TR/28. OCTOBER 1973.

CUBIC SPLINE INTERPOLATION OF HARMONIC FUNCTIONS

BY

N. PAPAMICHAEL and J.R. WHITEMAN.

W9260748

ABSTRACT

It is shown that for the two dimensional Laplace equation a univariate cubic spline approximation in either space direction together with a difference approximation in the other leads to the well-known nine-point finite-difference formula.

For harmonic problems defined in rectangular regions this property provides a means of determining with ease accurate approximations at any point in the region.

1. Introduction

The use of spline approximations for partial differential equations, and their relationship to finite-difference schemes is considered by Hoskins (1970) and Sakai (1970). Hoskins considers the two dimensional Poisson equation and shows that a bivariate cubic Spline approximation leads to a nine-point difference formula In the more general work of Sakai multi-dimensional cardinal splines are used for the approximation of various elliptic and parabolic partial differential equations. In particular Sakai shows that his spline approximation for the two dimensional Poisson equation leads to the nine-point difference formula of Birkhoff, Schultz and Varga (1968). For the heat conduction equation in one space dimension the Sakai spline approximation produces a particular case of the difference scheme obtained by Papamichael and Whiteman (1973) in which a cubic spline approximation for the space derivative is combined with a difference approximation for the time derivative.

In the present paper a technique, similar to the above of Papamichael and Whiteman for heat conduction problems, is developed for harmonic problems in rectangular domains. In this the domain is covered with a square mesh, and it is shown that for the two dimensional Laplace equation a univariate cubic spline approximation in either space direction together with a standard central difference approximation in the other leads to the well-known nine-point difference formula. Solution of the resulting linear difference system produces a numerical approximation at each of the grid points. The cubic spline/difference replacement of the Laplace equation is then used to construct a doubly cubic spline which interpolates the solution of the discretised harmonic problem. We remark that the parameters which determine this spline are given at once in terms of known values at the grid points. Thus, once the finite-difference solution at the mesh points is determined, the technique does not require the solution of any other linear system.

Interpolation of harmonic functions is usually required when conformal transformation methods are used to solve numerically harmonic boundary value problems which have curved boundaries and/or contain boundary singularities. In particular the cubic spline technique described in the present paper may be used in conjunction with a conformal transformation method which maps the domain R of the problem onto a rectangle R'. In some cases the transformed harmonic problem in R' has a simple analytic solution which is determined by inspection; see Whiteman and Papamichael (1972). In general however, a standard finite-difference technique is used to determine the solution of the transformed problem at the grid points of a finite-difference mesh covering R'. The final solution, which is of course required at particular points of R', is then obtained by interpolation between the known values at the mesh points of R'. An example on the use of the cubic spline technique in conjunction with a conformal transformation method is given in Section 5.

2. Dirichlet Problems

To establish the relationship between the cubic spline/difference approximation of Laplace's equation and the nine-point difference formula we consider the harmonic Dirichlet problem,

$$\Delta u(x,y) = 0 \quad , \qquad (x,y) \; \varepsilon \; R, \tag{1}$$
$$u(x,y) = f(x,y), \qquad (x,y) \; \varepsilon \; \partial R,$$

In (1) Δ is the Laplacian operator, R is an open domain with a rectangular boundary ∂R so that

$$R \cup \partial R = \{(x,y): \quad 0 \le x \le a, \quad 0 \le y \le b\},$$

a and b being positive integers, and f(x,y) is a given function continuous on ∂R.

In order to discretise the problem we cover $RU\partial R$ with the square mesh,

$$(x_i, y_j) = (ih,jh), \quad i = 0,1, \text{--------}, n \; ; \; j = 0,1,\ldots.,m,$$

where nh = a and mh = b, and let $U_{i,j}$ be an approximation to u(x,y) at the point $(x_., y_.)$. We denote by $S_j(x)$ the cubic spline interpolating the values of $U_{.\;.}$ at the jth, j = 1,2,...,m - 1, mesh row and approximate Laplace's equation at the points (x_i,y_j) by

$$M_{i,j} = -\frac{1}{h^2}(U_{i,j-1} - 2U_{i,j} + U_{i,j+1}), \tag{2}$$

$$i = 1,2, \ldots., n-1 \; ; \; j = 1,2, \ldots.., m-1,$$

where $M_{.\;.} = s_j''(x_.)$. We assume that (2) is also satisfied when i = 0,n, and thus take

$$M_{i,j} = \frac{1}{h^2}(f_{i,j-1} - 2f_{i,j} + f_{i,j+1}) \quad , \tag{3}$$

$$i = 0,n \; ; \; j = 1,2, \ldots.,m - 1,$$

Where $f_{i,j} = f(X_i, Y_j)$.

