TR/29   September 1973.

COMPATIBLE SMOOTH INTERPOLATION

IN TRIANGLES

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ABSTRACT

 Boolean sum smooth interpolation to boundary data on

a triangle is described. Sufficient conditions are given

so that the functions when pieced together form a $C^{N-1}(\Omega)$

function over a triangular subdivision of a polygonal

region Ω and the precision sets of the interpolation functions

are derived. The interpolants are modified so that the

compatability conditions on the function which is interpolated

can be removed and a C^1 interpolant is used to illustrate the

theory. The generation of interpolation schemes for discrete

boundary data is also discussed.

1 . <u>Introduction</u>.

This note presents a method of removing the compatibility

conditions from the smooth rational Boolean sum interpolants

described in Barnhill, Birkhoff, and Gordon [l]. The resulting

interpolants match a function $F \varepsilon C^{N-1} (\partial T)$ and its first N-1 normal

derivatives along the boundary ∂T of a triangle T. The interpolants

can be used to define a piecewise function which is $C^{N-1} (\Omega)$ over a

triangulated polygon Ω.

For algebraic simplicity, the triangle T with vertices at

$V_1 = (0,1)$, $V_2 = (1,0)$, and $V_3 =0,0)$ is considered, where the side

opposite the vertex V_k is denoted by E_k . Any other triangle can

be obtained by an affine transformation of this "standard" triangle.

Hermite interpolation projectors P_k ,k = 1 ,2,3,are defined so that

on two sides of the triangle T, the function $P_k F(x,y)$ interpolates

$F \varepsilon C^{N-1} (\partial T)$ and its first N-1 directional derivatives along the

line through (x,y) parallel to the third side E_k . Such a function

will interpolate all partial derivatives up to order N-1 on the

two sides, since tangential derivatives along the sides are automatically

given. Boolean sum operators of these projectors are defined by

(1.1) $P_i \oplus P_j = P_i + P_j - P_i P_j$, $i \neq j$.

Barnhill and Mansfield [3] show that the function $(P_i \oplus P_j) F$

interpolates $F \varepsilon C^{N-1} (\partial T)$, and all its partial derivatives of order

N-1 and less on ∂T, provided that certain derivatives of F are

compatible at a vertex. In Section 2, we show how these compatibility

conditions may be removed, whilst preserving the interpolation

properties and precision set of the resulting interpolant. This is

an alternative approach to that proposed by Mansfield [5]. The

precision set is the set of polynomials for which the interpolant is

exact, and it is important in that it indicates the order of accuracy

of the interpolant.

Smooth interpolation functions have applications to computer

aided geometric design and finite element analysis. Finite element

basis functions can be generated by taking the boundary data as

polynomials which interpolate functionals along a side and this

possibility is discussed in Section 4. An alternative application

to finite element analysis is that data along the boundary of a

triangulated polygon can be matched exactly with smooth interpolation.

2. Hermite Boolean sum interpolation to boundary data.

The Hermite interpolation projectors on the standard triangle T

are defined by

$$(2.1) \quad P_1 F = \sum_{i=0}^{N-1} \varphi_i\left(\frac{x}{1-y}\right)(1-y)^i F_{i,0}(0,y) + \sum_{i=0}^{N-1} \psi_i\left(\frac{x}{1-y}\right)(1-y)^i F_{i,0}(1-y,y),$$

$$(2.2) \quad P_2 F = \sum_{i=0}^{N-1} \varphi_i\left(\frac{y}{1-x}\right)(1-x)^i F_{0,i}(x,0) + \sum_{i=0}^{N-1} \psi_i\left(\frac{Y}{1-x}\right)(1-x)^i F_{0,i}(x,1-x),$$

$$(2.3) \quad P_3 F = \sum_{i=0}^{N-1} \varphi_i\left(\frac{x}{x+y}\right)(x+y)^i \left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i F\right)(0,x+y)$$

