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CONTINUED FRACTION SOLUTIONS

OF

LINEAR DIFFERENTIAL EQUATIONS

by

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CONTINUED FRACTION SOLUTIONS OF

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INTRODUCTION

We will develop a general theory to unify a number of the results presented in TR/25, and which will enable us to discuss more complex examples. The theory is a refinement of a paper by E. Laguerre published in 1885. First we construct the theory for J fractions for a certain class of functions, and then we show that the theory readily extends to deal with M fractions. Although numerically we can generate both J fractions and M fractions from suitable series for a function, in general, depending on the number of significant figures carried in the computation, the rounding errors eventually terminate the procedure. The purpose of this paper then is to develop the theory so that these fractions may be obtained directly from more compact descriptions of the functions, in our case differential equations. We will construct relations which successively generate coefficients of the C.Fs. Further we indicate by examples how asymptotic formulae for the coefficients in a C.F can be constructed, and how an almost periodic C.F is approximating the positions of the branch points. Two notions that generalise.

The theory as presented is very limited dealing only with second order linear differential equations which are, or can be reduced to, equations in which one solution is a polynomial. Nevertheless the results can be extended certainly numerically to other differential equations closely related to these. We briefly indicate this aspect, but our work is by no means complete.

The method of constructing our theory is directly applicable to a variety of first order problems. We conclude by deriving the J fraction for the Laplace transform of a function.

This paper is a sequel to TR/25, and relations in that technical report are referenced directly. We recall that in TR/25, we showed that the f_i in the linear relations

$$f_1 = d_1 f + c_1(-1)$$

$$f_2 = d_2 f_1 + c_2 f$$

$$\cdots\cdots$$

$$f_n = d_n f_{n-1} + c_n f_{n-2}$$

where c_i d_i are polynomials in x,
can be expressed as rational functions by terminating the C.F

$$f_{i-1} = \frac{A_i}{D_i +} \frac{D_{i-1} c_{i+1}}{d_{i+1} +} \cdots \frac{c_n}{+d_n -} \frac{f_n}{f_{n-1}} ,$$

where A. = $(-1)^{i-1}$ $c_1 c_2$ ... c_i and D. is a polynomial.
In particular the n^{th} approximant to f is

$$f_{o/n} = \frac{c_1}{d_1 +} \frac{c_2}{d_2 +} \cdots \frac{c_n}{+d_n} = \frac{c_n}{D_n},$$

the error being

$$f - \frac{C_n}{D_n} = \frac{f_n}{D_n} .$$

11. Laguerre's Problem

Apart from some minor refinements which we will discuss later, Laguerrre in his paper (1885) set out to determine and study the J fraction.

$$c_o(x) + \frac{c_1}{x+d_1 -} \, \frac{c_2}{x+d_2 -} \cdots \frac{c_n}{x+d_n -} \qquad (11.1)$$

for a function f that satisfies a first order differential equation

$$w \frac{df}{dx} = 2vf + U , \qquad (11.2)$$

where U, V, W and $c_0(x)$ denote polynomials in x.

To do this, set

$$f = \frac{C_n}{D_n} + \frac{f_n}{D_n} \qquad (11.3)$$

and choose the polynomials C_n (x) and D_n (x) so that their ratio $\frac{C_n}{D_n}$ matches the terms in the series for f up to and including the term $\frac{1}{x^{2n}}$. Then

$$w\frac{d}{dx}\left[\frac{c_n}{D_n}\right] - 2v\left[\frac{c_n}{D_n}\right] - U = \left\{ - w\frac{\partial}{dx}\left[\frac{f_n}{D_n}\right] + 2v\left[\frac{f_n}{D_n}\right] \right\} \qquad (11.4)$$

$$= A_{n+1} \left\{ w\frac{(2n+1)}{x^{2n+2}} + \frac{2v}{x^{2n+1}} + \text{lower terms} \right\} .$$

as $\frac{f_n}{D_n} = \frac{A_{n+1}}{x^{2n+1}} + \ldots .$

Performing the differentiation

$$W\left[D_n C_n' - C_n D_n'\right] - 2vC_n D_n - UD_n^2 = A_{n+1} D_n^2 \{ \qquad \}$$

$$= A_{n+1} \theta_n \qquad (11.5)$$

where θ_n (x) is a polynomial, for the left hand side is a polynomial.

Further θ_n can be regarded as being of fixed degree μ, the degree of the highest term in $W\left(\frac{1}{x^2}\right) + 2V\left(\frac{1}{x}\right)$, only its coefficients depend on n.

Thus we can now split (11.4) into two equations ,

$$W \frac{d}{dx}\left[\frac{C_n}{D_n}\right] - 2v\left[\frac{C_n}{D_n}\right] - U = \frac{A_{n+1}\,\theta_n}{D_n^2}\ , \qquad (11.6)$$

$$W \frac{d}{dx}\left[\frac{f_n}{D_n}\right] - 2v\left[\frac{f_n}{D_n}\right] = -\frac{A_{n+1}\,\theta_n}{D_n^2}\ . \qquad (11.7)$$

The. latter is particularly important because of the second order differential equations that we will derive from it, and because integrated it gives the error in approximating f by $\frac{C_n}{D_n}$.

To construct (11.1) we must derive the recurrence relations between the f_n and this we can do in the following manner.

Recurrence Relations for D_n and f_n .

Replacing n in (11.6) by (n—1) and subtracting the result from (11.6) gives

$$W \frac{d}{dx}\left[\frac{C_n}{D_n} - \frac{C_{n-1}}{D_{n-1}}\right] - 2V\left[\frac{C_n}{D_n} - \frac{C_{n-1}}{D_{n-1}}\right] = \frac{A_{n+1}\,\theta_n}{D_n^2} - \frac{A_n\,\theta_{n-1}}{D_{n-1}^2}\ .$$

As $C_n D_{n-1} - C_{n-1} D_n = A_n$,

$$W \frac{d}{dx}\left[\frac{A_n}{D_n\ D_{n-1}}\right] - 2V\left[\frac{A_n}{D_n\ D_{n-1}}\right] = \frac{A_{n+1}\,\theta_n}{D_n^2} - \frac{A_n\,\theta_{n-1}}{D_{n-1}^2}\ .$$

But $A_{n+1} = c_{n+1} A_n$, so that

$$W\left[D'_n D_{n-1} - C'_{n-1} D_n\right] - 2VD_n D_{n-1} = - - c_{n+1} = \theta_n D^2_{n+1} + \theta_{n+1} D^2_n$$

which rearranged gives

$$[WD'_n \;;+VD_n + c_{n+1}\ \theta_n\ D_{n-1}]\ D_{n-1} = [W\ D'_{n-1} + V\ D_{n-1} - \theta_{n-1}\ D_n]\ D_n \quad (11.8)$$

These are two expressions for the same polynomial. D_n and D_{n-1} are

clearly factors of that polynomial. We can therefore equate both sides to $\Delta_n\ D_n\ D_{n-1}$. where $\Delta_n(x)$ is a polynomial.

This yields the two relations

$$W D'_n + (V - \Delta_n) D_n + c_{n+1}\ \theta_n\ D_{n-1} = 0, \quad (11.9)$$

$$WD'_{n-1} + (V + \Delta_n)\ D_{n-1} \qquad \theta_{n-1}\ D_n = 0 \ . \quad (11.10)$$

To eliminate D we step n by one

$$W\ D'_n + (V+\Delta_{n+1})D_n - \theta_n\ D_{n+1} = 0 \ . \quad (11.11)$$

and subtract (11.11) - (11.9)

$$D_{n+1} = \frac{1}{\theta_n}\ (\ \square_{n+1} + \square_n)\ D_n - c_{n+1}\ D_{n-1}, \quad (11.12)$$

the required recurrence relation between the denominator polynomials. To obtain the recurrence relation between the f_n , we simply multiply by f_n and use the result (11.13) , $D_n\ f_{n-1}$,- D_{n-1}, $f_n = A_n$,

$$D_{n+1}\ f_n = \frac{1}{\theta_n}\ .\square_{n+1} + \square_n)\ D_n\ f_n - c_{n+1}\ [D_n\ f_{n-1}]$$

$$D_n\ f_{n+1} = \frac{1}{\theta_n}\ .\square_{n+1} + \square_n)\ D_n\ f_n - c_{n+1}\ [D_n\ f_{n-1}]$$

$$\therefore \quad f_{n+1} = \frac{1}{\theta_n}\ .(\square_{n+1} + \square_n)\ f_n - c_{n+1}\ f_{n-1} \quad (11.13)$$

We can also deduce relations similar to (11.9) and (11.11) for the f_n , we simply state them

$$W\ f'_n - (\square_n + v)\ f_n + c_{n+1}\ \theta_n\ f_{n-1} = 0 \quad (11.14)$$

$$W\ f'_n + (\square_{n+1} - V)\ f_n - \theta_n\ f_{n+1} = 0 \ . \quad (11.15)$$

Let us elaborate on the way (11.13) generates (11.1).
To solve Laguerre's problem, given

$$W\frac{df}{dx} = 2Vf + U$$

first we extract the polynomial c_o (x) part of the solution

$$f = c_0(x) + f_o$$

so that $\quad W\frac{df_o}{dx} = 2vf_o - c_1\,\theta_o \quad \times$

In fact □$_o$ is V, so rewriting this equation as

$$\theta_o f_1 = W\frac{df_o}{dx} + (\Delta_1 - v)\,f_0 = (\Delta_1 + V)\,f_o - c_1\,\theta_o \tag{11.16}$$

we get $f_1 = \frac{1}{\theta_o}\,(\Delta_1 + V)\,f_o - c_1$

$$f_2 = \frac{1}{\theta_o}\,(\Delta_2 + \Delta_1)\,f_1 - c_2\,f_o$$

etc.

These linear equations are equivalent to (11.1). All that remains is to deduce a convenient means of generating successively the c_n and the polynomials θ_n (x) and Δ_n (x).

But first we digress and note that $D_n(x)$ and f_n (x) are essentially the two solutions of some interesting second order differential equations.

* NOTE if $\theta_0 \neq$ constant U cannot be chosen arbitrarily

$$U = W\frac{dc_o}{dx} - 2Vc_o - c_1\,\theta_o$$

Second Order Differential Equations

We consider the two cases θ_n=constant and $\theta_n \neq$ constant separately.

$$Wf_n' \; D_n - W \; f_n \; D_n' - 2V \; f_n \; D_n' = - A_{n+1} \; \theta_n, \tag{11.17}$$

this follows from equation (11.7). Differentiating we get

$$(wf_n' - 2Vf_n) \; D_n' + \frac{d}{dx} + [Wf_n' - 2Vf_n] \; D_n +$$

$$- Wf_n \; D_n' - W'f_n \; D_n' - Wf_n' \; D_n' \; A_{n+1} \; \theta_n' \tag{11.18}$$

For θ_n = constant, $\theta_n' = 0$

$$\therefore \; W \; D_n' + (W' + 2V) \; D_n' - \frac{1}{f_n} \frac{d}{dx} [W \; f - 2V \; f_n] \; D_n = 0$$

which reduces to

$$W \; D_n' + (W' + 2V) \; D_n' + K_n \; D_n = 0 \; , \tag{11.19}$$

where $K_n(x)$ is necessarily a polynomial of fixed degree. In addition

$$\frac{d}{dx}[W \; f'_n - 2Vf_n] + K_n f_n = 0 \tag{11.20}$$

Now put $f_n = e^{\int \frac{2v}{W}dx.} y$,

as $Wf_n' = 2Ve^{\int} y + We^{\int} y'$, we find

$$\frac{d}{dx}\left[We^{\int \frac{2V}{W}dx} y'\right] + k_n e^{\int \frac{2V}{W}dx} y = 0.$$

Hence the second order differential equation

$$\underline{W \; y'' + (W' + 2v) y' + k_n y = 0} \tag{11.21}$$

is satisfied by both D_n and $e^{-\int \frac{2V}{W}dx} f_n$.