For the jth, j = 1,2,,..., m - 1, row the results of Ahlberg, Wilson and Walsh (1967) show that

$$S_j(x) = M_{i-1,j}\frac{(x_i - x)^3}{6h} + M_{i,j}\frac{(x - x_{i-1})^3}{6h}$$

$$+ \left(U_{i-1,j} - \frac{h^2}{6}M_{i-1,j}\right)\frac{(x_i - x)}{h}$$

$$\left(U_{i,j} - \frac{h^2}{6}M_{i,j}\right)\frac{(x - x_{i-1})}{h} \tag{4}$$

$$x_{i-1} \le x \le x. \ , \ i = 1,2,...., n.$$

Hence,

$$S'_j(x_i +) = -\frac{h}{3}M_{i,j} - \frac{h}{6}M_{i+1,j} + \frac{U_{i+1,j} - U_{i,j}}{h} \ , \tag{5}$$

$i = 0,1,\ldots,n - 1,$

$$S'_j(x_i -) = \frac{h}{3}M_{i,j} + \frac{h}{6}M_{i-1,j} + \frac{U_{i,j} - U_{i-1,j}}{h} \ , \tag{6}$$

$i = 1,2, ..., n,$

so that continuity of the first derivatives implies

$$\frac{h}{6}M_{i-1,j} + \frac{2h}{3}M_{i,j} + \frac{h}{6}M_{i+1,j} = \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h} \ , \tag{7}$$

$i = 1,2, ,. . . , n-1.$

The elimination of the M's in (7), by means of equation (2), gives the finite-difference equation

$$D_h U_{i,j} = 0, \quad i = 1,2,\ldots\ldots\ldots\ldots,n - 1 \; ; \; j = 1,2,\ldots\ldots\ldots\ldots,m - 1, \tag{8}$$

where Dh is the nine-point difference operator defined by

$$\begin{aligned} D_h\, a(x,y) = {} & a\{x-h,y+h) + 4a(x,y+h) + a(x+h,y+h) \\ & + 4a(x-h,y) - 20a(x,y) + 4a(x+h,y) \\ & + a(x-h,y-h) + 4a(x,y-h) + a(x+h,y-h). \end{aligned}$$

We now let $T_i(y)$ be the cubic spline interpolating the values $U_{i,j}$ at the ith, i - 1,2,....,n - 1, mesh column and approximate Laplace's equation at (x_i,y_j) by

$$N_{i,j} = -\frac{1}{h^2}\left(U_{i-1,j} - 2U_{i,j} + U_{i+1,j}\right) , \tag{9}$$

$$i = 1,2,\ldots, n - 1 \; ; \; j = 1,2,\ldots, m - 1,$$

where $N_{i,j} = T''_i(y.)$. Then, corresponding to (3), (4) and (7) respectively, we have

$$N_{i,j} = -\frac{1}{h^2}\left(f_{i-1,j} - 2f_{i,j} + f_{i+1,j}\right) , \tag{10}$$

$$i = 1,2, \ldots, n - 1 \; ; \; j = 0, m,$$

$$T_i(y) = N_{i,j-1}\frac{(y_j - y)^3}{6h} + N_{i,j}\frac{(y - y_{j-1})^3}{6h}$$

$$+ \ (U_{i,j-1} - \frac{h^2}{6}N_{i,j-1})\frac{y_j - y}{h}$$

$$+ \ (U_{i,j} - \frac{h^2}{6}N_{i,j})\frac{y - y_{j-1}}{h} \ , \qquad (11)$$

$$Y_{j-1} \le Y \le Y_j \ , \quad J=1,2,\ldots,m,$$

and

$$\frac{h}{6}N_{i,j-1} + 2\frac{h}{3}N_{i,j} + \frac{h}{6}N_{I,J+1} = \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{h}$$

(12)

$j = 1,2,\ldots, m-1.$

The elimination of the N's in (12), by means of (9), gives again the equation (8). It is thus shown that the cubic spline/difference replacements (2) and (9) to the Laplace equation both produce the well-known nine-point difference approximation (8) which has local truncation error of order h^6 ; see e.g. Forsythe and Wasow (1960, p.194) .

3. Cubic Spline Interpolation.

The application of (8) at each internal mesh point, together with the boundary conditions of (1), leads to a positive definite diagonally dominant linear system of {(n-l)x(m-l)} equations which is solved for the unknowns $U_{i,j}$, i = 1(l)n - 1, j = 1(l)m - 1. Formulae (2)- (U) and (9)- (11) then produce the cubic splines $S_j.(x)$ and $T_i.(y)$ which approximate respectively the solution $u(x,y_j)$

at the jth mesh row and the solution $u(x_i, y)$ at the ith mesh column.

We note that,

$$D_h S_j(x) = D_h U(x, y_j) = 0, \qquad (13)$$
$$x_{i-1} \le x \le x_i, \; i = 2,3, \ldots, n-1; \; j = 2,3, \ldots, m-2,$$

and, (14)

$$D_h T_i(y) = D_h U(x_i, y) = 0,$$

$$y_{j-1} \le y \le y_j, \; j = 2,3, \ldots, n-1; \; j = 2,3, \ldots, m-2,$$

These follow at once from (4), (11) and (8) since,

$$D_h M_{i,j} = -{}^{1}/_{h^2}\{D_h U_{i,j-1} - 2D_h U_{i,j} + D_h U_{i,j+1}\} = 0,$$

$$i = 1,2, \ldots, n-1; \; j = 2,3, \ldots, m-2,$$

and,

$$D_h N_{i,j} = -{}^{1}/_{h^2}\{D_h U_{i-1,j} - 2D_h U_{i\,j} + D_h U_{i+1,j}\} = 0,$$

$$i = 2,3, \ldots, n-2; \; j — 1,2, \ldots, m-1.$$

We now describe a method for interpolating the solution of (1) at any point $(x,y) \varepsilon R$. For this we let $r_{i,j}$ be the square

$$r_{i,j} = \{(x,y): x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$$
$$i=1,2\ldots,n, ; j=1,2,\ldots,m,$$

and consider three procedures, each dealing with a different part of R.

