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SUMMARY

A regression model for type 1 censored exponentially distributed observations with an exponential link function for the means is considered. Three methods of estimation are examined. The first method is maximum likelihood using the uncensored times to failure. The second and third methods are maximum likelihood and weighted least squares, respectively, using numbers of failures only. The moment properties of the estimators of the regression coefficients are obtained by simulation for the case of a single regressor variate. Small sample variance efficiencies are compared with asymptotic results.

CONTENTS

1. INTRODUCTION

Suppose that we have g groups of 'individuals'or 'components',the ith group containing n_i individuals. The response variable of interest is time to failure. We shall assume that all individuals enter the investigation at time zero and that some right censoring of the data occurs because of the need for early termination of the experiment. Specifically we shall assume that type 1 censoring within groups occurs. Thus if t_i. denotes the fixed censoring time for all individuals in the ith group, the lifetime of an individual in the ith group is known exactly only if it is less than t_i. Let Y^*_{ij}. and Y_{ij}. be r.v's representing the time to failure and recorded survival time, respectively, for the jth individual in the ith group, $i = 1,\ldots,g, j=1,\ldots,n_i$. We have

$$Y_{ij} = \begin{cases} y^*_{ij} & \text{if } y^*_{ij} < y_{ji} \\ t_i & \text{if } y^*_{ij} > y_{ij} \end{cases} \qquad (1.1)$$

A more complicated form of type 1 censoring occurs if the individuals within the groups have a staggered entry into the investigation, but this will not be considered here.

We now suppose that measurements are available on k regressor or explanatory variables and that the individuals in the ith group have the same values $x_{i1},\ldots,x_{ik}$ for these variables. If the underlying distribution is assumed to be exponential, $Y^*_{i1},\ldots,Y^*_{in_i}$ are taken to be independently and identically distributed with p.d.f.

$$f_i(y) = \mu_i^{-1}\exp(y/\mu_i), \qquad y > 0 \qquad (1.2)$$

and zero otherwise. An exponential link function for the mean is assumed with

$$\mu_i = \exp(\underset{\sim}{x}_i' \underset{\sim}{\beta}), \qquad i = 1,\ldots\ldots, g \qquad (1.3)$$

where = $(1,x_{ij},\ldots,x_{ik})$ and = $(b_0, b_1\ldots, b_k)$ is a vector of parameters with unknown values. This form of model for the means has been discussed by Glasser(1967), Prentice(1973), Lawless(1976) and others and has the advantage over other models, such as the linear model $\mu_i = \underset{\sim}{X}_i' \underset{\sim}{\beta}$, , that the requirement $\mu_i . > 0$ is automatically satisfied for all $\underset{\sim}{x}_i$ and $\underset{\sim}{\beta}$.

In this report, we consider three methods of estimation for the vector of regression coefficients, $\underset{\sim}{\beta}$. The first method is maximum likelihood (ML) using the observed times to failure. The second method is ML using the observed numbers of failures in the groups, denoted by $r_1,r_2,\ldots,r_g$. The $\{r_i\}$ are distributed as independent binomial random variables with $r_i \sim b(n_i,P_i)$, where $P_i = 1 - \exp(-t_i/\mu_i)$. For the binomial observations, an alternative method of estimation is weighted least squares (WLS) with empirically estimated weights. Reviews of the three methods of estimation and the methods for generating numerical solutions are given in sections 2, 3 and 4. Asymptotic variance efficiency results for the two ML methods of estimation relative to ML with uncensored data are given in section 5. Finally, in section 6 the results and findings are given of a Monte Carlo investigation of the moment properties of the estimators when a single regressor variate is present.

2. ML ESTIMATION USING TIMES TO FAILURE

Since the survival function for an individual in the ith group is $S_i(y) = \exp(-y/\mu_i)$, the likelihood for the ith group is

$$\ell_i = \binom{n_i}{r_i} \prod_{j=1}^{r_i} f_i(y) \left\{S_i(t_i)\right\}^{n_i - r_i} = \binom{n_i}{r_i} \mu_i^{-r_i} \exp\left[-\mu_i^{-1}\left\{\sum_{j=i}^{r_i} y_{ij} + (n_i - r_i)t_i\right\}\right] \tag{2.1}$$