Alternatively we could verify that $e^{\int \frac{2v}{w}dx}.D_n$ as well as f_n satisfied (11.20).

<u>For $\theta_n \neq$ constant,</u> we use (11.17) to eliminate A_{n+1} in 11.18), then

$$W D_n' + (W' + 2v - W\frac{\theta_n'}{\theta_n}) D_n' - \frac{1}{f_n}\left\{\frac{d}{dx}[W f_n' - 2V f_n] - (W f_n' - 2V f_n)\frac{\theta_n'}{\theta_n}\right\} D_n = 0$$

$$W \theta_n D_n'' + [(W' + 2V)\theta_n - W \theta_n'] D_n' + k_n^* D_n = 0 , \qquad (11.22)$$

where $K_n^*(x)$ is a polynomial of fixed degree.

The appropriate generalisation of (11.21) is that the equation

$$\underline{W \theta_n y'' + [(w' + 2V)\theta_n - W \theta_n^*] y' + k_n^* 0} \qquad (11.23)$$

has the complete solution

$$y = A D_n(x) + B e^{\int \frac{2V}{W}dx} f_n \qquad (11.24)$$

<u>Interrelation between $\theta_n \Delta_n$ and K_n.</u>

In the preceeding analysis we have introduced three polynomials of fixed degrees θ_n , D_n and K_n. To obtain the relation between them we now derive (11.22) directly from (11.9) and (11.10). Eliminating D_{n-1} from (11.10) using (11.9)

$$-c_{n+1} \theta_n W D'_{n-1} + (V + D_n) W D_n' + [V^2 - D_n^2 + c_{n+1} \theta_n \theta_{n-1}] D_n = 0$$

W must be a factorof the polynomial in the square brackets, [ ] = W S_n say,

$$\therefore\ -c_{n+1} \theta_n D_{n-1}' + (V + \underline{\Delta_n}) D_n^* + S_n D_n = 0 . \qquad (11.25)$$

Substituting (11.9)

$$\theta_n \frac{d}{dx}\left[\frac{W D_n' + (V - \Delta_n) D_n}{\theta_n}\right] + (V + \Delta_n) D_n' + S_n D_n = 0$$

and rearranging, we find

$$w\,\theta_n D_n'' + [(w' + 2v)\,\theta_n - w\,\theta_n']\,D_n' + [(v' - \Delta_n + s_n)\,\theta_n - (v - \Delta_n)\,\theta'_n]\,D_n = 0\,. \tag{11.26}$$

With θ_n=constant, comparing this equation with (11.19)

$$\underline{k_n = V' - \Delta_n' + S_n}, \tag{11.27}$$

Where $\underline{W\,S_n = V^2 - \Delta_n^2 + c_{n+1}\theta_n\theta_{n-1}}$ $\quad n \geq 1.$ (11.28)

with $\theta_n \neq$ constant, comparing with (11.19) gives

$$\underline{k_n^* = (V' - \Delta_n' + s_n)\theta_n - (V - \Delta_n)\theta_n'} \tag{11.29}$$

Connection between θ_n and S_n

$$W\,S_n = V^2 - \Delta_n^2 + c_{n+1}\,\theta_n\,\theta_{n-1}\,.$$

$$W\,S_{n+1} = V^2 - \Delta^2_{n+1} + c_{n+2}\,\theta_{n+1}\,\theta_n\,,$$

subtracting

$$w(s_n - s_{n+1}) = \Delta^2_{n+1} - D^2_n + c_{n+1}\,\theta_n\,\theta_{n-1} - c_{n+2}\,\theta_{n+1}\,\theta_n\,. \tag{11.30}$$

With our definitions $\theta_n(x + d_{n+1}) = \Delta_{n+1} + \Delta_n$,

$$\therefore\ W(S_n - S_{n+1}) = \theta_n[(x + d_{n+1})(\Delta_{n+1} - \Delta_n) + C_{n+1}\,\theta_{n-1} - c_{n+2}\,\theta_{n+1}], \tag{11.31}$$

and θ_n is therefore a factor of $w(S_n - S_{n+1})$.

In fact as Laguerre shows the square bracket is W, that is for $n \geq 1$

$$\underline{(x + d_{n+1})(\Delta_{n+1} - \Delta_n) = W + c_{n+2}\,\theta_{n+1} - c_{n+1}\,\theta_{n-1}}$$

(11.32)

and

$$\underline{\theta_n = S_n - S_{n+1}}\,. \tag{11.33}$$

Thus we have a polynomial relation (11.32) connecting our unknowns. We shall find that from it, or (11.28), knowing little more than the form of θ_n and Δ_n we can deduce the coefficients in our C.F. Before we apply our results let us say therefore a little about θ_n and $\square_n$, in particular about $\square_n$.

The Polynomials θ_n and $\square_n$.

First let us recall that θ_n is a polynomial of fixed degree μ, the term in x^μ being given by the first term in the expansion

$$(2n+1)\frac{W}{x^2}+\frac{2V}{x}\ . \tag{11.34}$$

Then observe that from (11.9)

$$\Delta_n = W\frac{D_n'}{D_n}+V+C_{n+1}\,\theta_n\,\frac{D_{n-1}}{D_n}\ ,$$

and write $D_n = x^n+(d_1+d_2+\ldots.+d_n)x^{n-1}+\ldots.\equiv x^n+\alpha_n x^{n-1}+\beta_n x^{n-2}+\ldots$

so that

$$\Delta_n = \frac{W}{x}\left[n-\frac{\alpha_n}{x}+\frac{(\alpha_n^2-2\beta_n}{x^2}+\ldots\right]+V+c_{n+1}\frac{\theta_n}{x}\left[1-\frac{d_n}{x}+\ldots\right] \tag{11.35}$$

where, because Δ_n is a polynomial, the expansion on the right must terminate. For most purposes all we will require is the leading term of A and the form of Δ_n . We will simply substitute θ_n and D_n in (11.32), or (11.28), equate coefficients of x and hence determine our unknowns and in particular the coefficients of our C.F. The other polynomials that we have introduced, S_n , K_n and K^*_n can then be determined by the appropriate expression (11.27) to (11.29).

To clarifv the preceeding analysis we will construct the recurrence relations and hence the C.Fs. when W is quadratic in x. There are essentially two cases. The first is when f has two distinct singularities these we will take at ± 1 by taking $W = x^2 - 1$, *the* denominator polynomials will turn out to be the Jacobi polynomials. Then we will consider the case when the singularities coincide by taking $W = x^2$, the resulting C.F gives useful approximations to the error function and related functions.

12. Jacobi Functions

Let us consider the differential equation

$$(x^2 - 1)\frac{df}{dx} = 2(x + \lambda)f + U(x) \ . \qquad (12.1)$$

For this equation

$W = x^2 - 1$, $V = (\lambda x + \mu)$

and the weight function

$$w(x) = e^{\int \frac{2V}{W} dx} = \exp\left\{\int \frac{\lambda+u}{x-1} + \frac{\lambda-u}{x+1} dx\right\} = (1-x)^{\lambda+u}, (1+x)^{\lambda-u} \qquad (12.2)$$

which is the weight function for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$

if we take $\alpha = \lambda + \mu$, $\beta = \lambda - \mu$.

From (11.34) and (11.35) we readily deduce that

$$\theta_n = 2n+1 + 2\lambda$$

$$\Delta_n = (n+\lambda)\ x + \delta_n$$

The key information is now obtained by equating coefficients in the identity (11.28) ,

$$WS_n = V^2 - \Delta_n^2 + c_{n+1}\ \theta_n\ \theta_{n-1}$$

$$(x^2 - 1)S_n = (\lambda x + \mu)^2 - [(n + \lambda)\ x + \delta_n]^2 + c_{n+1}\ (2n + 1 + 2\lambda)(2n - 1 + 2\lambda) .$$

S_n is necessarily a constant,

$$\therefore\ S = \lambda^2 - (n + \lambda^2) = - n(n + 2\lambda)$$

$$\delta_n = \frac{\lambda\mu}{n + \lambda}$$

and

$$c_{n+1} = \frac{n(n + 2\lambda\lambda[(n + \lambda)^2 - \mu^2]}{(n - \lambda)^2[4(n + \lambda)^2 - 1].} \quad n \geq 1$$

Also from the expression for Δ_n,

$$\Delta_{n+1} + \Delta_n = (n + 1 + \lambda)x + (n + \lambda)x + \frac{\lambda\mu}{n + 1 + \mu} + \frac{\lambda\mu}{n + \lambda}$$

$$= (2n + 1 + 2\lambda\lambda \left[x + \frac{\lambda u}{(n + 1 + \lambda)\ (n + \lambda)} \right]$$

where $(2n + 1 + 2\lambda)$ is θ_n Thus from (11.13) the J fraction for the particular integral of (12.1) is generated by the recurrence relation

$$f_{n+1} = \left[x + \frac{\lambda u}{(n + 1 + \lambda)(n + \lambda)} \right] f_n - \frac{n(n + 2\lambda\lambda[(n + \lambda)^2 - u^2]}{(n + \lambda)^2 [4(n + \lambda)^2 - 1]} f_{n-1} \quad . (12.3)$$

In particular with $U(x) = \theta_0,\ \lambda > -$,we would obtain the J fraction for

$$f = W(x) \int_{\infty}^{x} \frac{2\lambda + 1}{(x^2 - 1) w(x)} dx,$$

the first partial numerator c_{-1} being unity.
The denominators of these J fractions by (11.12) satisfy the same recurrence relations as the f and are readily shown to be the Jacobi polynomials $P_n\ (\alpha, \beta)(x)$ arranged so that the coeffioient of x^n is one.
For by (11.27)

$$K_n = V' - \Delta_n' + S_n = -n\ (n + 2\lambda + 1),$$

thus by (11.21) the denominator $D_n(x)$ and $\frac{1}{w(x)} f_n(x)$ are both solutions of

$$(x^2 - 1)\ y'' + (2x + 2Ax + 2\mu)\ y' - n(n + 2\lambda + 1)\ y = 0 \qquad (12.4)$$

which is the differential equation satisfied by Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$; $2\lambda = (\alpha, \beta)$ $2\mu = \alpha - \beta$.

<u>Particular Cases</u>

In passing we observe that the recurrence relation (12.3) simplifies in the following three cases:-

i) $\lambda = 0 \quad f_{n+1} = x f_n + \dfrac{\mu^2 - n^2}{4n^2 - 1} f_{n-1} \quad n \geq 1$

(12.5)

which generates the C.P (4.22) for the Associated Legendre functions

ii) $\mu = 0 \quad f_{n+1} = x f_n + \dfrac{n(n + 2\lambda)}{4(n + \lambda)^2 - 1} f_{n-1}$ (12.6)

which, with $\lambda = \nu - \frac{1}{2}$ generates the C.F (5.6) *tor* the Laplace transforms of Bessel functions.

iii) $\mu = \frac{1}{2} \quad f_{n+1} = \left[x + \dfrac{\frac{1}{2}\lambda}{(n+1+\lambda)(n+\lambda)} \right] f_n - \frac{1}{4} \dfrac{n(n+2\lambda\lambda}{(n+\lambda)^2} f_{n-1}$ (12.7)

Written as a Laplace transform the function f becomes

$$\int_0^\infty e^{-(x+1)t} \, {}_1F_{1b} \ (a;b;2t)\,dt \qquad \text{where } \begin{array}{l} a = 1 + \lambda - u \\ b = 2(1+\lambda) \end{array} \tag{12.8}$$

and consequently is related to the Kummer and Whittaker functions.