$$+ \sum_{i=0}^{N-1} \psi_i\left(\frac{x}{x+y}\right)(x+y)^i \left(\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^i F\right)(x+y,0),$$

where the ϕ_i *(t)* and $\psi_i(t) = (-1)^1 \phi_i(1-t)$ are the cardinal basis

functions for Hermite two point Taylor interpolation on the

interval [0,1].Thus

(2.4) $\phi_i^{(j)}(o) = \delta_{ij}$ and $\phi_i^{(j)}(1) = 0$, $0 \leq j < N$

It is convenient to express these projectors as

(2.5) $P_i = P_i^j + P_i^k$, $i \neq j \neq k$,

where P_i^ℓ. denotes the part of the projector involving the side E_ℓ

Since $P_i^k P_j = P_i^k$, the Boolean sum of two projectors can then be

expressed as

(2.6) $P_i \oplus P_j = P_i + P_j - P_i P_j$

$$= P_i^j + P_j - P_i^j P_j$$

Boolean sum interpolation properties: The following theorem

on Boolean sum interpolation is Lemma 4.2 of [3]. At the end of

Section 2, we show how the Boolean sum function can be modified so

that the compatibility conditions (2.7) can be relaxed.

Theorem 2.1. The Boolean sum functions

$(P_i \oplus P_j) F = (P_i + P_j - P_i P_j)F$, $i \neq j$; $i,j = 1,2,3$,

interpolate F ε $C^{N-1}(\partial T)$ and its derivatives of order N-1 and less

on ∂T, provided that F satisfies the compatibility conditions

(2.7) $(\dfrac{\partial^{m+n}F}{\partial s_i^m \partial s_j^n})(V_k) = (\dfrac{\partial^{n+m}F}{\partial s_j^n \partial s_i^m})(V_k)$,   m, n < ; m + n $\geq$ N,

4.

where V_k is the vertex with adjacent sides E_i and E_j , and $\partial/\partial s_\ell$ denotes differentiation along the side E_ℓ .

Proof. It is sufficient to consider the case $(P_1 \oplus P_2)F$ as the other cases then follow by affine transformation and symmetry. Firstly, the interpolation properties hold on $x=0$ and $1-x-y=0$ since

$$F - (P_1 \oplus P_2)F = (I-P_1) \ (I-P_2) \ F \ ,$$

where I is the identity operator, and $I-P_1$. and its partial derivatives of order N-1 and less are null on those sides. Secondly, from (2.6),

$$F - (P_1 \oplus P_2) F = (I-P_2)F - P_1^2 \ (I-P_2)F$$

and hence we require that

$$(\frac{\partial P_1^2(I - P_2)F}{\partial y j})(x,0) = 0, \quad 0 \le j \le N - 1,$$

for the interpolation properties to hold on y = 0.Now

(2.8) $\quad P_1^2(I - P_2)F = \sum_{i=0}^{N-1} \varphi_i(\frac{x}{1 - y}) (1 - y)^i [F_{i,0}(0, y) - (\frac{\partial^i P_2 F}{\partial x^i}) (0, y)].$

Also

$$(\frac{\partial^{k+i} P_2 F}{\partial y^k \partial x^i}) (0,0) = (\frac{\partial^{i+k} P_2 F}{\partial x^i \partial y^k}) (0,0) = (\frac{\partial^i F_{0,k}}{\partial x^i}) (0,0), 0 \le i, k \le N - 1,$$

where the change of order of differentiation is permissible, as for $F \ \varepsilon \ C^{N-1} \ (\partial T)$, it can be shown that $P_2 F$ is N-1 times continuously differentiable at (0,0).