If no regression model is imposed on the $\{\mu_i\}$, the ML estimate of μ_i is

$$\tilde{\mu}_i = r_i^{-1}\left\{\sum_{j=1}^{r_i} y_{ij} + \left(n_i - r_i\right)t_i\right\} = r_i^{-1} z_i \text{ say, } i = 1, \ldots g \tag{2.2}$$

where z_i is the total recorded survival time for individuals in the ith group. If the regression model for the means is given by (1.3), the log-likelihood over all groups is

$$L_1 = \sum_i \log\binom{n_i}{r_i} - r_i \underset{\sim}{x}_i' \underset{\sim}{\beta} - z_i e^{-\underset{\sim}{x}_i' \underset{\sim}{\beta}}$$

We have

$$\frac{\partial L_1}{\partial \beta_s} = \sum_i x_{is}(z_i e^{-\underset{\sim}{x}_i' \underset{\sim}{\beta}} - r_i), \qquad s = 0,1,\ldots, k \tag{2.3}$$

where $x_{i0} = 1$ for $i = 1,\ldots,g$. The ML estimate $\hat{\underset{\sim}{\beta}}_I$ is given by the solution of the $k + 1$ equations

$$\sum_i x_{is} z_i \hat{\mu}_{iI}^{-1} = \sum_r r_i x_{is}, \qquad s = 0,1, \ldots ,k \tag{2.4}$$

where $\hat{\mu}_{iI} = \exp(\underset{\sim}{x}_i' \hat{\underset{\sim}{\beta}}_I)$, $i = 1,\ldots g$.

The second order derivative is

$$\frac{\partial^2 L_I}{\partial \beta_s \partial \beta_t} = -\sum_i x_{is} x_{it} z_i \mu_{il}^{-1}, \quad s, t = 0,1, \ldots k \tag{2.5}$$

From Bartholomew (1957) we have

$$E(Z_i)=n_i P_i \mu_i \tag{2.6}$$

where $P_i = 1 - \exp(-t_i/\mu_i)$. The elements in the information matrix are

$$E\left(-\frac{\partial^2 L_1}{\partial \beta_s \partial \beta_t}\right) = \sum_i n_i P_i x_{is} x_{it}, \quad s, t = 0,1, \ldots k \tag{2.7}$$

The maximum likelihood equations (2.4) are readily solved by the Fisher scoring method. The following procedure due to Aitken and Clayton (1980) shows how the estimates can be obtained using the statistical package GLIM. Setting $m_i = z_i/\mu_i$, $i = 1,\ldots,g$, the Log-likelihood is

$$L_1 = \sum_i \left\{ \log \binom{n_i}{r_i} - r_i \log z_i \right\} + \sum_i \log m_i - \sum_i m_i$$

The kernel of the log-likelihood is

$$\sum_i r_i \log m_i - \sum_i m_i \tag{2.8}$$

which is equivalent to treating $r_1, r_2 \ldots, r_g$ as independent Poisson random variables with means $m_1, m_2, \ldots, m_g$. Writing

$$\log m_i = \log z_i + \underset{\sim}{x}_i^{*'} \underset{\sim}{\beta} \tag{2.9}$$

where $\underset{\sim}{x}_i^{*} = -\underset{\sim}{x}$ a logarithmic link function is used with $\log z_i$ as an offset variable.

A useful distribution result for the ML estimator $\hat{\underset{\sim}{\beta}}$ can be derived from the likelihood equations (2.4) which have the form

$$\sum_i x_{is} \left\{ \sum_{j=1}^{r_i} y_{ij} + (n_i - r_i) t_i \right\} e^{-\underset{\sim}{x}_i' \underset{\sim}{\beta}_i} = \sum_i r_i x_{is}, \tag{2.10}$$

$s = 0,1,\ldots,k$. The solution may be written as

$$\hat{\beta}_{is} = h_s\left(\{y_{ij}\}, \{r_i\}, \{t_i\}\right), \qquad s = 0,1, \ldots, k \tag{2.11}$$

although the form of $h_s(.)$ is not known explicitly. We define

$$Y^*_{ij} = y_{ije} - \underset{\sim}{x}'_i \underset{\sim}{\beta}, \quad t^*_i = t_i e - \underset{\sim}{x}'_i \underset{\sim}{\beta}, \quad \underset{\sim}{\beta}^*_i - \underset{\sim}{\beta} \tag{2.12}$$

and rewrite (2.10) as

$$\sum_i x_{is} \left\{ \sum_{j=1}^{r_i} y^*_{ij} + \left(n_i - r^*_j\right) t^*_i \right\} e^{-\underset{\sim}{x}'_i \underset{\sim}{\beta}^*_1} = \sum_i r^*_i x_{is}, \tag{2.13}$$