Error Analysis for Jacobi functions.

With $U(x) = -e$, our J fraction for f is

$$f = \frac{1}{x + \dfrac{u}{1+\lambda} -} \quad \frac{c_2}{x + d_2 -} \quad --- \frac{c_n}{x + d_n -} \quad \frac{f_n}{f_{n-1}}$$

where c_{n+1} and d_{n+1} are given by (12.3).

Now the error in terminating this expression after n terms is

$$f - \frac{C_n}{D_n} = \frac{f_n}{D_n}$$

and this expression satisfies, by (11.7), the differential equation

$$\frac{d}{dx}\left[\frac{f_n}{D_n}\right] - \frac{2V}{W}\left[\frac{f_n}{D_n}\right] = - \frac{A_{n+1}\,\theta_n}{W D_n^2}.$$

Hence the error can be written as an integral

$$\frac{f_n}{D_n} = w(x) \int_x^\infty \frac{A_{n+1}\,\theta_n}{W\,w(u) D_n^2(u)}\,du$$

and the asymptotic value of this integral for large n we have deduced is 15.

$$\frac{f_n}{D_n} = \frac{\pi\, \Gamma(\alpha+\beta+1)(x-1)^{\alpha}}{2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}(x+1)^{\beta}\, e^{-i(\alpha+\frac{1}{2})\pi}\left[1 - \tanh\left(\frac{N\varsigma}{2} - \frac{i\phi}{2}\right)\right]\left\{1+0\left(\frac{1}{n}\right)\right\}$$

$$= \frac{\pi\,\Gamma(\alpha+\beta+1)(x-1)^{\alpha}}{2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}(x+1)^{\beta}\;\frac{2e^{-N\varsigma}}{1+e^{-N\varsigma+i\phi}}\left\{1+0\left(\frac{1}{n}\right)\right\} \qquad (12.9)$$

where $x = \cosh\varsigma$, $N = (2n + 1 + \alpha + \beta)$ and $\phi = (\alpha + \frac{1}{2})\pi$.

This expression for the error ia of a similar form to the error estimates obtained in TR.25 and in fact generalises a number of these results, in particular (5.28) for the Laplace transforms of Bessel functions.

Error Function and Related Functions

For our second example we consider the equation

$$x^2 \frac{df}{dx} = 2(\lambda x + \mu)f + Ux) \qquad (12.10)$$

where

$W = x^2$, $V = \lambda x + \mu$

An almost identical analysis with that for the Jacobi functions produces the recurrence relation ,

$$f_{n+1} = \left[x + \frac{\lambda u}{(n+1+\lambda)} (n-\lambda) \right] f_n + \frac{u^2 n(n+2\lambda\lambda}{(n+\lambda)^2 [4(n+\lambda)^2 - 1]} f_{n-1} , \qquad (12.11)$$

as the relation which generates the J fraction for the particular integral of (12.10).

Now the weight function for this set of

functions $w(x) = e^{\int \frac{2V}{W} dx} = \exp\left\{ \int \frac{2\lambda}{x} + \frac{2u}{x^2} dx \right\} = x^{2\lambda} \; e^{\frac{-2u}{x}}$ (12.12)

is clearly closely related to that for the generalised Laguerre functions. However the C.F generated by (12.11) is <u>not</u> to be confused with (2.17); it is matching a different series. It provides a powerful sequence of approximations to the error and related functions.

When $U(x) = -2\lambda x$, the solution of (12.10) is

$$f(x) = x^{2\lambda} e^{-\frac{2u}{x}} \int_{\infty}^{x} -\frac{2\lambda}{x^{2\lambda+1}} e^{\frac{2u}{x}} dx ,$$

we take $\mu = ½$. As an example we put $\mu = \frac{1}{4}$ so that

$$f(x) = x^{\frac{1}{2}} e^{-\frac{1}{2}} \int_{\infty}^{x} \frac{-\frac{1}{2}}{x^{3/2}} e^{\frac{1}{2}} dx$$

and thus

$$f\left(\frac{1}{E^2}\right)=\frac{e^{-z^2}}{z}\int_o^z e^{t2}\,dt=1-\frac{\frac{2z^2}{3}}{1+\frac{2}{3}z^2}+\frac{\frac{24}{25.21}z4}{1+\frac{2}{45}z^2}+\text{----}\,. \quad (12.13)$$

Luke [6.4] derives an estimate for the error in terminating the C.F for

$$f(x) = x^{2\lambda} e^{-\frac{1}{2}} \int_x^\infty \frac{2\lambda}{x^{2\lambda+1}} e^{\frac{1}{x}} dx \qquad (12.14)$$

after n terms. Making the zero convergent one, he estimates the error behaves as

$$E_n(x) \sim \frac{(-1)^{n+1} e^{-\frac{1}{2}} r\left(\frac{1}{2}\right) r(2\lambda+1)}{2^{2\lambda}(2x)^{2n+1} r(2n+2\lambda+1)\; n^{\frac{1}{2}}} \;. \qquad (12.15)$$

Luke's figures for these formulae are impressive, so we will reproduce them.

Putting z = 1 in (12.13) $\qquad f(l) = 0.5380\ 7951$

n	$f_{0/n}(1)$	$f - f_{0/n}$	E_n
0	1.0	-	-
1	0.5238 0952	0.14(-1)	0.15(-l)
2	0.5382 4561	-0.17(-3)	-0.17(-3)
3	0.5330 7854	0.97(-6)	0.98(-6)
4	0.5380 7951	0	0

For our next example we increase the degree of W and hence increase the number of singularities in f. We will find that no longer can we derive the c and d explicitely, but that we must be content with producing a set of equations which successively generate the c_n and d_n.

13. Three Distinct Singularities

We take W = x - 1 so that the three singularities are symmetrically placed about the origin. Consider

$$(x^3-1)\,\frac{dy}{dx} = 3\ (\alpha x^2 + ßx + y)y + U(x) \qquad (13.1)$$

For this equation $W = x^3 - 1$, $2V = 3(\alpha x^2 + ßx + \gamma)$ (13.2)
and the weight function

$$W(x) = e^{\int \frac{2V}{W}dx} \equiv \exp\left\{\int \frac{3(\alpha(^2 + \beta x + y}{(x^3-1)}dx\right\} = \exp\left\{\int \frac{A}{x-1} + \frac{B}{x-w} + \frac{C}{x-w^2}\,dx\right\}$$

giving

$$w(x) = (x-1)^{\alpha+\dot{\beta}+\gamma}\ (x-w)^{\alpha+ß\dot{w}^2+\gamma w}\ (x-w^2)^{\alpha+ßw+\gamma w2} \qquad (13.3)$$

From (11.34) and (11.35) we can readily establish the form of θ_n and □$_n$

$$\theta = (2n+1+3\alpha)x + \phi_n \qquad (13.4)$$

$$D_n = (d + \frac{3}{2}\alpha)\,x^2 + ß_n x + y_n \qquad (13.5)$$

where θ_n, $ß_n$ and y are to be determined.
The interrelation of these quantities with the coefficients c_n and d_n in the continued fraction can be obtained directly by equating coefficients in (11.32)

$$(x + d_{n+1})(\Delta_{n+1} - \Delta_n) = W + c_{n+2}\ \theta_{n-1}$$

i.e. for n≥1

$$(x+d_{n+1})\ [x^2 + (\beta_{n+1} - \beta_n)\ x + (y_{n+1} - y_n)] =$$
$$x^3-1 + c_{n+2}\,[(2n+3+3\alpha)x + \phi_{n+1}] - c_{n+1}\,[(2n-1+3\alpha)x + \phi_{n-1}]$$

$$\left.\begin{aligned} d_{n+1} + \beta_{n+1} - \beta_n &= 0 \\ d_{n+1}(\beta_{n+1} - \beta_n) + y_{n+1} - y_n &= c_{n+2}(2n+3+3\alpha\alpha - c_{n+1}(2n-1+3\alpha) \\ d_{n+1}(y_{n+1} - y_n) &= -1 + c_{n+2}\,\Phi_{n+1} - c_{n+1}\,\Phi_{n-1}\end{aligned}\right\}(13.6)$$

In addition we have that

$\theta_n (x + d_{n+1}) = \Delta_{n+1} + \Delta$

i.e.

$$[(2n + 1 + 3\alpha)x + \phi_n](x + d_{n+1}) - (2_n +_1 + 3\alpha))x^2 + (ß_{n+1} + ß_n)\ x + (\gamma_{n+1} + \gamma_n)$$

so that

$$\left.\begin{aligned} (2n + 1 + 3\alpha)\, d_{n+1} + \phi_n &= \beta_{n+1} + \beta_n \\ \iota_n\, d_{n+1} &= \gamma_{n+1} + \gamma_n . \end{aligned}\right\} \qquad (13.7)$$

A simple rearrangement of these five equations gives the following scheme for successively computing d_{n+1}, $ß_{n+1}$, γ_{n+1}, c_{n+2}, Φ_{n+1} :-

$$\left.\begin{aligned} (2n+2+3\alpha)\, d_{n+1} &= 2\beta_n - \phi_n \\ \beta_{n+1} &= \beta_n - d_{n+1} \\ \gamma_{n+1} &= \gamma_n + \phi_n\, d_{n=1} \\ (2_n + 3 + 3\alpha)\, c_{n+2} &= (2n - 1 + 3\alpha)\, c_{n+1} + (\gamma_{n+1} - \gamma_n) - d_{n+1}^2 \\ c_{n+2}\ \phi_{n+1} &= c_{n+1}\, \phi_{n-1} + 1 + d_{n+1}\, (\gamma_{n+1} - \gamma_n) \end{aligned}\right\} \qquad (13.8)$$

where the last two equations can be used with n = 0 if the terms containing c_{n+1} are dropped.

The initial conditions are partly determined by U(x). We can eliminate any polynomial part to y by taking U(x) = - θ_0 and to keep the number of parameters to three, let us put U(x) = - [(1 + 3α)x + 3β].