Procedure I,

Used for $(x,y)\ \varepsilon\ r_{i,j}$, $i - 1,2,\ldots,n$; $j = 2,3,\ldots,m - 1$,

We determine the six values

$$T_{i-1}(y + kh) \quad , \qquad k = -1,0,1 \ ,$$
$$T_i(y + kh),$$

and hence, using the approximations of Section 2, we calculate $U(x,y)$ from the cubic spline interpolating the values $U(x_i,y)$, $i - 0,1,2,\ldots,n$. Thus, we take (see equation (4)) ,

$$U(x,y) = Q_1(x,y)$$

$$= M_{i-1}(y)\frac{(x_i - x)^3}{6h} + M_{i(y)}\frac{(x - x_{i-1})^3}{6h}$$
$$+ (T_{i-1}(y) - \frac{h^2}{6}M_{i-1}(y))\frac{(x_i - x)}{h}$$
$$+ (T_{i(y)} - \frac{h^2}{6}M_i(y))\frac{(x - x_{i-1})}{h} \quad ,$$
$$(x,y)\ \varepsilon\ r_{i,j} \ ,$$

where

$$M_i(y) = -\frac{1}{h^2}\{T_i(y-h) - 2T_i(y) + T_i(y+h)\},$$

with

$$T_0(y) = f(0,y) \quad \text{and} \quad T_n(y) = f(a,y).$$

We note that ,

$Q_1(x, y_j) = S_j(x)$ and $Q_1(x_i, y) = T_i(y)$.

Procedure II.

Used for $(x,y) \,\varepsilon\, r_{i,j}$, $i = 2,3,\ldots,n-1$; $j = 1,2,\ldots,m$.

The value of $U(x,y)$ is obtained by interchanging the roles of x and y in the technique of Procedure I. Thus, we determine the six values

$$S_{i-1}(x + kh) , \qquad S_j(x + kh) , \qquad k = -1,0,1,$$

and hence calculate $U(x,y)$ from

$$U(x,y) = Q_2(x,y)$$

$$= N_{j-1}(x)\frac{(y_j - y)^3}{6h} + N_j(x)\frac{(y - y_{j-1})^3}{6h}$$

$$+ \left(S_{j-1}(x) - \frac{h^2}{6}N_{j-1}(x)\right)\frac{(y_j - y)}{h}$$

$$+ \left(S_j(x) - \frac{h^2}{6}N_j(x)\right)\frac{(y - y_{j-1})}{h} , \quad (x, y) \in r_{i,j} ,$$

where,

$$N_j(x) = -{}^1/_{h^2}\{S_j(x-h) - 2S_j(x) + S_j(x+h)\} ,$$

with

$$S_0(x) = f(x,0) \text{ and } S_m(x) = f(x, b).$$

Again we note that,

$$Q_2(x, y) = S_j(x) \text{ and } Q_2(x, y) = T_i(y) .$$

The functions $Q_1(x,y)$ and $Q_2(x,y)$ are bicubic in each mesh square $r_{i,j}$ of the rectangles

$$R_1 = \{(x,y): \ 0 \le x \le a, \ h \le y \le b - h\} \quad ,$$

and

$$R_2 = \{(x,y): \ h \le x \le a - h, \ 0 \le y \le b\},$$

respectively. By use of the continuity properties of $S_j(x)$ and $T_j(y)$, it can be shown that $Q_1(x,y)$ is in $C^4{}_2(R_1)$ and $Q_2(x,y)$ is in $C^4{}_2(R_2)$. (By the terminology of Ahlberg et al.(1967, P-235) $C^n{}_r(R)$ is the family of functions $F(x,y)$ on R whose n th order partial derivatives, involving no more than r th order differentiation with respect to a single variable, exist and are continuous.) It follows that $Q_1(x,y)$ and $Q_2(x,y)$ are "simple double cubic splines" interpolating the values $U_{i,j}$ at the mesh points of R_1 and R_2 respectively; see Ahlberg et al. (1967, p.235-39). Also, it can be shown that,

$$Q_1(x,y) = Q_2(x,y), \qquad (x,y) \in R_1 \cap R_2 ,$$

and, by means of (13) and (14), that

$$D_h U(x,y) = 0, \qquad (x,y) \in r_{i,j} ,$$
$$i = 3,4, \ldots ,n - 2 \ ; \ j = 3, 4,\ldots ,m - 2. \tag{15}$$

<u>Procedure III .</u>

<u>Used for $(x,y) \in r_{i,j}$., $i = 1,n$; $j = 1,m$.</u>

We assume that (15) holds for $(x,y) \in R_1 \cap R_2$ and use this nine-point formula to express $U(x,y)$ in terms of values that can be interpolated by means of Procedure 1 or II. Thus, if

$$(x,y) = (ph,qh) \in r_{i,j} \quad , \quad i = 1,n \ ; \ j = 1,m,$$

we take

$$\begin{aligned} U_{p,q} = & - (4U_{p+\alpha,q} + U_{p+2\alpha,q}) \\ & - (4U_{p,q+\beta} - 20U_{p+\alpha, q+\beta} + 4U_{p+2\alpha, q+\beta}) \\ & - (U_{p,q+2\beta} + 4U_{p+\alpha, q+2\beta} + U_{p+2\alpha, q+2\beta}) \quad , \end{aligned}$$

where $U_{k,\ell} = U(kh,\ell h)$ and $\alpha = 1$ when $i = 1$, $\alpha = -1$ when $i = n$, $\beta = 1$ when $j = 1$ and $\beta = -1$ when $j = m$.