$s = 0,1,...,k$, where $r^*_i \sim b(n_i.P^*_i)$ with P^*_i - $1\text{-}\exp(-t^*_i)$. Since $P^*_i = P_i$, r_i and r^*_i are identically distributed binomial random variables. A comparison of (2.10) and (2.13) shows that

$$\beta^*_{1s} = h_s\left(\left\{y^*_{ij}\right\}, \left\{r^*_i\right\} \left\{t^*_i\right\}\right), \qquad s = 0,1, \ldots, k \tag{2.14}$$

This result shows that the distribution of $\underset{\sim}{\beta}^*_1$ is the same as that of the ML estimator of $\underset{\sim}{\beta}$ when the observations $\{y^*_{ij}\}$ are distributed as standard exponential (i.e. with $\underset{\sim}{\beta} = \underset{\sim}{0}$) and the adjusted censoring times are $\{t^*_i\}$.

3. ML ESTIMATION USING NUMBERS OF FAILURES

We now suppose that the times to failure are not recorded and that the only data available are the numbers of individuals failing within the groups. The $\{r_i\}$ are independent binomial r.v's with $r_i \sim b(n_i,P_i)$, where $P_i = 1\text{-}\exp(-t_i/\mu_i)$ and the log-likelihood based on the $\{r_i\}$ is

$$L_2 = \sum_i \left\{ \log \binom{n_i}{r_i} + r_i \log\left(1 - e^{-t_i/\mu_i}\right) - \left(n_i - r_i\right) t_i \mu_i^{-1} \right\}$$

We have

$$\frac{\partial L_2}{\partial \beta_s} = -\sum_i \frac{\left(r_i - n_i p_i\right)}{p_i} \frac{t_i}{\mu_i} x_{is}, \qquad s = 0,1, \ldots ,k \tag{3.1}$$

The ML estimate $\underset{\sim}{\hat{\beta}}_2$ is the solution of the k+1 equations

$$\sum_i \frac{\left(r_i - n_i \bar{p}_{i2}\right)}{\hat{p}_{i2}} \frac{t_i}{\hat{\mu}_{i2}} x_{is} = 0, \qquad s = 0,1, \ldots ,k \tag{3.2}$$

where $\hat{\mu}_{i2} = \exp\left(\underset{\sim}{x}'_i \underset{\sim}{\hat{\beta}}_2\right)$ and $\hat{p}_{i2} = 1 - \exp\left(-t_i/\hat{\mu}_{i2}\right)$.

The second order derivative is

$$\frac{\partial^2 L_2}{\partial \beta_s \partial \beta_t} = \sum_i \frac{t_i}{\mu_i} \left\{ \frac{\left(r_i - n_i P_i\right)}{P_i} - \frac{r_i t_i Q_i}{\mu_i P_i^2} \right\} x_{is} x_{it} \tag{3.3}$$

for s,t = 0,1,..., k and we have

$$E\left(-\frac{\partial^2 L_2}{\partial\beta_s \partial\beta_t}\right) = \sum_i n_i \frac{Q_i}{P_i}\left(\frac{t_i}{\mu_i}\right)^2 x_{is} x_{it} \tag{3.4}$$

The likelihood equations (3.3) can again be solved using Fisher's scoring method. Since

$$\log\{-\log(1-P_i\} = \log t_i + \underset{\sim}{x}_i^{*'} \underset{\sim}{\beta}, \qquad i = 1,\ldots,g \tag{3.5}$$

the fit can be made using GLIM with a complementary log-log link function and $\log t_i$ as an offset variable.