Since $\Delta_0 = V = \frac{3}{2}$ $(ax^2 + ßx + \gamma)$, we then have the initial conditions

$$c_1 = 1\ ,\ \phi_0 = 3\beta\ ,\ \beta_0 = \frac{3}{2}\beta\ ,\ \gamma_0 = \frac{3}{2}\gamma \qquad (13.9)$$

From (13.8) we then obtain

$$d_1 = \mathbf{0}\ ,\ \beta_1 = \frac{3}{2}\beta\ ,\ y_1 = -\frac{3}{2}y\ ,\ 0_2 = -\frac{y}{1+\alpha}\ ,\ \phi_1 = \frac{1}{c_2} = -\frac{1+\alpha}{y}$$

$d_2 = \dfrac{3\beta - \phi_1}{4+3a}$, etc.

c and d are rational functions of α, ß, γ.
In general the above technique applied to the linear first order differential equation (11.2) will yield a set of interrelations like (13.8) from which the coefficients of c_n and d_n of (11.1) can be derived. The process is superior to simply determining a series solution of the equation and then converting it into a C.F by the method indicated in (1.17) of TR.25 in that it is numerically more stable.
The error in truncating the C.F. after n terms is still given by the integral of (11.7)

$$\frac{f_n}{D_n} = w(x)\int_x^\infty \frac{A_{n+1}\,\theta_n(t)}{W(t)\;D_n^2(t)\;W(t)}\;dt$$

$$= A_{n+1}\,W(x)\int_x^\infty \frac{(2n+1+3\alpha)t+\phi_n}{W(t)D_n^2(t)(t^3-1)}\,dt\,, \qquad (13.10)$$

where $w(x) = (x-1)^A\ (x-w)^B\ (x-w^2)^C$. A_{n+1} we can if necessary compute, although it is worth noting $c_n\ c_{n+1}\ \ c_{n+2} \rightarrow \frac{1}{16}$ as we shall observe later, A. very crude estimate for $D_n(x)$ would be x^n , for a rather better estimate we could use the differential equation satisfied by D_n for x large and n large

$$x^2y' \rightarrow (2+3\alpha)xy' - n^2y = 0\ .$$

What we have tried to do and what ideally we would like to do is to obtain asymptoticc estimates depending on n for c and d_n (or simple combinations of the cs and/or ds). For unless we can do this there seems little possibility of deriving a suitable error estimate that depends on n in the form we obtained when W was linear or quadratic; further such asymptotic estimates might guide us in our handling of a much wider class of problems.

We observe that there are certain combinations of our unknowns which remain constant as n increases. Equating coefficients in the identity (11.28)

$$WS_n = V^2 - D_n^2 + c_{n+1}\ \theta_n\ \theta_{n-1}$$

we find

$$\left.\begin{aligned} y_n^2 + (2n+3\alpha)\,\beta_n - c_{n+1}\ \phi_n\ \phi_{n-1} &= \frac{9}{4}(y^2+2\alpha\beta) \\ \beta_n^2 + (2n+3\alpha)\,y_n - c_{n+1}\ [(2n+3\alpha)^2-1] &= \frac{9}{4}(\beta^2+2\alpha y) \\ 2\beta_n\,y_n - c_{n+1}\,[(2n+1+3\alpha)\,\phi_{n-1} + (2n+1+3\alpha)\,\phi_n] + n(n+3\alpha\alpha &= \frac{9}{4}\ (2\beta\ y) \end{aligned}\right\} \quad (13.11)$$

This set of equations can be used instead of the equations (13.6) and should be regarded as a first integral of that set of finite difference equations.

Quasi-Periodicity

To gain insight into these approximations we have computed the values of $c_n\ d_n\ \phi_n\ \beta_n\,\gamma_n$ using the formulae (13.8) for various values of α ,ß ,γ. These formulae seem to be numerically remarkably stable, the results tabulated in Tables A were obtained carrying ten figures. Table A for $a = 2ß = 2\gamma = \frac{1}{3}$ clearly indicates the almost periodic nature of the partial numerator c_n and of the partial denominator $x + d_n$ of the C.F. This three term periodicity results from the distributing of the poles and zeros of the convergents along the three branch cuts. The distribution along the three branch cuts is disturbed by the introduction of one extra pole and zero, is improved by a further pole and zero and finally regains a position similar to the first when the third pole and zero are added. For plot3 of poles and zeros see graphs on page V7. As more and more poles and zeros are introduced, as n is increased, so these poles and zeros etch out the branch cuts of the function y which satisfies the differential equation (13.1).

For other values of α ,ß ,γ.. the C.F settles to being periodic, but often only after an initial disturbance caused by the initial conditions has died away. Table A for $\alpha = \frac{1}{3}$, $\beta = 0$, $y = \frac{2}{3}$ shows a large leap in the value of c_n to - 1009, before c_n and d_n slowly settle towards their periodic values. Usually the effect of the initial conditions is less dramatic taking longer to be absorbed. In all cases computed, the three term periodicity eventually dominated.

How can we take advantage of this three term quasi-periodicity in the coefficients c_n and d_n ? For a suitable large n

$$R = \frac{-c_n}{x+d_n} - \frac{c_{n+1}}{x+d_{n+1}} - \frac{c_{n+2}}{x+d_{n+2}} + R^* \qquad (13.12)$$

where R* is almost the same function of x as R. Treating (13.12) as a C.F and writing its convergents

$$\frac{C_1}{D_1} = \frac{-c_n}{x+d_n} \quad , \quad \frac{C_2}{D_2} = \frac{-c_n(x+d_{n+1})}{x^2+(dn+d_{n+1})x+(d_n d_{n+1} - c_{n+1})}$$

$$\frac{C_3}{D_3} = \frac{-c_n[x^2+(d_{n+1}+d_{n+2})x+(d_{n+1}d_{n+2}-c_{n+2})]}{x^3+(d_n+d_{n+1}+d_{n+2})x^2+(d_n d_{n+1}+d_n d_{n+2}+d_{n+1}d_{n+2}-c_{n+1}-c_{n+2})x + +(d_n d_{n+1} d_{n+2} - d_n {}^{o}n+2 - d_{n+2}c_{n+1})}$$

we can then write

$$R = \frac{C_3 + R^* C_2}{D_3 + R^* D_2} \quad . \qquad (13.13)$$

If we replace R* lay R, R satisfies the quadratic equation

$$D_2 R^2 + (D_3 - C_2) R - C_3 = 0 \qquad (13.14)$$

from which we can determine R. When the problem is simple one of calculating y that satisfies the differential equation (13.1), this R can be used to plug our C.F ,

$$\frac{1}{x} - \frac{c_2}{x+d_2-} \frac{c_3}{x+d_3-} \;---\; \frac{c_{n-1}}{x+d_{n-1}} + R(x) \tag{13.15}$$

and will go a long way towards inserting the singularities of y(x). The branch points of R are contained in the discriminant of (13.14), a simple manipulation gives the discriminant.

$$\Delta = [C_2 + D_3]^2 - 4c_n\ c_{n+1}\ c_{n+2}$$
$$= [x^3 + \sigma x^2 + \tau x^2 + \upsilon] - 4c_n\ c_{n+1}\ c_{n+2} \tag{13.16}$$

where $\sigma = d_n + d_{n+1} + d_{n+2}$

$\tau = d_n\ d_{n+1} + d_n\ d_{n+2} + d_{n+1}\ d_{n+2} - c_n - c_{n+1} - c_{n+2}$

$\nu = d_n\ d_{n+1}\ d_{n+2} - c_n\ d_{n+1} - c_{n+1}\ d_{n+2} - c_{n+2}\ d_n.$

To indicate how closely the branch points of y(x) are being approximated by those of R(x), we have calculated the discriminant (13.16) using our numerically results for o and d in Tables A.

For $\alpha = \frac{1}{3}, \beta = \frac{1}{5}, y = \frac{1}{6}$ and taking n = 40

$\sigma = \cdot 00010$ $\quad$ $\Delta = [x^3 + .00010x^2 + .00007x - '50001]^2 - 0.24997$

$\tau = \cdot 00007$ branch points $x^3 + \cdot 0001\ 0x^2 + .00007x - .50001 = \pm\ 0.49997$

$\upsilon = -.50001$ $\quad$ hence $x^3 + .00010x^2 + .00007x - 0.99998 = 0$

$4c_n\ c_{n+1}\ c_{n+2} = 0.24997$ or $x^3 + .00010x^2 + .00007x - 0.00004 = 0$

For $\alpha = \frac{1}{3}, \beta = 0, y = \frac{2}{3}$ and taking n = 54

$\sigma = 0.00003$ $\quad$ $\Delta = [x^3 + 0.00003x^2 + .00007x - .49996]^2 - 0.24994$

$\tau = 0.00007$ branch points $x^3 + .00003x^2 + .00007x - .49996 = \pm\ 0.49994$

$\nu = -.49996$ $\quad$ hence $x^3 + .00003x^2 + .00007x -.\ 99990 = 0$

$4c_n\ c_{n+1}\ c_{n+2} = 0.24994$ or $x^3 + .00003x^2 + .00007x - .00002 = 0$

For $\alpha = 3$, $\beta = 2$, $.\gamma = 1$ and taking $n = 28$

$\sigma = 0.0153$ .... $\square = [x^3 + 0.015x^2 + 0.0066x - 0.4849]^2 - 0.2348$

$\tau = 0.0066$ branch points $x^3 + 0.015x^2 + 0.0066x - 0.4849 = \pm$ 0-4845

$\nu = -0.48489$ hence $x^3 + 0.015x^2 + 0.0066x - 0.969 = 0$

$4c_n\, c_{n+1}\, c_{n+2} = 0.23480$ or $x^3 + 0.015x^2 + 0.0066x - 0.0004 = 0$

In each case increasing n by multiples of three we find the branch points of R(x) tend to those of y(x) as we would expect. In this particular problem the branch points of y(x) are given directly by the polynomial $W = x^3 - 1$ of the differential equation. The 'artificial' triple branch point at the origin arises because we are forcing the branch cuts towards the origin by our approximations at infinity. Our numerical considerations suggest that we write

$$[x^3 + \sigma x^2 + rx + v - 2\sqrt{c_n c_{n+1} c_{n+2}}\,] \to x^3 - 1\,, \qquad (13.17)$$

as n increases, this implies

$\sigma \to 0$ $\qquad d_n + d_{n+1} + d_{n+2} \to 0$

$\tau \to 0$ $\quad d_n d_{n+1} + d_n\, d_{n+1} + d_{n+1}\, dn{+}2 - c_n - c_{n+1} - c_{n+2} \to 0$.

In addition, the forcing of the branch cuts towards the origin suggests

$$V + 2\sqrt{c_n c_{n+1} c_{n+2}} \to 0$$

$\therefore\ \nu \to ½$ $\qquad d_n\, d_{n+1}\, d_{n+2} - c_n\, d_{n+1} - c_{n+1}\, d_{n+2} - c_{n+2}\, d_n \to -\frac{1}{2}$

$$4c_n\, c_{n+1}\, c_{n+2} \to \frac{1}{4}\,.$$

Now it is not a simple matter to take advantage of these conditions as n increases in the non-linear generating relations (13.8). Instead they more naturally line up with our linear method of generating the C.F outlined in (1.17) of TR25, and strongly suggest fitting the series six terms at a time once the disturbance due to the initial conditions has died away.

Our analysis in section 11 can be extended in various ways. First we will look at another solution to the first order differential equation (11 .2). Then we extend our considerations to functions defined by second order linear differential equations. Finally we will derive a general continued fraction matching the series in 1/s for the Laplace transform of a function.

14. M Fraction Solution

The particular function f satisfying the differential equation (11.2) usually possesses a series for both x large and x small, so that besides our J fraction for f we can also construct an M fraction for f. A set of approximations derived from an M fraction will tend to give good approximations near the origin, at infinity and also along particular lines in the complex plane joining them. For a discussion of Murphy's M fraction see McCabe (1971).