Thus differentiation of (2.8) by Leibniz' rule gives

$$(2.9) \quad \left(\frac{\partial^j P_1^2 (I - P_2) F}{\partial y^j}\right)(x,0)$$

$$= \sum_{k=1}^{j} \sum_{i=N-k}^{N-1} \binom{j}{k} \left(\frac{\partial^{j-k}}{\partial y^{j-k}} \varphi_i\left(\frac{x}{1-y}\right)(1-y)^i\right)(x,0) \left[\left(\frac{\partial^k}{\partial y^k}\left(\frac{\partial^i F}{\partial x^i}\right)\right)(0,0)\right.$$

$$\left. - \left(\frac{\partial^i}{\partial x^i}\left(\frac{\partial^k F}{\partial y^k}\right)\right)(0,0)\right],$$

Where the original summation $\sum_{i=0}^{N} \sum_{k=0}^{1} \sum_{j}$ has been reduced to that in (2.9)

Since the cross derivatives are compatible for $0 \leq I+k \leq N-1$ if $F \varepsilon C^{N-1}(\partial T)$

The requirement that (2,9) be zero for all $1 \leq j \leq N-1$ is true if and only if

$$(2.10) \quad \left(\frac{\partial^k}{\partial y^k}\left(\frac{\partial^i F}{\partial x^i}\right)\right)(0,0) = \left(\frac{\partial^i}{\partial x^i}\left(\frac{\partial^k F}{\partial y^k}\right)\right)(0,0), \qquad i,k < N; i+k \geq N,$$

which completes the proof of the theorem.

Continuity at vertices: The previous theorem indicates the behaviour

of the Boolean sum functions along $\P T$. The following theorem gives

sufficient conditions that the apparent singularities at two of the

vertices of T are removable and also that the Boolean sum function

be N-1 times continuously differentiable.

Theorem 2.2. Let the derivative values of F in $(P_i \oplus P_j) F$, as

functions of their single variables, be N-1 times continuously

differentiable on the sides of T and the function values of F be

N times continuously differentiable on the sides of T. Then $(P_i \oplus P_j) F$

is N-1 times continuously differentiable on T excluding the vertices

V_i. and V_j. where, for $F \varepsilon C^{N-1}(\partial T)$, it is N-1 times continuously

differentiable as the limit to the vertices is approached from T.

Proof. It is sufficient to consider the case

(2.11) $(P_1 \oplus P_2)F = P_2F + P^2_1(I-P_2)F$

$$= \sum_{i=0}^{N-1} \varphi_i(\frac{y}{1-x})(1-x)^i F_{0,i}(x,0) + \sum_{i=0}^{N-1} \psi_i(\frac{y}{1-x})(1-x)^i F_{0,i}$$

$$+ \sum_{i=0}^{N-1} \varphi_i(\frac{y}{1-y})(1-Y)^i[F_{i,0}(0,y) - (\frac{\partial^i P_2 F}{\partial x^i})(0,y)].$$

Since the boundary functions are N-1 times continuously differentiable with respect to their single variables x or y on [0,1], then (2.11) is also N-1 times continuously differentiable on T excluding the vertices $V_1=(0,1)$ and $V_2=(1,0)$, where, for $(x,y) \varepsilon T$, the functions Φ and ψ in (2.11) have singularities.

Consider first $V_2 = (1,0)$ where from (2.11), it follows that

(2.12) $\underset{(x,y)=(1,0)}{\text{limit}} \frac{\partial^{m+n}(P_1 \oplus P_2)F}{\partial x^m \partial y^n} = \underset{(x,y)=(1,0)}{\text{limit}} \frac{\partial^{m+n}P_2F}{\partial x^m \partial y^n}, 0 \le m+n \le N-1.$

The differentiability of the function and derivative values of F in P_2F implies that, for $F \varepsilon C^{N-1}(\partial T)$, they have the following Taylor expansions about x = 1 :

(2.13) $F_{0,i}(x,0) = \sum_{j=0}^{N-1-i}(x-1)^{(j)}F_{j,i}(1,0) + \int_1^X (x-\tilde{x})^{(N-1-i)}\frac{d^{N-i}F_{0,i}(\tilde{x},0)}{d\tilde{x}^{N-i}}d\tilde{x}$

