} F_{i,0} (g(y), y), \tag{5.1}$$

$$T_x^3 F = \sum_{j \leq N} [y - f(x)]^{(j)} F_{0,j} (x, f(x)). \tag{5.2}$$

The cardinal functions $\alpha_{0,0}(x,y)$ and $\beta_{0,0}$ of Section 3 have the properties

$$[\alpha_{0,0}(l\text{-}f(x),y) + \beta_{0,0}(l\text{-}f(x),y)](E_3) = 1$$

and

$$[D^\alpha \alpha_{0,0}(l\text{-}f(x),y) + D^\alpha \beta_{0,0}(l\text{-}f(x),y)](E_3) = 0$$

for $1 \leq |\alpha| \leq N$ . Thus

$$P_1 F = \alpha_{0,0} (1 - f(x), y) \; T_y^3 F + \beta_{0,0} (1 - f(x), y) \; T_x^3 F \tag{5.3}$$

is a suitable projector on $E_3$ The dual projector is

$$P_1 F = \alpha_{0,0}\ (x,1 - g(y))\ \ T_y^3 F + \beta_{0,0}\ (x,1 - g(y))\ \ T_x^3 F \qquad (5\text{-}4)$$

or alternatively an average of these two can be considered.

The Boolean sum function $(P_1 \oplus P_2)F$, where $P_2$ is defined by (3.2), gives a blending function interpolant on the curved triangle,

<u>EXAMPLE,</u> For the case $N = 0$ , (5.3) gives the projector

$$P_1F = y\ F(x,f(x)) + [l\text{-}f(x)]\ F(g(y),y)$$

and

$$P_2F = F(0,y) + F(x,0) - F(0,0)\,.$$

Then

$$(P_1 \oplus P_2)F = y\ F(x,f(x)) + [1\text{-}f(x)]\ F(g(y),y) + F(0,y) + F(x,0)$$

$$- F(0,0) - y[F(0,f(x)) + F(x,0) - F(0,0)]$$

$$- [l\text{-}f(x)][F(0,y) + F(g(y),0) - F(0,0)] \quad .$$

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