Suppose then that

$$f = \begin{cases} a_o + a_1 x + a_2 x^2 + \quad ---- & \text{for } x \text{ small} \\ \dfrac{b_1}{x} + \dfrac{b_2}{x^2} + \dfrac{b_3}{x^3} + \quad ---- & \text{for } x \text{ large} \end{cases} \tag{14.1}$$

and that f satisfies the linear differential equation (11.2)

$$W \frac{df}{dx} = 2Vf + U \tag{14.2}$$

We begin by writing

$$f = \frac{P_n}{Q_n} + \frac{f_n}{Q_n} \tag{14.3}$$

and choosing the polynomials P_n (x), Q_n (x) so that their ratio $\frac{P_n}{Q_n}$ matches n terms in each of the series (14.1). We can write this ratio as the M fraction

$$\frac{P_n}{Q_n} = \frac{p_1}{1 + q_1 x} + \frac{p_2 x}{1 + q_2 x} + \quad ---- \quad + \frac{p_n x}{1 + q_n x} \; . \tag{14.4}$$

Proceeding as we did in section 11, we obtain

$$W \frac{d}{dx}\left[\frac{f_n}{Q_n}\right] - 2v \frac{f_n}{Q_n} = -W \frac{d}{dx}\left[\frac{P_n}{Q_n}\right] + 2V \frac{P_n}{Q_n} + U$$

which on differentiating gives

$$Q_n^2\left\{w\frac{d}{dx}\quad -2V\frac{f_n}{Q_n}\left\{w\frac{d}{dx}\left[\frac{f_n}{Q_n}\right]-2V\frac{f_n}{Q_n}\right\}=-W[Q_nP_n'-P_nQ_n]+2VP_nQ_n+UQ_n^2\right.$$

$$= \text{a polynomial,} \qquad (14.5)$$

as W(x), V(x) and U(x) are polynomials. Further as

$$\frac{f_n}{Q_n}=\begin{cases} A_{n+1}x^n+0(x^{n+1}) & x \text{ smal} \\ \dfrac{A_{n+1}}{(q_1q_2--q_n)^2\ q_{n+1}}\dfrac{1}{x^{n+1}}+0\left(\dfrac{1}{x^{n+2}}\right) & x \text{ large} \end{cases} \qquad (14.6)$$

from the L.H.S we deduce that the degree of the terms in this polynomial lie between n-1 and n-1+υ where the term of largest degree in $\frac{w}{x}+2V$ is *v*, *v* is fixed.

Thus the corresponding result to the key relation (11.7) is

$$\underline{W\frac{d}{dx}\left[\frac{f_n}{Q_n}\right]-2V\frac{f_n}{Q_n}=\frac{A_{n+1}x^{n-1}\phi_n(x)}{Q_n^2}} \qquad (14.7)$$

where $\phi_n(x)$ is of fixed degree υ, and $A_{n+1} = (-1)^n P_1 P_2 ---- P_{n+1}$.
The factor x^{n-1} is the only difference between this result and (11.7), it removes some of the elegance of the subsequent analysis of section 11 , but nevertheless it does not prevent parallel results being derived.
We will not repeat the analysis but simple indicate our results.
(11.9) and (11.10) become

$$Wx\,Q_n' + \left[Vx-(n-1)\frac{W}{2}+\Delta_n\right]Q_n + P_{n+1}\,x\phi_n\,Q_{n-1} = 0 \qquad (14.8)$$

$$Wx\,Q_{n-1}' + \left[Vx-(n-1)\frac{W}{2}+\Delta_n\right]Q_{n-1} + \phi_{n-1}\,Q_n = 0 \qquad (14.9)$$

and so eliminating Q'_n, the recurrence relation for the Qs (and fs) is

$$\underline{Q_{n+1} = \frac{1}{\phi_n}\left[\frac{W}{2}+\Delta_{n+1}+\Delta_n\right]Q_n + P_{n+1}\,x\,Q_{n-1}\ .} \qquad (14.10)$$

Eliminating Q_{n-1} from (14.9) we obtain the differential equation for Q_n,

$$Wx\,Q_n' + \left[W'x + 2Vx - (n-1)\,W - \frac{\phi_n'}{\phi_n}\,Wx\right]Q_n + H_n(x)Q_n = 0 \qquad (14.11)$$

where

$$H_n(x) = \left[V'x + \frac{(n-1)}{2}\left(\frac{W}{x} - w'\right) + \left(\Delta_n' - \frac{\Delta_n}{x}\right) + S_n\right] +$$
$$-\frac{\phi_n'}{\phi_n}\left[\,Vx - (n-1)\,\frac{W}{2} + \Delta_n\right] \qquad (14.12)$$

and the crucial relation corresponding to (11,28) is

$$\underline{Wx\,S_n = \left[\,Vx - (n-1)\,\frac{W}{2}\right]^2 - \Delta_n^2 - P_{n+1}\,x\,\phi_n\phi_{n-1}\,.} \qquad (14.13)$$

where again S_n and Δ_n are polynomials whose degree does not depend on n.

To determine the M fraction (14.4) we therefore must first find the polynomials ϕ and Δ_n . Now we are approximating f for both x small and x large and both of these considerations will yield information on the coefficients of these polynomials ϕ_n and Δ_n .

ϕ_n is defined by (14.7)

$$A_{n+1}\,x^{n-1}\,\phi_n(x) = Q_n^2\left\{W\frac{d}{dx}\left[\frac{f_n}{Q_n}\right] - 2V\,\frac{f_n}{Q_n}\right\},$$

substituting (14.6) we find

$$\phi_n(x) = \begin{cases} (nW - 2Vx) + \text{other terms} & x\ \text{small} \qquad (14.14) \\ -\dfrac{1}{Q_{n+1}}\left\{\dfrac{(n+1)}{x}W + 2V\right\} + \text{other terms} & x\ \text{large} \qquad (14.15) \end{cases}$$

where the dominant terms are contained in these expressions.
Prom the relation (14.8), the polynomial Δ_n can be written

$$\Delta_n = (n-1)\frac{W}{2} - Vx - Wx\frac{Q_n'}{Q_n} - P_{n+1}\,x\,\phi_n\,\frac{Q_{n-1}}{Q_n}$$

and again we cons ider x both small and large.

For x small, $Q_n = ^1 + \alpha_n x + - - - ---------$ where $\alpha_n = q_1 + (p_2+q_2) + - - + (p_n+q_n)$,

$$\therefore \Delta_n = (n-1)\frac{W}{2} - Vx - Wx\frac{\alpha_n + --}{1+\alpha_n x + -} - P_{n+1} x \phi_n \frac{1+\alpha_{n-1}x + --}{1+\alpha_n x + --}$$

$$= (n-1)\frac{W}{2} - Vx - Wx\,\alpha_n - P_{n+1}\,x\,\phi_n + \text{ higher terms .} \quad (14.16)$$

For x large, $Q_n = (q_1\ q_2\ (x^n + w_n x^{n-1} + - -)$

$$\text{Where } w_n = \frac{1}{q_1} + \frac{(q_1+p_2)}{q_1 q_2} + -- \frac{(q_{n-1}+p_n)}{q_{n-1}q_n},$$

$$\therefore \Delta_n = (n-1)\frac{W}{2} - Vx - W(n - \frac{w_n}{x}) - \frac{p_{n+1}}{q_n}\phi_n + \text{ lower terms.} \quad (14.17)$$

These conditions,together with the relation

$$\frac{w}{2} + \Delta_{n+1} + \Delta_n = \varphi_n\ (1+q_{n+1}{}^x), \quad (14.18)$$

will be sufficient to determine Φ_n and Δ_n in the two simple examples which we will now use to illustrate the theory.

<u>Dawson's Integral</u>

One of the neatest M fractions belongs to the function which satisfies the differential equation

$$2x\ f' + (1+x)\ f = 1 \qquad f(0) = 1 \quad (14.19)$$

With W = 2x, 2V = - 1+x), we find

$$\phi_n = \begin{cases} 2nx + (1+x)x + --- = (2n+1)x + --- & x \text{ small} \\ -\frac{1}{q_{n+1}}\{2(n+1) - (1+x)\} + --- = \frac{1}{q_{n+1}}x + --- & x \text{ large} \end{cases}$$

consequently $\quad \phi_n = (2n+1)x$ and $q_{n+1} = \frac{1}{2n+1}$.

For x small $\quad \Delta_n = (n-1)x + \left(\frac{1+x}{2}\right)x - 2x^2\alpha_n - P_{n+1}\,x\phi_n + ---$

$$= (n-\frac{1}{2})x + \text{ higher terms}$$

For x large $\Delta_n = (n-1)x + \left(\frac{1+x}{2}\right)x - 2x\left[n - \frac{W_n}{x}\right] - \frac{P_{n-1}}{q_n}\varphi_n + ---$

$$= \frac{x^2}{2} + \left[n - \frac{1}{2} - 2n + P_{n+1}\left(4\frac{2}{n} - 1\right)\right]x$$

But the coefficient of x is (n - ½), therefore

$$\Delta_n = \frac{x^2}{2} + (n - \frac{1}{2})x \quad \text{and } P_{n+1} = -\frac{2n}{4n^2-1} \quad .$$

Hence the M fraction solution of (14.19) is

$$f(x) = \frac{1}{1+x} - \frac{\frac{2}{3}x}{1+\frac{1}{3}x} - \frac{\frac{4}{15}x}{1+\frac{1}{5}x} - \frac{\frac{6}{35}x}{1+\frac{1}{7}x} - ---$$

which can be written

$$f(x) = \frac{1}{1+x} - \frac{2x}{3+x} - \frac{4x}{5+x} - \frac{6x}{7+x} - --- \frac{2nx}{(2n+1)+x-} ---. \quad (14.20)$$

Further from (14.13) we can readily deduce that

$S = nx \equiv H_n\ (x)$

and that the denominator polynomials $Q_n(x)$ satisfy the differential equation

$$2x\, Q_n'' - [x + (2n-1)]\, Q'_n + n\, Q_n = 0 \ .$$

The integral

$$e^{-z^2}\int_0^z e^{t^2}\,dt = z\,f(2z^2)$$

is known as Dawson's Integral. The accuracy of the approximations obtained by simply truncating the C.F after a given number of terms is indicated below. The approximations are of course good for z <1 and z>4.

Z	1	2	3	4
Accuracy 8 terms	5D	2D	3D	5D
Accuracy 12 terms	9D	4D	4D	8D

<u>$f = \cot^{-1}x$</u>

A simple function having the correct behaviour at infinity as well as a Taylor expansion at the origin is $f = \cot^{-1}x$.

$$f = \begin{cases} \pi/2 - \tan^{-1}x = \pi/2 - x + \dfrac{x^3}{3} - \dfrac{x^5}{5} + --- & x \text{ small} \\ \tan^{-1}\dfrac{1}{x} = \dfrac{1}{x} - \dfrac{1}{3x^3} + \dfrac{1}{5x^5} - \dfrac{1}{7x^7} + --- & x \text{ large}. \end{cases}$$

This function satisfies the differential equation

$$(1+x^2)\frac{df}{dx} = -1 \qquad f(0) = \pi/2 \tag{14.21}$$

Now

$$\phi_n = \begin{cases} n(1+x^2) + --- \\ -\dfrac{1}{q_{n+1}}(n+1)\dfrac{(x^2+1)}{x} + --- \end{cases} \qquad \text{giving } \phi_n = n - \frac{n+1}{q_{n+1}}x.$$

For x snail $\quad \Delta_n = \frac{1}{2}(n-1)(1+x^2) - (1+x^2)x\alpha_n - p_{n+1}x\Phi n + - - - -$

$$= \frac{1}{2}(n-1) - (\alpha + n\, p_{n+1})\, x + \text{higher terms}.$$

For x large $\quad \Delta_n = \frac{1}{2}(n-1)(x^2+1) - (x^2+1)(n - \frac{w_n}{x}) - \frac{p_{n+1}}{q_n}\phi_n + ---$

$$= \frac{1}{2}(n+1)x^2 + \left(W_n + (n+1)\frac{p_{n+1}}{q_n q_{n+1}}\right)x + \text{lower terms}.$$

Hence $\Delta_n = \frac{1}{2}(n+1) + \delta_n x - \frac{1}{2}(n+1)x^2$.