(2.14) $F_{0,i}(x,1-x) = \sum_{j=0}^{N-1-i}(x-1)^{(j)}(\frac{d^j F_{0,i}(x,1-x)}{dx^j})x = 1$

where

$$+ \int_1^x (x-\tilde{x})^{(N-1-i)}\frac{d^{N-i}F_{0,i}(\tilde{x},1-\tilde{x})}{d\tilde{x}^{N-i}}d\tilde{x}$$

Now P_2 is exact for the function

$$\sum_{i+j<N} (x-1)^{(i)} y^{(j)} F_{i,j}(1,0)$$

[see proof of Theorem 2.3, equation (2.21)], and the integral remainder terms in (2.13) and (2.14] are zero for such a function. Thus substitution of the Taylor expansions into equation (2.2) for $P_2 F$ gives

$$(2.15) \quad P_2 F = \sum_{i+j<N} (x-1)^{(i)} y^{(j)} F_{i,j}(1,0)$$

$$+ \sum_{i=0}^{N-1} \varphi_i\left(\frac{y}{1-x}\right)(1-x)^i \int_1^x (x-\tilde{x})(N-1-i)\frac{d^{N-i}F_{0,i}(\tilde{x},0)}{d\tilde{x}^{N-i}} d\tilde{x}$$

$$+ \sum_{i=0}^{N-1} \psi_i\left(\frac{y}{1-x}\right)(1-x)^i \int_1^x (x-\tilde{x})(N-1-i)\frac{d^{N-i}F_{0,i}(\tilde{x},(\tilde{x},1-\tilde{x}))}{d\tilde{x}^{N-i}} d\tilde{x}$$

We note that the behaviour of the singular terms are like

$$\varphi_i\left(\frac{y}{1-x}\right)(1-x)N \quad \text{and} \quad \psi_i\left(\frac{y}{1-x}\right)(1-x)^N$$

Thus, since $0 \leq y/(1-x) \leq 1$ for $(x,y) \varepsilon T$, it is simple to show by differentiating (2.15), and using the continuity and hence boundedness of the derivatives, that

$$(2.16) \quad \lim_{(x,y)=(1,0)} \frac{\partial^{m+n} P_2 F}{\partial x^m \partial y^n} = F_{m,n}(1,0), \quad 0 \leq m+n \leq N-1; \quad (x,y) \varepsilon T,$$

which by (2.12) is desired result at the vertex $V_2 = (1,0)$.

At $V_1 = (0,1)$, since

$$(P_1 \oplus P_2)F = P_1 F + (I-P_1)P_2 F,$$

the dual argument applied to $P_1 F$ and $P_1 [P_2 F]$ gives that

$$(2.17) \quad \lim_{(x,y)=(0,1)} \frac{\partial^{m+n}(P_1 \oplus P_2) F}{\partial x^m \partial y^n} = F_{m,n}(0,1) \quad 0 \leq m + n \leq N - 1; (x,y) \varepsilon T.$$

<u>Remark</u> Weaker conditions on $F(0,x)$ and $F(x,1-x)$ are that their N-1'st derivative be absolutely continuous and their N'th derivative be bounded in the L norm, $p > 1$, on the interval $[1-\varepsilon, 1]$.

Theorem 2.1 and Theorem 2.2 give sufficient conditions that the interpolation functions can be pieced together to form a $C^{N-1} (\Omega)$ function over a triangulated polygon ft. Theorem 2.2. holds for boundary functions which are polynomials.

<u>Precision</u>: The following simple lemma is useful in establishing the set of polynomials for which the Boolean sum interpolants are exact.