The likelihood equations

$$\sum_i \left\{ \frac{r_i}{1 - \exp(-t_i e^{-\underset{\sim}{x}_1^i \underset{\sim}{\beta}_2})} - n_i \right\} \frac{t_i x_{is}}{e^{-\underset{\sim}{x}_1^i \underset{\sim}{\beta}_2}} = 0, \quad s = 0,1, \ldots, k \tag{3.6}$$

may be written as

$$\sum_i \left\{ \frac{r_i^*}{1 - \exp(-t_i e^{-\underset{\sim}{x}_1^i \underset{\sim}{\beta}_2^*})} - n_i \right\} \frac{t_i^* x_{is}}{e^{-\underset{\sim}{x}_1^i \underset{\sim}{\beta}_2^*}} = 0, \quad s = 0,1, \ldots, k \tag{3.7}$$

where $\underset{\sim}{\beta}_2^* = \hat{\underset{\sim}{\beta}}_2 - \underset{\sim}{\beta}$. It follows that the distribution of $\underset{\sim}{\beta}_2^*$ is the same as that of the ML estimator of $\underset{\sim}{\beta}$ when the observations are r_i^* and the adiusted censoring times are $\{t_i^*\}$.

4. WLS ESTIMATION USING NUMBERS OF FAILURES

An alternative method of estimation to ML for binomial data is to use WLS as follows. Let $p_i = r_i./n_i$ denote the observed proportion of failures in the ith group. The complementary log-log transform of p_i is

$$z_i = \log\{-\log(1-p_i)\}, \qquad i = 1,\ldots,g \tag{4,1}$$

Writing $z_i = g(p_i)$, where $g(p) = \log\{-\log(1-p)\}$ and using the standard first order approximations $E(z_i) \approx g\{E(p_i)\}$ and var(z.) w $\mathrm{var}(p_i)\,[g^1\{E(p_i)\}]^2$, we have

$$E(z_i) \approx \log t_i + \underset{\sim}{x}_i^* \underset{\sim}{\beta}, \qquad \mathrm{var}(z_i) \approx (1-Q_i)/\{n_i Q_i \log^2 Q_i\} \tag{4.2}$$

Define

$$Z_i^* = z_i - \log t_i, \qquad i = 1,\ldots,g \tag{4.3}$$

We have the approximate linear model $E(z_i^*) = \underset{\sim}{x}_i^{*\prime} \underset{\sim}{\beta}$. The approximate variance of Z depends on Q_i and hence is unknown. Using $q_i = 1-p_i$ to estimate Q_i, the estimated large sample variance of z_i^* is

$$(1-q_i)/(n_i q_i \log^2 q_i) = w_i^{-1}, \qquad \text{say} \tag{4.4}$$

Applying WLS with empirical weights $\{w_i\}$, the WLS estimates are found by minimising $\sum_i (z_i^* - \underset{\sim}{x}_i^{*\prime} \underset{\sim}{\beta})^2$ and the solution is

$$\underset{\sim}{\hat{\beta}}_3 = \left(\underset{\sim}{X}^{*\prime} \underset{\sim}{W} \underset{\sim}{X}^*\right)^{-1} \underset{\sim}{X}^{*\prime} \underset{\sim}{W} \underset{\sim}{X}^* \tag{4.5}$$

where $\underset{\sim}{W} = \text{diag}(w_1,\ldots,w_g)$, $\underset{\sim}{z}^{*\prime} = (Z_1^*, \ldots Z_g^*)$ and $\underset{\sim}{x}^* = ((-x_{ij}))$ .We note that although z_i^* is unfined when $p_i=0$ or 1, no problem arise as $w_i=0$ in these cases and z_i^* is discarded in the fit.

5. ML SYMPTOTIC EFFICIENCY COMPARISONS

In this section, we examine the asymptotic efficiency of the ML estimators with censoring, relative to the ML estimators when no censoring occurs and all failure times are observed. We shall restrict attention to the case $t_i = t$, $i = 1,\ldots,g$, when the same censoring time applies to all groups.

Consider first the case when no regressor variates are present so that all groups have the same exponential lifetime distribution with mean $\mu = \exp(\beta_0)$. If $\hat{\beta}_{10}$ denotes the ML estimate of b_0 using the observed times to failure when censoring occurs, use of (2.7) gives the asymptotic variance

$$\text{var}_a(\hat{\beta}_{10}) = 1/E(-\partial^2 L_1 / \partial\beta_0^2) = (NP)^{-1} \tag{5.1}$$

where $N = \sum n_i$. and $P = 1-\exp(-t/\mu)$ is the common probability of failure before time t . "Similarly if $\hat{\beta}_{20}$ denotes the ML estimate of β_o using the numbers of failures within the groups, use of (3.4) gives

$$\text{var}_a(\hat{\beta}_{20}) = 1/E(-\partial^2 L_1 / \partial b^2{}_0) = \{N(l-P)\log^2(l-P)/p\}^{-1} \tag{5.2}$$