To summarize the above we note that the value $U(x,y)$ approximating the solution $u(x,y)$ at any point $(x,y) \in R$ may be calculated by using,

(a) Procedure I, if $(x.,y) \in R_1$,

(b) Procedure II, if $(x,y) \in r_{i,j}$, $i = 2,3,\ldots,n - 1$; $j = 1,m$,

(c) Procedure III, if $(x,y) \in r_{i,j}$, $i = 1,n$; $j = 1,m$,

or by using,

(a) Procedure II, if $(x,y) \in R_2$,

(b) Procedure I, if $(x,y) \in r_{i,j}$, $i = 1,n$; $j = 2,3, \ldots ,m-1$,

(c) Procedure III, if $(x,y) \in r_{i,j}$, $i = 1,n$; $j = 1,m$.

Since when $(x,y) \in R_1 \cap R_2$, $Q_1(x,y) = Q_2(x,y)$, it is clear that for any $(x,y) \in R$ both the above two methods produce the same result.

For $(x,y) \in R_1 \cup R_2$ approximations to $\dfrac{\partial u(x,y)}{\partial x}$ may be determined from,

$$\frac{\partial U(x,\ y)}{\partial x} = \frac{\partial Q_1(x,\ y)}{\partial x} = -M_{i-1}(y)\,\frac{(x_i - x)^2}{2h} + M_i(y)\,\frac{(x - x_{i-1})^2}{2h}$$

$$- (T_{i-1}(y) - \frac{h^2}{6}M_{i-1}(y))\,\frac{1}{h} + (T_i(y) - \frac{h^2}{6}M_i(y))\,\frac{1}{h}, \tag{16}$$

if $(x,y)\ \varepsilon\ r_{i,j},\ r_{i,j}\ \varepsilon\ R_1$, and from.

$$\frac{\partial U(x,y)}{\partial x} = \frac{\partial Q_2(x,y)}{\partial x} = -N'_{j-1}(x)\frac{(y_j - y)^3}{6h} + N'_j\ (x)\frac{(y - y_{j-1})^3}{6h} \tag{17}$$

$$- (S'_{j-1}(x) - \frac{h2}{6}N'_{j-1}(x))\,\frac{y_j - y}{h} + (S'_j\ (x) - \frac{h^2}{6}N'_j\ (x))\,\frac{(y - y_{j-1})}{h}\quad ,$$

if $(x,y)\ \varepsilon\ r_{i-j}$, $i = 2,3,$ ---------------, $n - 1$; $j = 1,m$. In (17)

$$N'_j\ (X) = -\frac{1}{h^2}\ \{S'_j\ (x\text{-}h)\text{-}\ 2S'_j\ (x) + S'_j\ (x\text{+}h)\}\ ,$$

and $S'_j.(x)$ is found by differentiating (4) with respect to x.

To determine an approximation to $\frac{\partial u(x,y)}{\partial x}$ at a point $(x,y)\varepsilon\ r_{i,j}$ $i = 1,n$; $j = 1,m$, we note that

$$D_h\,\frac{\partial u(x,\ y)}{\partial x} = 0, \qquad (x,y)\ \varepsilon\ r_{i,j} \tag{18}$$

$$i = 3,4,\ldots,n - 2\ ;\ j = 3,4,\ \ldots,m - 2.$$

we assume that formula (18) holds for any $(x,y)\varepsilon\ R_1 \cap R_2$ and use it, as in Procedure III, to express $\frac{\partial u(x,y)}{\partial x}$ in terms of values that can be calculated by means of (16) and (17).

Approximations to $\frac{\partial u(x,\ y)}{\partial x}$ are determined in a similar

manner from,

$$\frac{\partial U(x,\ y)}{\partial x} = \frac{\partial Q_2(x,\ y)}{\partial y},\ \text{if}(x,\ y) \in R_2,\ \frac{\partial U(x,\ y)}{\partial y} = \frac{\partial Q_1(x,\ y)}{\partial y},$$

$$\text{if } (x,y)\ \varepsilon\ r_{i,j}\ ,\quad i = 1,n\ ;\ j = 2,3,\ldots,m-1,$$

and by using,

$$D_h \frac{\partial U(x,\ y)}{\partial x} = 0,\ \text{if}(x,\ y) \in r_{i,\ j},\ i = 1,\ n;\ j = 1,\ m.$$

4. <u>Mixed Boundary Value Problems</u>

To illustrate the application of the technique to mixed boundary value problems we consider the problem (1) but on the side

$$\partial R^* = \{(0,y):\ 0 < y < b\}$$

of ∂R we replace the Dirichlet boundary condition by a Neumann condition. Thus, we consider the harmonic problem,

$$\begin{aligned} \Delta u(x,y) &= 0, && (x,y)\varepsilon\ R, \\ \frac{\partial u(x,y)}{\partial x} &= g(y), && (x,y)\varepsilon\ \partial R^*, \\ u(x,y) &= f(x,y), && (x,y)\varepsilon\ \partial R - \partial R^*, \end{aligned} \qquad (19)$$

where $g(y)$ is a given function continuous on ∂R^* .