<u>Lemma 2.1</u>, Let f and g be functions of one and two variables respectively. The

$$(2.18) \quad P_i.[f(\xi_i) g (x,y)] = f(\xi_i) p [g(x,y)] , i = 1, 2, 3,$$

where $\xi_i = 0$ is the side E_i of ∂T parallel to the projector P_i, i.e.

$$(2.19) \quad \xi_1 \equiv y, \quad \xi_2 \equiv x \text{ and } \xi_3 \equiv 1 - x - y.$$

Proof. Substitution of $F(x,y) = f(\xi_i) g (x,y)$ into the function $P_i F$ gives the desired result.

Theorem 2.3. The operator $P_i \oplus P_j$ is exact for the set of

monomials

(2.20) $\qquad \xi_j^m \xi_i^n \qquad , \qquad \begin{cases} m \geq 0 \text{ for } 0 \leq n \leq 2N-1 \\ 2N \leq m+n \leq 3N-1 \text{ for } 2N \leq n \leq 3N-1. \end{cases}$

Proof. By affine transformation and symmetry it is sufficient

to consider the case $P_1 \oplus P_2$.

Firstly, the operator $P_1 \oplus P_2$ has at least the precision set of P_2 since

$$P_1 \oplus P_2 = P_2 {}^+ P_1 (I-P_2)$$

and $I-P_2$ is the null operator on that precision set. Now from

Lemma 2.1, $P_2 x^m y^n = x^m P_2 y^n$ for all m, and by the precision of the

Hermite projector P_2, it follows that $P_2 y^n = y^n$ for $0 \leq n \leq 2N-1$,

Thus

(2.21) $\qquad P_2 x^m y^n = x^m y^n, \quad m \geq 0; \ 0 \leq n \leq 2N-1.$

Secondly, using Lemma 2.1, we have that

(2.22) $\qquad (P_1 \oplus P_2) x^m y^n = y^n p1 x^m + x^m p_2 y^n - P_1 [x^m P_2 y^n]$

Now for $n \geq N$

$$P_2 y^n = \sum_{i=0}^{N-1} \psi_i (\frac{y}{1-x}) (1-x)^i \frac{n!}{(n-i)!} (1-x)^{n-i}$$

and each $\psi_i(t)$ is a polynomial of degree 2N-1 with a factor t^N.

Thus, by cancellation for $n \geq 2N$, $x^m P_2 y^n$ is a polynomial in x and y

which is of degree at most m+n-N in the variable x.

Hence, using the precision of P_1, it follows that $P_1 [x^m p_2 y^n] = x^m P_2 y^n$

for m+n - N ≤ 2N-1 and n ≥ 2N. Substituting in (2.22)

and since P x = x for 0 ≤ m ≤ 2N-1 we have that

(2.23) $(P_1 \oplus P_2)$ x^{m}y^n = x^{m}y^n ,   m+n ≤ 3N-1;   2N ≤ n ≤ 3N-1

It follows from Theorem 2.3 that Boolean sum interpolation on
a general triangle is exact for all polynomials of degree 3NH
or less, since this set is affine invariant, More precisely we
have

<u>Corollary 2.1</u> Let P. and P. be two Hermite projectors along
parallels to the sides E$_i$ and E$_j$ of a general triangle in the
(x,y) plane. Let ξ = ξ(x,y) and n = n(x,y) be the affine
transformation which transforms this triangle onto the standard
triangle in the (ξ,n)plane, with the side E. on the ξ axis and
E. on the n axis. Then the Boolean sum interpolant $(P_i \oplus P_j)$F
is exact for the set of polynomials

(2.24) $\xi^m \eta^n$, $\begin{cases} m \geq 0 \, for \, 0 \leq n \leq 2N-1 \\ 2N \leq m+n \leq 3N-1 \, for \, 2N \leq n \leq 3N-1. \end{cases}$

<u>Removal of compatibility conditions</u>: Without loss of
generality the Boolean sum function $(P_1 \oplus P_2)$F is considered.
By Theorem 2.1, the remainder function