Further ϕ_n and D are related by (14.18),

$$\frac{W}{2} + \Delta_n + \Delta_{n-1} = n-1\ \phi_{n-1}\ (1+q_n x),$$

and from the coefficient of x we deduce

$$\delta n^{+}\ \delta n-1 = (n-1)q_n - \frac{n}{q_n}. \tag{14.22}$$

This formula and our two expressions for δ_n ,

$$\delta_n = -\ (\alpha_n + np_{n+1}) \quad , \quad \delta_n = w_n + (n+1)\frac{p_{n+1}}{q_n q_{n+1}},$$

are clearly sufficient to enable us to successively calculate δ_n , p_{n+1} and q_{n+1} .;in fact we could eliminate δ_n. For definitions of α_n and w_n see (14.16) and (14.17). However this is by no means the end of the story. From (14.13) besides showing that

$$S_n = -\ (n-1)\ (\delta_n + np_{n+1}) - nx \tag{14.23}$$

we also find that

$$\delta_n^2 = n^2 + p_{n+1}\left(\frac{n^2-1}{q_{n+1}} + \frac{n^2}{q_n}\right), \tag{14.24}$$

and

$$2\delta_n = (n+1)\frac{p_{n-1}}{q_n q_{n+1}} - (n-1)\,p_{n+1}\,. \tag{14.25}$$

Immediately we see that $w_n = \alpha_n + p_{n+1}$., and hence deduce

$$p_{n+1} = \frac{1}{q_n} + \frac{p_n}{q_{n-1}q_n} - q_n \tag{14.26}$$

and

$$\underline{(n+1)\frac{p_{n+1}}{q_n q_{n+1}} = (n-2)(p_n + q_n) - p_{n+1} - \frac{n}{q_n}} \tag{14.27}$$

a simple pair of formulae which successively generate the coefficients p_{n+1} q_{n+1} , in the C.F. In practice we find that p_n tends steadily to $-\frac{1}{2}$, while a tends steadily to 1 as $n \to \infty$ (which agrees with one solution of letting $p_n \to p$ and $q_n \to q$ simultaneously in these formulae). A close examination of the computed values of p_n and q_n, see Table B , suggested that $P_n + \frac{1}{2} \sim 0\left(\frac{1}{n^2}\right)$ and $1 - q_n \sim 0\left(\frac{1}{n^3}\right)$.

Putting $$P_n = -\left(\frac{1}{2} + \frac{\rho}{n^2} + \frac{\tau}{n^3} + ---\right)$$

and $$q_n = 1 + \frac{\sigma}{n^2} + ---$$

in the formulae (14,26), (14.25) and (14.24) successively; we find

$$\alpha = -2p$$

$$\delta_n = -\frac{1}{2} + \frac{\sigma}{n^2} + ---$$

$$p = \frac{1}{8} \text{ and } r = p.$$

Hence the asymptotic forms of p_n , q_n and δ_n are

$$\left.\begin{aligned} P_n &= -\left(\frac{1}{2} + \frac{1}{8n^2} + \frac{1}{8n^3} + ---\right) \\ q_n &= 1 - \frac{1}{4n^3} + --- \\ \delta_n &= -\frac{1}{2} - \frac{1}{4n^2} + --- \quad , \end{aligned}\right\} \quad (14.28)$$

higher terms could be found.

As p_n and q_n tend steadily to limiting values we can of course plug this C.F after n terms with R such that

$$R_n = -\frac{P_{n+1}}{1 + q_{n+1}x} + R_n \quad ,$$

or simply with R

$$R = -\frac{\frac{1}{2}x}{1 + x + R}$$

$$R^2 + (1+x)\ R + 1/2x = 0$$

i.e. $R = -\dfrac{(1+x) + \sqrt{1 + x^2}}{2}$.

The two M fractions that we have considered have both been treated numerically by J. McCabe [p54, p143] so we will not pursue then further.

15. C.F Solutions of 2nd Order Differential Equations

In the preceeding sections we have been exclusively concerned with linear first order differential equations and with second order linear differential equations directly related to them by advancing an integer parameter n. The differential equation, for example,

$$(x^2 - 1)y' = -2 \qquad (15.1)$$

naturally leads to Legendre's differential equation

$$(x^2 - 1)y''_n + 2x\, y'_n - n(n + 1)y_n = 0 \qquad (15.2)$$

with solutions

$$y_n = AP_n(x) + B\, Q_n(x)$$

Where Q_n is expressibl e as the C.F, see (4.6), (15.3)

$$Q_n(x) = \frac{1}{(n+1)}\; P_{n+1}(x) \quad - \quad \frac{P_n(x)(n+1)^2}{(2n+3)x} \quad - \quad \frac{(n+2)^2}{(2n+5)x} \quad - \quad --- \qquad (15.4)$$

in the complex plane of x cut from [—1, 1] along the real axis. By truncating this C.F we obtain rational approximations to the second kind solution of Legendre's equation.

The obvious question and the one to which we now turn our attention is 'what happens if n is replaced by a non-integer parameter λ?' The classical answer is simply A. replaces n in the three term recurrence relation giving

$$(\lambda+1)\; Q_{\lambda+1} = (2\lambda+1)\; x\; Q_\lambda - \lambda\; Q_{\lambda-1}\;, \qquad (15.5)$$

and hence we are able to develop a C.F for the ratio of two successive Qs. How to develop useful rational function approximations for Q_λ. is still an open question, of course simple expressions for the coefficients are unlikely. What we are looking for is a solution which will be generally applicable to most linear differential equations, and perhaps other differential equations. A C.F which produces Padà approximants see Wall [p380] is the main contender. Our discussion will revolve round Legendre's differential equation,

$$(x^2-1)y'' + 2xy' - \lambda(\lambda+1)y = 0 \tag{15.6}$$

although the techniques are applicable to a much wider class of problems. We will consider the singular point at infinity and the regular point at the origin.

<u>The Singular Point at Infinity</u>

At infinity, the series solution for the Legendre function of the second kind of degree λ is

$$Q_\lambda(x) = \frac{r(\frac{1}{2})r(\lambda+1)}{2^{\lambda+1}\, r(\lambda+\frac{3}{2})} - \frac{1}{x^{\lambda+1}}\ {}_2F_1\left(\frac{1}{2\lambda}+\frac{1}{2},\ \frac{1}{2}\lambda+1\ ;\ \lambda+\frac{3}{2}\ ;\ \frac{1}{x^2}\right). \tag{15.7}$$

Consider $0 \leq \lambda < 1$, and in particular take A =

$$Q_{0+25}(x) = \frac{r(\frac{1}{2})r(\lambda+1)}{2^{\lambda+1}\ r(\lambda+\frac{3}{2})} - \frac{1}{x^\lambda}\left[\frac{1}{x} - \frac{.4018}{x} - \frac{0.2260}{x} - \frac{.2814}{x} --- \right. . \tag{15.8}$$

The coefficients of this C.P. with those for Q_0 (x) are listed in Table C;

$$Q_o(x) = \frac{1}{2}\log\left(\frac{1+\frac{1}{2}}{1-\frac{1}{x}}\right) = \frac{1}{x} - \frac{.3333}{x} - \frac{.2667}{x} - \frac{.2571}{x} - --- \tag{15.9}$$

(15.8) is of the desired form in the sense that the coefficients are shifted but still tend to - 0.25.

Next consider rational approximations for when $\lambda > 1$, n is an integer,

$$Q_{n+\Lambda}(x) = \frac{r(\frac{1}{2})\,r(n+1+\Lambda)}{2^{n+1+\Lambda}\ r(n+\frac{3}{2}+\Lambda)} - \frac{1}{x^{n+1+\Lambda}}\ {}_2F_1\left(\frac{1}{2}(n+\Lambda+1),\frac{1}{2}(n+\Lambda)+1;\ n+\Lambda+\frac{3}{2};\frac{1}{x^2}\right)$$

the position is rather different.

The Pad method certainly starts the denominator with a polynomial of degree (n+1), as we would expect from (15.4); the form being

$$Q^{(x)}_{n+\Lambda} = \frac{\Gamma(\frac{1}{2})\,\Gamma(n+1+\Lambda)}{2^{n+1+\Lambda}\ \Gamma(n+\frac{3}{2}+\Lambda)} \cdot \frac{1}{x^\Lambda\left[\text{poly. of degree } (n+1)+\text{C.F}\right]}.$$

In particular

$$Q_{4.25}(x) = \frac{\Gamma(\frac{1}{2})\,r(5.25)}{2^{5.25}\,r(5.75)}\;\frac{1}{x^{\frac{1}{4}}\left[x^5 - 1.4266\,x^3 + 0.45508\,x - \frac{0.00587}{x} + ---\right]}$$

the details, the coefficients in the C.F, are listed in Table C. Numerically these approximations for $Q_{4\cdot 25}(x)$ are quite good, at $x^2 = 2$ the tenth approximant gives 6 significant figures. However, the coefficients in the C.F are in no sense comparable to those in (15.4) for $Q_4(x)$. Padé approximants do not naturally generalise the convergents of our elementary C.F (15.4) to non-integer values of λ. We will return to this point in a later paper; for the moment we will be content with a generalisation of the solutions about the origin.

The Regular Point at the Origin

For Legendre's differential equation

$$(x^2-1)\,y'' + 2xy' - \lambda(\lambda+1)y = 0 \tag{15.10}$$

the origin is a regular point, so that in terms of series its complete solution can be written

$$y = A_1 y_1 + A_2 Y_2 \tag{15.11}$$

$$\equiv A_1\left[1 - \frac{\lambda(\lambda+1)}{2!}x^2 + ---\right] + A_2\left[x - \frac{(\lambda-1)(\lambda+2)}{3!}x^3 + ---\right],$$

the coefficients being generated by the recurrence relation

$$A_{r+2} = -\left[\frac{(\lambda-r+1)(\lambda+r)}{r(r+1)}\right]A_r\,.$$

The first series y_1 reduces to a polynomial when λ = an even integer ≥ 0, while the series y_2 terminates when λ = a positive odd integer; we shall consider λ positive. Now suppose we increase λ from an integer n through non-integer values we expect that the effect will be to modify the polynomial solution by the addition of a C.F. To investigate this hypothesis we construct the Padé approximants to the fir3t series y_1 as λ is increaaed from one to four. Similar results apply to the second solution y_2.