All approximations and results of Section 2 still hold except (3) for which we have ,

$$M_{0,j} = -\frac{1}{h^2} (U_{0,j-1} - 2U_{o,j} + U_{o,j+1}) , \tag{20}$$

and,

$$M_{n,j} = -\frac{1}{h^2} (f_{n,j-1} - 2f_{n,j} + f_{n,j+1}) , \quad j = 1,2 , \ldots , m-1 ,$$

where the $U_{o,j}$, $j = 1(1)m - 1$, are not known and must be determined.

The boundary condition (19) is approximated at the point (0, y.) by

$$S'_j(0) = g_j , \qquad j = 1,2, \ldots ,m - 1,$$

or, on using (5), by

$$-\frac{h}{3}M_{0,j} - \frac{h}{6}M_{1,j} + \frac{U_{1,j} - U_{0,j}}{h} = g_j . \tag{21}$$

The elimination of the M's in (21), by means of (2) and (20), gives the finite-difference equation

$$2U_{o,j+1} + U_{1,j+1} - 10U_{o,j} + 4U_{1,j} + 2U_{o,j-1} + U_{1,j-1} = 6hg_j , \tag{22}$$

$$j = 1,2,\ldots ,m - 1 ,$$

approximating the solution of the problem at the boundary points $(0, y_j)$. The approximation (22) has local truncation error of order h^4 and is equivalent to using the difference approximation

$$\frac{1}{12h}\{U_{1,j+1} - U_{-1,j+1}) + 4(U_{1,j} - U_{-1,j}) + (U_{1,j-1} - U_{-1,j-1})\},$$

for the derivative in (19), in conjunction with the nine-point formula applied at the point $(0,y_j)$.

The application of (22) at the $(m - 1)$ boundary points $(0, y)$, $j = 1(1)m - 1$, and of the nine-point formula at the internal mesh points leads to a linear system of $n \times (m - 1)$ equations which is solved for the unknowns $U_{i,j}$, $i = 0(1)n - 1$; $j = 1(1)m - 1$. The technique of Section 3 is then used to interpolate $U(x,y)$ at any point $(x,y) \varepsilon R \cup \partial R^*$ from the values $U_{i,j}$ at the mesh points. However, for $(x,y) \varepsilon r_{1,j}$, $r_{1,j} \varepsilon R$, the determination of $U(x,y)$ from the double cubic spline $Q_1(x,y)$ requires the knowledge of

$$T_o(y) = U(0,y)$$

and

$$M_o(y) = -\frac{1}{h^2}\{U(0,y-h) - 2U(0,y) + U(0,y+h)\}.$$

Since $T_i(y)$ is not defined for $i = 0$, the unknowns $U(0,y)$ and $M_o(y)$ are determined as follows.

The boundary condition (19) is approximated at the point $(0,y)$ by

$$\frac{\partial Q_1(0,y)}{\partial x} = g(y)$$

or, on using (16),by

$$-\frac{h^2}{3}M_0(y) - \frac{h^2}{6}M_1(y) + T_1(y) - U(0, y) = hg(y). \tag{23}$$

Equation (23) together with

$$\frac{h^2}{6}M_0(y) + 2\frac{h^2}{3}M_1(y) + \frac{h^2}{6}M_2(y) - U(0, y) + 2T_1(y) - T_2(y) = 0, \tag{24}$$

then gives the two relations

$$3U(0,y) = {}^7/_6 h^2 M_1(y) + \frac{h^2}{3}M_2(y) + 5T_1(y) - 2T_2(y) - hg(y),$$

and

$$3h^2M_o(y) = -5h^2M_1(y) - h^2M_2(y) -6T_1(7) + 6T_2(y) - 6hg(y),$$

which express the unknowns $U(0,y)$ and $M_o(y)$ in terms of values that can be determined from the cubic splines $T_1(y)$ and $T_2(y)$. We remark that (24), which follows at once from the construction of $Q_1(x,y)$ and can be verified easily by means of (14), is the continuity relation which shows that

$$\frac{\partial Q_1(x_1-, y)}{\partial x} = \frac{\partial Q_1(x_1+, y)}{\partial x}$$

The application of the technique to problems with Neumann conditions on any of the other three sides of ∂R is clear.

5. Numerical Results,

Problem 1, (Dirichlet Problem)

$$\Delta u(x,y) = 0, \qquad (x,y)\varepsilon\ R,$$
$$u(x,y) = \cos x \sinh y, \qquad (x,y)\varepsilon\ \partial R,$$

where, $R \cup \partial R = \{(x,y) : 0 \le x \le 1, \quad 0 \le y \le 1\}$.

A square mesh of size $h = 0.2$ is used. The finite-difference solution computed by applying the nine-point formula at the interior mesh points,

$$(0.2i,\ 0.2j)\ , \qquad i = 1(1)4 \quad ; \quad j = 1(1)4 \quad ,$$

is accurate to eight significant figures.