     R = F - $(P_1 \oplus P_2)$F

is zero, and has zero derivatives of order NH and less, on the

sides of the triangle T, except on y = 0 where

$$R_{0,j}(x,0) = -\sum_{k=1}^{j}\sum_{i=N-k}^{N-1} \binom{j}{k} \left(\frac{\partial^{j-k}}{\partial y^{j-k}} \varphi_i\left(\frac{x}{1-y}\right)\right)(1-y^i$$

$$\left[\left(\frac{\partial^{k+i}F}{\partial y^k \partial x^i}\right)(0,0) - \left(\frac{\partial^{i+k}F}{\partial x^i \partial y^k}\right)(0,0)\right], \quad 1 \le j \le N-1,$$

It is easily shown that $R \varepsilon C^{N-1}$ ($\P T$) and that at the vertex

$V_1 = (0,1)$, R satisfies the compatibility requirements (2.7) of

Theorem 2.1. Thus the Boolean sum functions

$$(P_2 \oplus P_3)R \text{ and } (P_3 \oplus P_2)R$$

interpolate the function R and its derivatives of order N-1

and less on the sides of T. Either of these functions when added

to $(P_1 \oplus P_2)F$ remove the compatibility conditions of Theorem 2.1.

By affine transformation and symmetry, we have thus proved the

following theorem.

Theorem 2.4. The functions

$$(2.25) \quad (P_i \oplus P_j)F + (P_j \oplus P_k)R ,$$
$$i \ne j \ne k ,$$
$$(2.26) \quad (P_i \oplus P_j)F + (P_k \oplus P_j)R ,$$

where

$$(2.27) \quad R = F - (P_i \oplus P_j)F ,$$

interpolate $F \varepsilon C^{N-1}$ (∂T) and its derivatives of order N-1 and

less on ∂T.

We note that the modified interpolant reduces to $(P_i \oplus P_j)F$ when the compatibility conditions of Theoren 2.1 are satisfied.

The precision set of Theorem 2.3 is true for the modified interpolants (2.25) and (2.26), since the corrective functions are zero for all polynomial F. Theorem 2.2 is also valid for the modified interpolants.

3. $\underline{C^1}$ interpolation to boundary data.

In this section the theory of Section 2 is illustrated with the case N= 2 of Boolean sum functions which interpolate $F \varepsilon C^1 (\partial T)$ and its first derivatives on ∂T. The basis functions for Hermite two point Taylor interpolation on $[0,1]$ are then defined by

$$\phi_0(t) = (t-1)^2(2t+1), \quad \phi_1(t) = (t-1)^2 t,$$

(3.1)

$$\psi_0(t) = t^2(-2t+3), \quad \psi_1(t) = t^2(t-1).$$

The Boolean sum function of P_1 and P_2 is

$$(3.2)\ (P_1 \oplus P_2)F = \sum_{i=0}^{1} \varphi_i(\frac{y}{1-x}) (1-x)^i F_{0,i}(x,0) + \sum_{i=0}^{1} \psi_i(\frac{y}{1-x}) (1-x)^i F_{0,i}$$

$$+ \sum_{i=0} \varphi_i(\frac{x}{1-y}) (1-y)^i [F_{i,0}(0,y) - (\frac{\partial^i P_2 F}{\partial x^i}) (0,y)],$$

where

$$(P_2F)(0,y) = \phi_0(y)F(0,0) + \phi_1(y)F_{0,1}(0,0) + \psi_0(y)F(0,1) + \psi_1(y)F_{0,1}(0,1)$$

$$(\frac{\partial P_2 F}{\partial x})(0,y) = y[\phi_0^{'}(y)F(0,0) + \phi_1^{'}(y)F_{0,1}(0,0) + \psi_0^{'}(y)F(0,1) + \psi_1^{'}(y)F_{0,1}(0,1)]$$

$$+ \varphi_0(y)F_{1,0}(0,0) + \varphi_1(y)[-F_{0,1}(0,0) + (\frac{\partial F_{0,1}(x,0)}{\partial x})x = 0]$$

$$+ \psi_0(y)[F_{1,0}(0,1) - F_{0,1}(0,1)] + \psi_1(y)[-F_{0,1}(0,1) + (\frac{\partial F_{0,1}(x,1-x)}{\partial x})x = 0]$$