Let $\hat{\beta}_0$ denote the ML estimator., of β_0 when no censoring occurs. Since $P_i = 1$ when t = □ , use of (5.1) gives

$$\text{var}_a(\hat{\beta}_0) = N^{-1} \tag{5.3}$$

We shall consider the limiting case with n_i □□ , $i = 1,\dots,g$ such that $\lim n_i./N \to \lambda_i$, where $0 < \lambda_i < 1$, $i = 1,\dots,g$. Define

$$E_{1a}^{(0)} = \lim \mathrm{var}_a(\hat{\beta}_0)/\mathrm{var}_a(\hat{\beta}_{10}),\quad E_{2a}^{(0)} = \lim \mathrm{var}_a(\hat{\beta}_0)/\mathrm{var}_a(\hat{\beta}_{20}) \qquad (5.4)$$

Then

$$E_{1a}^{(0)} = p, \qquad E_{2a}^{(0)} = (1-P)\log^2(1-P)/P \qquad (5.5)$$

Values of $E_{1a}^{(0)} = P$ and $E_{2a}^{(0)}$ are shown in table 1 for $P = 0.1(0.1)\ 0.9,\ 0.95,\ 0.99$. The results show that the efficiency differences are very small for $P \leq 0.5$. The efficiency values $E_{2a}^{(0)}$ reach a maximum of at $P = 0.8$ and approach zero slowly as P approaches 1.

Table 1.

Asymptotic variance efficiencies $E_{1a}^{(0)} = P$ and $E_{2a}^{(0)}$ of ML estimators

P	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
$E_{2a}^{(0)}$	0.10	0.20	0.30	0.39	0.48	0.56	0.62	0.65	0.59	0.47	0.21

We now consider the case when there is a single regressor variate. Let $\hat{\beta}_{10}, \hat{\beta}_{11}$ denote the ML estimates of β_0, β_1 using the observed times to failure with censoring present. From (2.7) the information matrix is

$$\underset{\sim}{I}_1 = \begin{bmatrix} \sum_i n_i P_i & \sum_i n_i x_i P_i \\ \sum_i n_i x_i P_i & \sum_i n_i x_i^2 P_i \end{bmatrix}$$

where $P_i = 1-\exp(-t/\mu_i)$. Inversion of the matrix gives

$$\mathrm{var}_a(\hat{\beta}_{10}) = \sum_i n_i x_i^2 P_i / D_1, \qquad \mathrm{var}_a(\hat{\beta}_{11}) = \sum_i n_i P_i / D_1 \qquad (5.6)$$

where $D_1 = (\sum_i n_i P_i)(\sum_i n_i x_i^2 P_i) - (\sum_i n_i x_i P_i)^2$.

Similarly let $\hat{\beta}_{20}$, $\hat{\beta}_{21}$ denote the ML estimates of β_0 and β_1 using the observed numbers of failures. From (3.4), the information matrix is

$$\underset{\sim}{I}_2 = \begin{bmatrix} \sum_i n_i \frac{Q_i}{P_i}\left(\frac{t}{\mu_i}\right)^2 & \sum_i n_i x_i \frac{Q_i}{P_i}\left(\frac{t}{\mu_i}\right)^2 \\ \sum_i n_i x_i \frac{Q_i}{P_i}\left(\frac{t}{\mu_i}\right)^2 & \sum_i n_i x_i^2 \frac{Q_i}{P_i}\left(\frac{t}{\mu_i}\right)^2 \end{bmatrix}$$

Substituting $\Phi_i = Q_i P_i^{-1} \log^2 Q_i$, we have

$$\mathrm{var}_a\left(\hat\beta_{20}\right) = \sum_i n_i x_i^2 \phi_i / D_2, \quad \mathrm{var}_a\left(\hat\beta_{21}\right) = \sum_i n_i \Phi_i / D_2 \tag{5.7}$$

where $D_2 = (\sum_i n_i \phi_i)(\sum_i n_i x_i^2 \phi_i) - (\sum_i n_i x_i \Phi_i)^2$.