The polynomial terms together with the first three terms of the C.F, as A varies, are indicated below.

$$\lambda = 1 \quad y_1 = 1 - \frac{x^2}{1} - \frac{0.3333x^2}{1} - \frac{0.2667x^2}{1} - --- = 1 - \left[\frac{x}{2}\log\left(\frac{1+x}{1-x}\right)\right]$$

$$\lambda = 1.5 \quad y_1 = 1 - \frac{1.875x^2}{1} - \frac{0.1875x^2}{1} - \frac{0.3542x^2}{1} - ---$$

$$\lambda = 2 \quad y_1 = 1 - 3x^2$$

$$\lambda = 2.5 \quad y_1 = 1 - 4.375x^2 + \frac{1.0026x^4}{1} - \frac{0.3750x^2}{1} - \frac{0.2187x^2}{1} - ---$$

$$\lambda = 3 \quad y_1 = 1 - 6x^2 + \frac{3x^4}{1} - \frac{0.2667x^2}{1} - \frac{0.2690x^2}{1} - ---$$

$$\lambda = 3.5 \quad y_1 = 1 - 7.875x^2 + \frac{6.3984x^4}{1} - \frac{0.1417x^2}{1} - \frac{0.3271x^2}{1} - ---.$$

$$\lambda = 4 \quad y_1 = 1 - 10x^2 + 11.6667x^4$$

The coefficients in the C.F part of y_1., for λ = 1, 1.5, 2.5, 3, 3.5 are tabulated in Tables D, they are rational functions of $\lambda(\lambda+1)$. The settling of these coefficents towards the value - 0.25 stands out in each case. But also observe how, as λ is increased, the C.F dies away as it extends one polynomial solution to the next of higher degree. In general we have, writing $\lambda = 2n + \Lambda$ and taking $A_1. = 1$,

$$y_1 = 1 + A_3x^2 + ---- + A_{2n+1}x^{2n} + \frac{A_{2n+3}x^{2n+2}}{1} + \frac{(\Lambda-2)(4n+\Lambda+3)\,x^2}{(2n+3)(2n+4)} + -- \qquad (15.12)$$

so that as $\Lambda \to 2$ we find y_1 reduces to a polynomial.

These expressions are fitting the derivatives of y_1 , at the origin and attempting to produce the branch cuts from $x = \pm 1$. To obtain some indication of the accuracy with which they are approximating y_1., we have computed the convergents at various values of x; at $x^2 = 0.9$ with $\lambda = 2.5, 3, 3.5$ some seventeen terms of the C.F are required to produce five significant figures.

Rational functions, and therefore continued fractions, are intimately bound to integral transfores, an aspect developed in TR/25. Consequently some of the continued fractions that we have obtained can usefully be interpreted as integral transforms and inverted to yield approximations to the originals. The technique we developed in section 11 is directly applicable to a variety of algebraic and first order differential problems. To conclude we will derive a formal continued fraction for the Laplace transform of a function.

16. *J* Fraction for the Laplace Transform of f(a+t).

We assume the Laplace transform $\mathcal{L}f(a+t)$ exists and that the function f(t) is such that we can use as our starting point the property

$$\mathcal{L}f' = s\,\mathcal{L}\,f - f(a) \tag{16.1}$$

where f = f(a+t). Our J fraction will match the series expansion

$$\ell f = \frac{f(a)}{s} + \frac{\ell f'}{s} = \frac{f(a)}{s} + \frac{f'(a)}{s^2} + \frac{f''(a)}{s^3} + \dots. \tag{16.2}$$

Now $$\ell f' = \int_0^\infty e^{-st}\,\frac{df}{dt}(a,t)\,dt = \int_0^\infty e^{-st}\,\frac{df}{dt}(a,t)\,dt = \frac{\partial}{\partial a}\ell f,$$

therefore (16.1) can be written

$$\frac{\partial}{\partial a}\ell f = s\ell f - f(a) \tag{16.3}$$

which is a suitable form for applying our method.

We set $$\ell f = \frac{C_n(s,a)}{D_n(s,a)} + \frac{F_n(s,a)}{D_n(s,a)} \qquad (16.4)$$

and arrange that $\frac{C_n}{D_n}$ matches the first *2n* terms of the series (16.2),

We Find that

$$\underline{\frac{\partial}{\partial a}\left[\frac{F_n}{D_n}\right] - s\left[\frac{F_n}{D_n}\right] = -\frac{A_{n+1}}{D_n^2}} \qquad (16.5)$$

where $A_{n+1} = c_1\ c_2\ \ldots\ c_{n+1}$., the partial numerators in the C.F being -c for n>1 see (16.9).

This determines not only the error in approximating $\mathcal{L}f$ by $\frac{C_n}{D_n}$ but

proceeding, as in section 11, we deduce the relations satisfied by D_n'

$$D'_n + c_{n+1}\ D_{n-1} = 0 \qquad (16.6)$$

$$D'_{n-1} + \left(s - \frac{A'_n}{A_n}\right) D_{n-1} - D_n = 0 \qquad (16.7)$$

where the prime indicates differentiation with respect to a.

Putting $d = \frac{A'_n}{A_n}$ and eliminating D'_n we obtain the recurrence relation

$$\underline{D_{n+1} = (s-d_{n+1})\ D_n - c_{n+1}\ D_{n-1}}\ . \qquad (16.8)$$

F_n satisfies this same three term recurrence relation, and hence

$$\mathcal{L}\,F(a+t) = \frac{f(a)}{s-d_1(a)} - \frac{c_2(a)}{s-d_2(a)} - \frac{c_3(a)}{s-d_3(a)} - \ldots \frac{c_n(a)}{s-d_n(a)} - \frac{F_n}{F_{n-1}} \qquad (16.9)$$

where the $c_n(a)$ and d_n (a) can be successively generated by (16. 10).

Using (16.6) and $d_n = \frac{A_n'}{A_n}$ we find

$$c_{n+1} = \frac{\partial d_n}{\partial a} + c_n \quad \text{and} \quad d_{n+1} = \frac{c_{n+1}'}{c_{n+1}} + d_n \; ; \tag{16.10}$$

for the initial conditions we take $d_1 = \frac{f'(a)}{f(a)}$ and $c_1 = 0$.

These relations can be used directly to generate C.F's, for example (2.17) and (3-5) are readily obtained, but essentially the result (16.9) is a formal one. Rutishauser [§ 4] in his investigations of the Q.D algorithm derived (16.9) by a limiting process. With this direct derivation our starting point is precise and more information is available on the D_n and the error term.

Conclusion

An algebraic structure centred around the J fraction for a function satisfying a first order differential equation has been constructed. It generalises and extends features developed in TR/25 for some special functions. We have concentrated on the J fraction, which approximates the function for large x, rather than other continued fractions for two reasons. The second order differential equations satisfied by the f_n are particularly important. Secondly the continued fractions for the f_n can often be usefully interpreted as Laplace transforms, the convergents being inverted by first expanding them in partial fractions. We have also shown that the analysis is applicable to other first order problems, the most significant being the derivation of M fractions for certain functions satisfying differential equations.

The theory presented is neat but limited. A. number of difficulties have been indicated and only partly resolved. We did not obtain an estimate of the error in the problem with three singularities, and we have only indicated how rational function approximations extend to second order differential equations in which the parameter n takes non-integer values. However, we have shown that asymptotic formulae for the coefficients in some continued fractions can be found; this and the concept of quasi-periodicity generalise.

Numerically it is often sufficient to simply derive a continued fraction for a function, or its transform, which fits derivatives of the function at one *(or* more) .regular point. In general, however, continued fractions whose coefficients are quasi-periodic are of most interest, indicating as they do the branch point structure of the function. It is upon these and in developing asymptotic formulae for generating their coefficients that we will concentrate.

TABLES A

COEFFICIENTS c_n AND d_n IN TEE C.F OF THE PARTICULAR INTEGRAL

OF $(x^3 - 1)\frac{dy}{dx} = 3(\alpha(^{2} + \beta x + \gamma)y - (1+3\alpha)x - 3\beta.$

$\alpha = \frac{1}{3}$ $\beta = \gamma = \frac{1}{6}$			$\alpha = \frac{1}{3}$, $\beta = 0$, $\gamma = \frac{2}{3}$		
n	c_n	d_n	n	c_n	d_n
1	1.000000	0.000000	1	1.000000	0.000000
2	-0.125000	1.700000	2	-0.500000	0.400000
3	-2.706667	-1.540209	3	0.006667	-31.828571
4	0.170248	-0.127784	4	-1009.921769	31.728902
5	-0.158664	1.571167	5	-0.008915	0.114929
6	-2.287107	-1.434432	6	-0.028545	7.901099
7	0.170092	-0.128807	7	-62.882730	-7.956774
8	-0.168036	1.536492	8	0.034327	0.058928
9	-2.178812	-1.404657	9	-0.042535	5.465697
10	0.169883	-0.128283	10	-30.069405	-5.499657
11	-0.172461	1.520536	11	0.048521	0.035309
12	-2.129528	-1.390739	12	-0.050045	4.727100
13	0.169744	-0.127784	13	-22.454871	-4.748690
14	-0.175039	1.511385	14	0.055384	0.022319
15	-2.101410	-1.382694	15	-0.054732	4.371630
16	0.169648	-0.127397	16	-19.176654	-4.385283
17	-0.176727	1.505456	17	0.059392	0.014109
18	-2.083248	-1.377455	18	-0.057915	4.162954
19	0.169579	-0.127099	19	-17.368894	-4.171094
20	-0.177917	1.501304	20	0.062009	0.008452
21	-2.070555	-1.373774	21	-0.060231	4.025828
22	0.169527	-0.126865	22	-16.227879	-4.029920
23	-0.178802	1.498235	23	0.063849	0.004319
24	-2.061185	-1.371047	24	-0.061988	3.928886
25	0.169487	-0.126678	25	-15.443645	-3.929881
26	-0.179486	1.495874	26	0.065212	0.001167
27	-2.053985	-1.368945	27	-0.063366	3.856744
28	0.169455	-0.126525	28	-14.872061	-3.855293
29	-0.180030	1.494001	29	0.066261	-0.001316
30	-2.048280	-1.367276	30	-0.064476	3.800975
31	0.169428	-0.126398	31	-14.437225	-3.797544
32	-0.180473	1.492480	32	0.067092	-0.003322
33	-2.043648	-1.365918	33	-0.065389	3.756578
34	0.169407	-0.126291	34	-14.095437	-3.751511
35	-0.180841	1.491220	35	0.067768	-0.004977
36	-2.039813	-1.364792	36	-0.066153	3.720400
37	0.169388	-0.126200	37	-13.819788	-3.713959
38	-0.181151	1.490159	38	0.068328	-0.006365
39	-2.036585	-1.363843	39	-0.066802	3.690353

TABLES A (cont)

COEFFICIENTS c_n AND d_n IN THE C.F OF THE PARTICULAR INTEGRAL

$$\text{OF } (x^3-1)\frac{dy}{dx}=3(\alpha(^2+\beta x+\gamma)y-(1+3\alpha)x-3\beta.+$$

$\alpha = \frac{1}{3}$ $\beta = \gamma = \frac{1}{6}$ $\alpha = \frac{1}{3}$, $\beta = 0$, $y = \frac{2}{3}$

n	c_c	d_n	n	c_n	d_n
40	0.169372	-0.126121	40	-13.592813	-3.682743
41	-0.181417	1.489254	41	0.068799	-0.007547
42	-2.033831	-1.363033	42	-0.067360	3.665003
43	0.169359	-0.126053	43	-13.402689	-3.656384
44	-0.181646	1.488472	44	0.069202	-0.008564
45	-2.031454	-1.362333	45	-0.067845	3.643328
46	0.169347	-0.125992	46	-13.241130	-3.633831
47	-0.181847	1.487789	47	0.069549	-0.009450
48	-2.029381	-1.361722	48	-0.068271	3.624583
49	0.169336	-0.125939	49	-13.102158	-3.614315
50	-0.182023	1.487189	50	0.069851	-0.010227
51	-2.027557	-1.361184	51	-0.068647	3.608213
52	0.169327	-0.125891	52	-12.981353	-3.597261
53	-0.182180	1.486657	53	0.070118	-0.010915
54	-2.025940	-1.360707	54	-0.068981	3.593793
55	0.169319	-0.125848	55	-12.875374	-3.582231
56	-0.182320	1.486182	56	0.070353	-0.011529
57	-2.024497	-1.360281	57	-0.069282	3,580995
58	0.169311	-0.129809	58	-12.781653	-3.568886
59	-0.182446	1.485755	59	0.070564	-0.012079
60	-2.023201	-1.359898	60	-0.069552	3.569559