The remainder function

$$(3.3) \qquad R = F - (P_1 \oplus P_2)F$$

is zero and has zero first derivatives on ¶T, except on y = 0

where

$$(3.4) \qquad R_{0,1}(x,0) = -\varphi_1(x)[(\frac{\partial}{\partial y}(\frac{\partial F}{\partial x}))(0,0) - (\frac{\partial}{\partial x}(\frac{\partial F}{\partial y}))(0,0)]$$

For this case it can be shown that

$$(3.5) \qquad (P_2 \oplus P_3)R = (P_3 \oplus P_2)R$$

$$= -\frac{x^2 y(x+y-1)^2}{x+y}[(\frac{\partial}{\partial y}(\frac{\partial F}{\partial x}))(0,0) - (\frac{\partial}{\partial y}(\frac{\partial F}{\partial x}))(0,0)].$$

14.

The function (3.5) clearly has the desired properties that

it is zero and its first derivatives are zero on ∂T, except on

y = 0 where its y partial derivative gives (3.4). Thus

(3.6) $(P_1 \oplus P_2)F + (P_1 \oplus P_2)R$

gives the compatibly corrected interpolant to $F \varepsilon C^1(\P T)$. The precision
set of this interpolant contains the set of all polynomials of
degree five and less.

The average of (3.6) and its dual for $(P_2 \oplus P_1)F$ gives a
more symmetric form on the standard triangle.

4. Interpolation to discrete boundary data.

   Smooth interpolation functions can be used to construct interpo-
lation schemes which involve only point functionals on $\P T$. (See for
example Barnhill and Gregory [2], and Birkhoff and Mansfield [4].)
This is achieved by defining the function and normal derivatives
along a side of the triangle T as polynomials which interpolate
the functionals on that side. This ensures continuity of the
function and derivatives across a side common to two adjacent triangles.
The requirement that $F \varepsilon C^{N-1}(\partial T)$, and that the interpolant has
the local support of the triangle, implies that the function F and
its partial derivatives of order N-1 and less must be specified
at each vertex. These 3N(N+1)/2 values are sufficient for the
modified smooth interpolation schemes of Theorem 2.4, but the
compatibility conditions of the unmodified scheme would impose
some additional higher order partial derivatives at each vertex.

The n'th normal derivative along a side can then "be defined
as   the polynomial which interpolates the 2(N-n) values given
at the ends of the side,i-e. the n'th normal derivative, and the
N-n-1 directional derivatives along the side of this n'th normal
derivative. The precision of the final interpolation function is
limited by the lowest precision achieved by the boundary data
interpolants and higher precision can be gained by including more
point functionals along the sides.

As an example, we consider the smooth C^1 interpolant of Section 3,
Defining the boundary functions of this interpolant as the cubic
polynomials which interpolate F , and its first directional derivative
along the side, at each vertex, and the normal derivatives as the
quadratic polynomials which interpolate the normal derivative at the
mid points of the sides and at each vertex, results in a twelve
parameter interpolation function with cubic precision. Alternatively
the normal derivative can be defined as the linear polynomials which
interpolate the normal derivative at each vertex, or equivalently a
linear variation on the normal derivative of the previous scheme can
be imposed. This gives a nine parameter interpolant with quadratic
precision.

It is important to note that the interpolation schemes for
discrete boundary data are not affine invariant and so cannot be
treated only on the standard triangle. This follows since no
direction is invariant under each of the affine transformations
which take two adjacent triangles onto the standard one, except
the direction along the common side. In particular, the normal

16.

derivative on the common side will in general be

transferred into two different and non-normal directions

by each of these affine transformations.

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