Numerical results obtained by the cubic spline technique at the points

$$(\ 0.1i,\ 0.1j)\ , \qquad i = 1(1)9 \quad ; \quad j = 1(2)9 \quad ,$$

are given in Table 1. They are compared with:

(i) values computed from the analytic solution

and,

(ii) values obtained by interpolating the results at the mesh points using the bivariate interpolation formula,

$$\begin{aligned}\tilde{F}(x_i + ph,\ y_j + qh) = {} & \frac{q(q-1)}{2}F_{i,j-1} + \frac{p(p-1)}{2}F_{i-1,j} \\ & + (1 + pq - p^2 - q^2)F_{i,j} + \frac{p(p-2q+1)}{2}F_{i+1,j} \\ & + \frac{q(q-2p+1)}{2}F_{i,j+1} + pq\ F_{i+1,j+1}, \end{aligned} \tag{25}$$

$$|\,p\,| < 1. \quad |\,q\,| < 1,$$

of Abramowitz and Stegun (1965, eqn.25.2.67). Formula (25), which determines an approximation $\widetilde{F}(x,y)$ to $F(x,y)$ in terms of values of $F(x,y)$ at six grid points of a square mesh, has truncation error

$$e_{p,q} = F(x_i + ph, y_j + qh) - \widetilde{F}(x_i + ph, y_j + qh) ,$$

of order h^3. However, when $F(x,y)$ is harmonic and $p = q = ½$ the order of the truncation error rises to h^4. In particular with,

$$F(x,y) = u(x,y) = \cos x \sinh y ,$$

and $h = 0.2$,

$$e_{\frac{1}{2},\frac{1}{2}} \underline{\Omega} - \frac{1}{32} h^4 \frac{\partial^4 u}{\partial x^2 \partial y^2} = 0.5 \;\; 10^{-4} \cos x \;\; \sinh y$$

whereas,

$$e_{0,\frac{1}{2}} \underline{\Omega} - \frac{1}{16} h^3 \frac{\partial^3 u}{\partial y^3} = -0.5 \;\; 10^{-3} \cos x \;\; \cosh y.$$

The last two equations explain why the results obtained "by the use of (25) at the points,

$$(0.11, \; 0.1j) \; , \; i = 1(2)\,9 \; 5 \; j = 1\,(2)\,9 ,$$

are much more accurate than those obtained at the points,

$$(0.2i, \; O.lj) \; , \; i = 1(1)4 \; ; \; j = 1(2)9.$$

<u>Problem 2</u> (Mixed Boundary Value Problem)

$$Au(x,y) = 0 , \qquad (x,y) \, \varepsilon \, R ,$$

$$\frac{\partial u(x, y)}{\partial x} = \cosh y, \qquad (x, y) \in \partial R^*,$$

$$u(x,y) = \sin x \cosh y, \qquad (x,y) \, \varepsilon \, \partial R - \partial R^* ,$$

where

$$\partial R^* = \{(0,y) : 0 < y < 1\} ,$$

and,

$$R \cup \partial R = \{(x,y) : 0 \le x \le 1; 0 \le y \le 1\}.$$

A square mesh of size h = 0.2 is used. The finite-difference solution computed by applying the nine-point formula at the interior mesh points,

$$(0.2i , 0.2j) , \quad i = 1(1)4 ; j = 1(1)4,$$

and formula (22) at the boundary points

$$(0, 0.2j) , \quad j - 1(1)4,$$

is accurate to five significant figures.

Numerical results obtained by the cubic spline technique at the points

$$(0.1i , 0.1j) , \quad i = 0(1)9 ; j = 1(2)9,$$

are given in Table 2 and are compared with values computed from the analytic solution.

To illustrate the use of the cubic spline technique in conjunction with a conformal transformation method we consider the following harmonic problem.

Problem 3.

$$\Delta u(x,y) = 0, \qquad (x,y) \varepsilon R,$$

$$u(x,y) = 1, \qquad (x,y) \varepsilon \partial R_1,$$

$$u(x,y) = 0, \qquad (x,y) \varepsilon \partial R_2,$$

where R is the semi-circular open domain

$$R = \{(x,y) : x^2 + y^2 < 1 , |x| < 1, y > 0\} ,$$

with boundary $\partial R = \partial R_1 \cup \partial R_2$ where,

$$\partial R_1 = \{(x,y) : x^2 + y^2 = 1 , jx| < 1, y > 0\} ,$$

and $\partial R_2 = \overline{ABC} - \{(x,0) : |x| < 1\}$,

with A = (1,0), B = (0,0) and C =(-1,0).

The three successive conformal transformations,

$$z = \left\{\frac{1 + w}{1 - w}\right\}^2 , \tag{26}$$

$$t = \frac{1}{1 - z} , \tag{27}$$

and

$$w' = \frac{1}{k(\frac{1}{\sqrt{2}})} sn^{-1} (t^{\frac{1}{2}}, \frac{1}{\sqrt{2}}), \tag{28}$$

where sn denotes the Jacobian elliptic sine and K(t//2) is the complete elliptic integral of the first kind with modulus $1/\sqrt{2}$,

are used to map $G = R \cup \partial R$ in the $w = x + iy$ - plane onto the square

$$G' = R' \cup \partial R' = \{(\xi, n) : o \leq \xi , \leq 1\} ,$$

in the $w' = \xi + in$ - plane with

w − plane	w′ − plane
A $\longrightarrow$	(0,0),
B $\longrightarrow$	(0,1),
C $\longrightarrow$	(1,0).