TABLES A (cont)

COEFFICIENTS c_n AND d_n IN THE C.F OF THE PARTICULAR INTEGRAL

OF $(x^3 - 1)\frac{dy}{dx} = 3(\alpha(^2 + \beta x + \gamma)y - (1 + 3\alpha\alpha) - 3\beta$

$\alpha = 3$, ß = 2 , $\gamma = 1$

n	c_n	d_n
1	1.000000	0.000000
2	-0.250000	0.769231
3	-0.226331	0.132730
4	0.046761	-2.731354
5	-0.272298	2.984785
6	-0,066225	0.086126
7	-0.024463	7.569739
8	-56.246359	-7.420071
9	0.028765	0.020952
10	-0.081550	2.620903
11	-6.565110	-2.483389
12	0.087568	-0.037118
13	-0.125498	1.864499
14	-3.238119	-1.713866
15	0.125873	-0.085528
16	-0.160432	1,562745
17	-2.208730	-1.392058
18	0.152020	-0.125579
19	-0.189027	1.403542
20	-1.727487	-1.211795
21	0.170597	-0.158948
22	-0.21 2971	1.306865
23	-1.452920	-1.095068
24	0.184247	-0.187052
25	-0.233374	1.242868
26	-1.276727	-1.012656
27	0.194564	-0.210989
28	-0.251002	1.197927
29	-1.154553	-0.951040
30	0.202551	-0.231595
31	-0.266407	1.164972
32	-1.065065	-0.903061
33	0.208862	-0.249505
34	-0.279996	1.139990
35	-0.996792	-0.864546
36	0.213938	-0.265210
37	-0.292081	1.120544
38	-0.943038	-0.832893
39	0.218084	-0.279088
40	-0.302905	1.105075

ZEROS AND POLES OF CONVERGENTS

$$\alpha = \frac{1}{3}, \beta = 0, \gamma = \frac{2}{3}$$

ZEROS OF NUMERATORS

ZERO OF DENOMINATORS

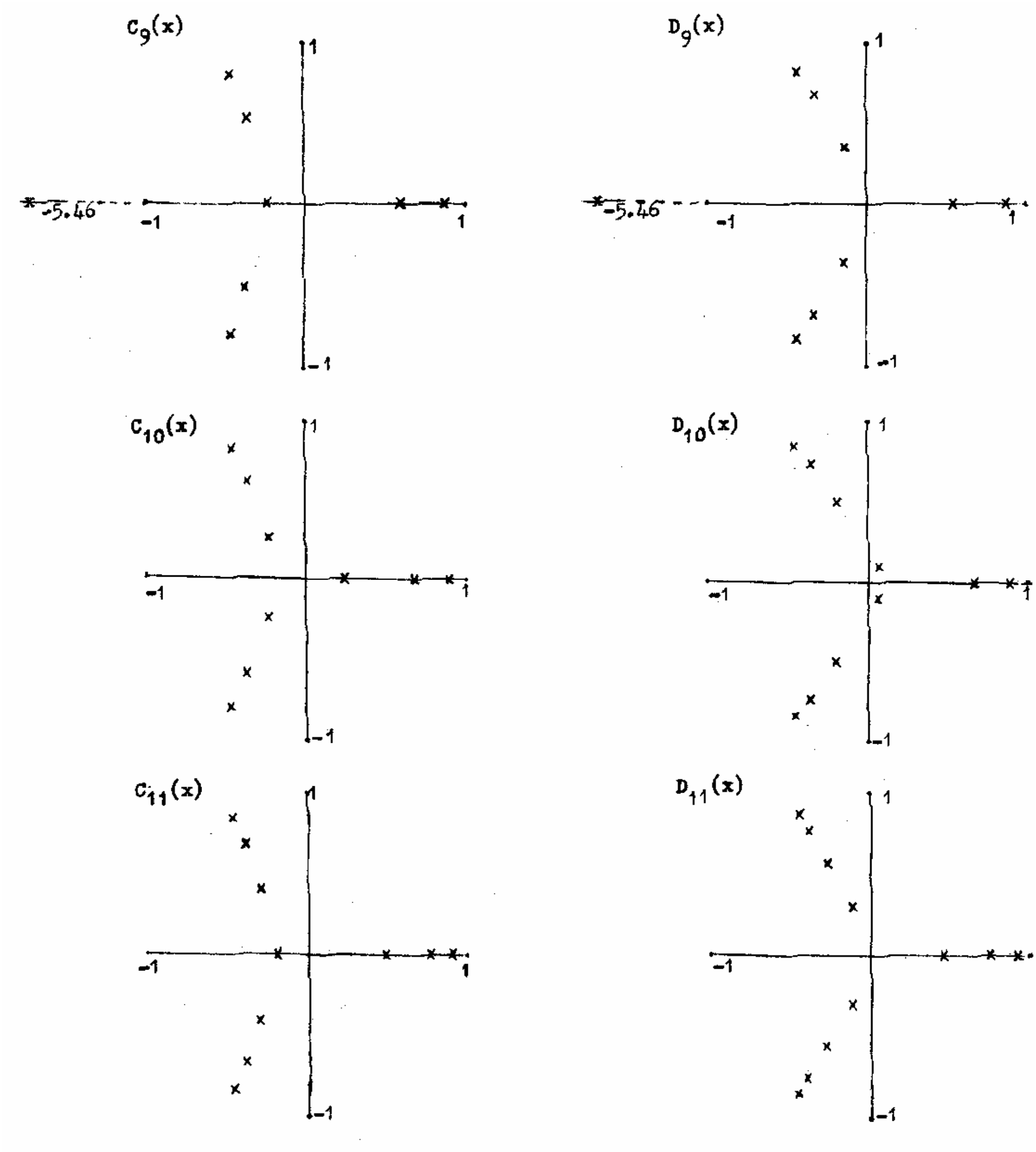

TABLE B

COEFFICIENTS p_n AND q_n IN THE M - FRACTION

FOR $f = \cot^{-1} x$

n	P_n	q_n.
1	1.5707963	1.5707963
2	-0.9341766	0.9341766
3	-0.5003349	0.9793851
4	-0.5051991	0.9925126
5	-0.5046930	0.9967252
6	-0.5036107	0.9983113
7	-0.5027398	0.9990151
8	-0.5021159	0.9993733
9	-0.5016723	0.9995755
10	-0.5013508	0.9996987
11	-0.5011122	0.9997783
12	-0.5009309	0.9998320
13	-0.5007902	0.9998696
14	-0.5006789	0.9998968
15	-0.5005894	0.9999169
16	-0.5005165	0.9999321
17	-0.5004562	0.9999438
18	-0.5004059	0.9999529
19	-0.5003634	0.9999600
20	-0.5003270	0.9999649

TABLES C

COEFFICIENTS IN THE CONTINUED FRACTION PART OF

THE LEGENDRE FUNCTION Q_λ (x) WHEN λ = 0 , 0.25 , 4.25.

$Q_0(x)$		$Q_{0\cdot25}(x)$		$Q_{4.25}(x)$	
Coeffs. of S Fraction		Coeffs. of S Fraction		Coeffs. of S Fraction	
j	P_i	j	P_j	j	P_j
1	1.0000000	1	-0.4017857	1	-0.0058743
2	-0.3333333	2	-0.2260552	2	0.2945793
3	-0.2666667	3	-0.2814200	3	-0.6165775
4	-0.2571429	4	-O.2361369	4	0.1896098
5	-0.2539683	5	-0.2664741	5	-1.1916838
6	-0.2525253	6	-0.2403340	6	-0.1085335
7	-0.2517483	7	-0.2610090	7	0.0837972
8	-0.2512821	8	-0.2425934	8	-0.6068143
9	-0.2509804	9	-0.2582261	9	0.1274644
1	-0.2507740	10	-0.2440005	10	-0.2147691
1	-0.2506266	11	-0.2565519	11	-0.8933427
1	-0.2505176	12	-0.2449599	12	0.0493964
1	-0.2504348	13	-0.2554378	13	-1.0053992
1	-0.2503704	14	-0.2456556	14	0.6043554
1	-0.2503193	15	-0.2546445	15	-0.0062869
1	-0.2502781	16	-0.2461829	16	-0.8266352
1	-0.2502444	17	-0.2540515	17	-0.7527953
1	-0.2502165	18	-0.2465963	18	0.9312590
1	-0.2501931	19	-0.2535919	19	-0.7789507
2	-0.2501733	20	-0.2469291	20	0.0057600
2	-0.2501563	21	-0.2532254	21	0.2643442

TABLES D

COEFFICIENTS IK THE CONTINUED FRACTION PART

OF y_1 FOR λ = 1, 1.5 , 2.5 , 3 , 3.5

$\lambda = 1$

Coeffs. of S Fraction

j	P_j
1	-1.0000000
2	-0.3333333
3	-0.2666667
4	-0.2571429
5	-0.2539683
6	-0.2525253
7	-0.2517483
8	-0.2512821
9	-0.2509804
10	-0.2507740
11	-0.2506266
12	-0.2505176
13	-0.2504348
14	-0.2503704
15	-0.2503193
16	-0.2502781
17	-0.2502444
18	-0.2502165
19	-0.2501931

$\lambda = 1.5$

Coeffs. of S Fraction

j	P_j
1	-1.8750000
2	-0.1875000
3	-0.3541667
4	-0.2162115
5	-0.2889640
6	-0.2275208
7	-0.2739839
8	-0.2333907
9	-0.2673302
10	-0.2368559
11	-0.2635672
12	-0.2391312
13	-0.2611470
14	-0.2407372
15	-0.2594596
16	-0.2419306
17	-0.2582159
18	-0.2428520
19	-0.2572613

$\lambda = 2.\ 5$

Coeffs. of S Fraction

3	P_j
	1.0026042
1	-0.3750000
2	-0.2187500
3	-0.2959325
4	-0.2273557
5	-0.2760975
6	-0.2329742
7	-0.2683100
8	-0.2365276
9	-0.2641333
10	-0.2388859
11	-0.2615157
12	-0.2405505
13	-0.2597187
14	-0.2417847
15	-0.2584080
16	-0.2427353
17	-0.2574093
18	

$\lambda = 3$

Coeffs. of S Fraction

j	P_j
	3.0000000
1	-0.2666667
2	-0.2690476
3	-0.2607459
4	-0.2558814
5	-0.2534745
6	-0.2522484
7	-0.2515702
8	-0.2511609
9	-0.2508949
10	-0.2507119
11	-0.2505802
12	-0.2504822
13	-0.2504071
14	-0.2503484
15	-0.2503016
16	-0.2502636
17	-0.2502324
18	

$\lambda. = 3.5$

Coeffs. of S Fraction

j	P_j
	6.3984375
1	-0.1416667
2	-0.3270833
3	-0.2239252
4	-0.2883854
5	-0.2307257
6	-0.2734508
7	-0.2347506
8	-0.2668644
9	-0.2375866
10	-0.2632180
11	-0.2395903
12	-0.2608863
13	-0.2410539
14	-0.2592600
15	-0.2421627
16	-0.2580590
17	-0.2430296
18	

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