Thus the combined effect of (26) , (27) and (28) is to transform the original problem in G into the problem

$$\begin{aligned} \Delta v(\xi,n) &= 0 , & (\xi,n) &\varepsilon R' , \\ v(\xi,n) &= 1 , & (\xi,n) &\varepsilon \partial R'_1 , \\ v(£,n) &= 0 , & (\xi,n) &\varepsilon \partial R'_2 , \end{aligned} \tag{29}$$

in $G' = R' \cup \partial R'_2$ where $\partial R' = \partial R'_1 \cup \partial R'_2$ and

$\partial R'_1 = \{(\xi,0) : 0 < \xi < 1\}$, $\partial R'_2 = \partial R' - \partial R'_1$
Full computational details for the implementation of the above conformal transformation method are given in Whiteman and Papamichael (1972).

The boundary values of the transformed problems have jump discontinuities at the corners (0,0) and (1,0) of $\partial R'$. A detailed discussion on techniques for removing such discontinuities from the boundary conditions of harmonic problem, in rectangular regions is given by Rosser (1973). In the present case we introduce the harmonic function

$$V(\varepsilon,n) = v(\varepsilon,n) - \frac{2}{\pi}\{\arctan(\frac{\varepsilon}{\eta}) + \arctan(\frac{1-\varepsilon}{\eta})\}, \tag{30}$$

and instead of (29) we consider the problem

$$\begin{aligned} \Delta V(\xi,n) &= 0 , & (\xi,n) &\varepsilon R' , \\ V(\xi n) &= -1 , & (\xi,n) &\varepsilon \partial R'_1 , \\ V(£,n) &= f(\xi,n), & (\xi,n) &\varepsilon \partial R'_2 , \end{aligned} \tag{31}$$

where

$$f(\varepsilon,\eta) = -\frac{2}{\pi}\{\arctan(\frac{\varepsilon}{\eta}) + \arctan(\frac{1-\varepsilon}{\eta})\}.$$

The nine-point formula is used to determine approximations $\tilde{V}_{i,j}$ to the solution of (31) at the mesh points

$$(0.2i, 0.2j) \quad , \quad i = 1(1)1+ \quad ; \quad j = 1(1)\ 4,$$

of a square mesh, of size h = 0.2, covering G'. The solution of the original problem is of course required in R. To determine an approximation to this solution at a specific point P = (x,y) ε R, P is mapped into the point $p' \varepsilon R'$. Since p' will not in general be a point of the finite-difference mesh covering G', the cubic spline technique is used to compute an approximation to $V(p')$ in terms of the known values $\tilde{V}_{i,j}$ at the mesh points in G'. An approximation to $u(P) = v(p')$ is then found by means of (30).

Numerical approximations to u(x,y) obtained by using the conformal transformation method (CTM) in conjunction with the cubic spline technique are given in Table 3. The results are given at the points,

$$(0.1i, 0.1j) \quad , \quad i = 0(1)9 \quad ; \quad j = 1(2)9,$$

of the quadrant

$$\{(x,y) : \quad x^2 + y^2 < 1, \quad x > 0, \quad y > 0\} \quad ,$$

of R and are compared with

(i) values computed from the analytic solution,

$$u(x,y) = \arctan\{2y/(1 - x^2 - y^2)\},$$

and,

(ii) values computed by using formula (25) to interpolate the finite-difference approximations $\tilde{V}_{i,j}$ in G'.

We remark that when a square mesh of size $h = 0.1$ is used to determine the finite-difference solution of (31) and to perform the cubic spline interpolation in G', the approximations $U(x,y)$ to $u(x,y)$ computed at the points of Table 3 are such that

$$|u(x,y) - u(x,y)| \leq 2 \times 10^{-6}.$$

At each point the numbers represent:

(i) <u>Upper entry</u>: Value computed, by Spline Interpolation

(ii)<u>Middle entry</u>: Value computed from the Analytic Solution

(iii) <u>Lover entry</u>: Value computed by using Interpolation Formula 25.

y		x = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	Upper	1.021361	1.006051	0.980663	0.945481	0.900848	0.847217	0.785119	0.715178	0.638074
	Middle	1.021388	1.006055	0.980669	0.945485	0.900853	0.847221	0.785123	0.715181	0.638093
	Lower	1.021342	1.006723	0.980624	0.946112	0.900811	0.847357	0.785086	0.715656	0.638062
0.7	Upper	0.754789	0.743459	0.724697	0.698699	0.665714	0.626084	0.580192	0.528508	0.471540
	Middle	0.754794	0.743463	0.724703	0.698702	0.665720	0.626086	0.580197	0.528510	0.471543
	Lower	0.754761	0.744053	0.724670	0.699257	0.665689	0.626277	0.580170	0.528930	0.471521
0.5	Upper	0.518489	0.510706	0.497817	0.479959	0.457300	0.430077	0.398552	0.363049	0.323916
	Middle	0.518492	0.510708	0.497821	0.479961	0.457304	0.430079	0.398556	0.363051	0.323918
	Lower	0.518470	0.511244	0.4 97800	0.480464	0.457284	0.430333	0.398538	0.363432	0.323903
0.3	Upper	0.302997	0.298449	0.290917	0.280481	0.267239	0.251330	0.232908	0.212160	0.189292
	Middle	0.302999	0.298450	0.290919	0.280482	0.267242	0.251331	0.232910	0.212161	0.189293
	Lower	0.302987	0.298954	0.290908	0.280955	0.267231	0.251659	0.232901	0.212519	0.189285
0.1	Upper	0.099662	0.098170	0.095692	0.092259	0.087904	0.082671	0.076611	0.069787	0.062262
	Middle	0.099666	0.098170	0.095693	0.092260	0.087905	0.082671	0.076612	0.069787	0.062265
	Lower	0.099658	0.097671	0.095685	0.091791	0.087897	0.082155	0.076605	0.069432	0.062259

<u>TABLE 1</u>

At each point the numbers represent:

(i) Upper entry:
Value computed by Spline Interpolation

(ii) Lower entry:

Value computed from the Analytic Solution,

0.9	0.000000	0.143062	0.284708	0.423503	0.558067	O.687054	0.809178	0.923214	1.028029	1.122545
	0.000000	0.143070	0.284710	0.423506	0.558070	0.687058	0.809181	0.923220	1.028033	1.122575
	-0.000004	0.125304	0.249361	0.370923	0.488783	0.601754	0.708718	0.808595	0.900399	0.983202
0.7	0.000000	0.125308	0.249364	0.370928	0.488786	0.601760	0.708722	0.808602	0.900403	0.983208
	-0.000004	0.112571	0.224022	0.333232	0. 439115	0.540607	0.636702	0.726430	0.808906	0.883294
0.5	0.000000	0.112575	0.224025	0.333236	0.439118	0.540612	0.636706	0.726437	0.808909	0.883300
	-0.000003	0.104357	0.207674	0.308915	0.4 07071	0.5401157	0.59024	0.673420	0.74 9877	0.818837
0.3	0.000000	0.104360	0.207677	0.308919	0.407074	0.501162	0.590243	0.673426	0.749880	0.818842
	0.000000	0.100328	0.199662	0.196997	0.391365	0.4 81821	0.567466	0.647437	0.720943	0.787225
0.1	0.000000	0.100333	0.199664	0.296999	0.391367	0.4 81825	0.567468	0.647441	0.720946	0.787247
0	0.1	0.2	0.3	04	0.5	0.6	0.7	0.8	0.9	1.0

TABLE 2.

TABLE 3

At each point the numbers were computed from;

(i) Upper entry: CTM and Cubic Spline Interpolation.

(ii) Middle entry: Analytic Solution.

(iii) Lover entry: CTM and Interpolation Formula (25).

Y \ X	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	0.933048 0.933049 0.932845	0.936548 0.936549 0.936650	0.947070 0.947071 0.947007	0.964668 0.964669 0.964195	0.989390 0.989391 0.988604					
0.7	0.777599 0.777600 0.777724	0.781623 0.781624 0.782069	0.793802 0.793804 0.793932	0.814450 0.814453 0.814371	0.844039 0.844042 0.843836	0.883101 0.883102 0.882845	0.932049 0.932050 0.931798	0.990906 0.990906 0.990701		
0.5	0.590334 0.590334 0.590285	0.594428 0.594428 0.594386	0.606946 0.606947 0.606917	0.628611 0.628613 0.628622	0.660658 0.660660 0.660775	0.704830 0.704833 0.705133	0.763263 0.763269 0.763553	0.838057 0.838064 0.838015	0.930249 0.930252 0.929809	
0.3	0.371093 0.371094 0.370909	0.374329 0.374334 0.374392	0.384354 0.384359 0.384217	0.402145 0.402147 0.401971	0.429551 0.429553 0.429615	0.469692 0.469708 0.469804	0.527631 0.527662 0.527675	0.611167 0.611200 0.611050	0.730782 0.730803 0.730423	0.894857 0.894863 0.894188
0.1	0.126906 0.126902 0.126880	0.128156 0.128162 0.128021	0.132089 0.132096 0.132744	0.139202 0.139209 0.139110	0.150529 0.150533 0.150448	0.168035 0.168045 0.168123	0.195649 0.195695 0.195759	0242178 0.242238 0.242282	0.330453 0.330499 0.330466	0.533457 0.533475 0.533287

References

1. ABRAMOWITZ,M. and STEGUN,I.A.: Handbook of mathematical functions. New York: Dover 1965.

2. AHLBERG, J.H., NILSGN,E.N. and WALSH,J.L.: The theory of splines and their applications London: Academic Press 1967.

3. BIRKHOFF,G., SCHULTZ ,M.H. and VARGA,R.S.: Piecewise Hermite interpolation in one and two variables with application to partial differential equations, Numer.Math. 11 , 232-256 (1968).

4. FORSYTHE, G.E. and WASOW,W.R.: Finite-difference methods for partial differential equations. New York: Wiley 1960.

5. HOSKINS, W.D.: Studies in spline approximation and variational methods. Doctoral Thesis, Brunei University 1970.

6. PAPAMICHAEL, N. and WHITEMAN, J.H.: A cubic spline technique for the one dimensional heat conduction equation, J.Inst.Maths Applies. 11, 111-113 (1973).

7. ROSSER, J.B.: Finite-difference solution of Poisson's equation in rectangles of arbitrary shape. Technical Report TR/27, Dept. of Mathematics, Brunei University 1973.

8. SAKAI, M. :Multidimensional cardinal spline function and its applications. Memoirs of Facility of Sciences, Kyushu University, Series A, XXIV, 40-46 (1970).

9. WHITEMAN,J.R. and PAFAMICHAEL,N.: Treatment of harmonic boundary value problems containing singularities by conformal transformation methods. Z.angev.Math.Phys.23, 655-664(1972) .