Finally let $\hat\beta_0$, $\hat\beta_1$, denote the ML estimates of β_0, β_1, when no censoring occurs- Putting $P_i = 1, i = 1,\ldots g$ in (5.6) we have

$$\mathrm{var}_a\left(\hat\beta_0\right) = \sum_i n_i x_i^2 / D, \qquad \mathrm{var}_a\left(\hat\beta_1\right) = \sum_i n_i / D \tag{5.8}$$

where $D = (\sum_i n_i)(\sum_i n_i x_i^2) - (\sum_i n_i x_i)^2$. Using (5.6), (5.7) and (5.8), the asymptotic efficiences for β_0 are

$$E_{1a}^{(1)} = \frac{\sum_i \lambda_i x_i^2 \{(\sum_i \lambda_i P_i)(\sum_i \lambda_i x_i^2 P_i) - (\sum_i \lambda_i x_i P_i)^2\}}{D^* \sum_i \lambda_i P_i x_i^2} \tag{5.9}$$

$$E_{2a}^{(0)} = \frac{\sum_i \lambda_i x_i^2 \{(\sum_i \lambda_i \Phi_i)(\sum_i \lambda_i x_i^2 \Phi_i) - (\sum_i \lambda_i x_i \Phi_i)^2\}}{D^* \sum_i \lambda_i \Phi_i x_i^2}. \tag{5.10}$$

and for β_1 are

$$E_{1a}^{(1)} \quad \frac{\sum_i \lambda_i \{(\sum_i \lambda_i P_i)(\sum \lambda_i x_i^2 P_i) - (\sum_i \lambda_i x_i P_i)^2\}}{D^* \sum_i \lambda_i P_i} \tag{5.11}$$

$$E_{2a}^{(1)} \quad \frac{\sum_i \lambda_i \{(\sum_i \lambda_i \Phi_i)(\sum \lambda_i \Phi_i x_i^2) - (\sum_i \lambda_i x_i \Phi_i)^2\}}{D^* \sum_i \lambda_i \Phi_i} \tag{5.12}$$

where $D^* = (\sum_i \lambda_i)(\sum_i \lambda_i x_i^2) - (\sum_i \lambda_i x_i)^2$.

The efficiencies given by (5.9) to (5.12) depend on the probabilities of failure before time t for individuals in each of the g groups. A convenient way to specify the probabilities is to set $P_1 = p$ and $P_g = \theta^{-1} p$, where p and θ are prescribed constants. Putting

$$v_i = \{\log(1-p)/\log(1-\theta^{-1}p)\}^{(x_i-x_1)/(x_g-x_1)}, \quad i = 1,\ldots,g \tag{5.13}$$

we have

$$P_i = 1 - (1-p)^{v_i} \quad i = 1,\ldots,g. \tag{5.14}$$

Values of the asymptotic variance efficiences have been computed for the case when there is a single explanatory variable with values $x_i = i - \frac{1}{2}(g+1)$ and equal sample sizes with $\lambda_i = g^{-1}$, for $g = 5$, 10, Values $p = 0.5, 0.6, 0.7, 0.8$ and $\theta = 1(1)5$ were used and the results are shown in tables 2 and 3.

It is seen that

(i) there is an appreciable loss of efficiency as the group probabilities of censoring increase with decreasing p and increasing θ,

(ii) the additional loss of efficiency through using ML based on numbers of failures instead of times to failure is quite marked when there is a small degree of censoring but becomes negligible as the degree of censoring increases,

(iii) the efficiencies for estimation of β_0 and β_1 are very similar and almost independent of the value of g.

Table 2

Asymptotic variance efficiencies $E^{(0)}_{1,a}$ and $E^{(0)}_{2,a}$ for β_0 for the case of a single regressor variate with $P_1 = p$ and $P_g = \theta^{-1} p$.

		$E^{(0)}_{1,a}$				$E^{(0)}_{2,a}$			
	θ\ p	0.8	0.7	0.6	0.5	0.8	0.7	0.6	0.5
	1.0	0.80	0.70	0.60	0.50	0.65	0.62	0.56	0.48
	1.5	0.66	0.57	0.49	0.41	0.59	0.53	0.47	0.40
	2.0	0.56	0.49	0.42	0.35	0.52	0.47	0.41	0.34
	2.5	0.50	0.43	0.37	0.31	0.47	0.42	0.36	0.30
g=5	3.0	0.45	0.39	0.33	0.27	0.43	0.38	0.33	0.27
	4.0	0.38	0.33	0.28	0.23	0.37	0.32	0.28	0.23
	5.0	0.34	0.29	0.24	0.20	0.33	0.28	0.24	0.20
	0	0.80	0.70	0.60	0.50	0.65	0.62	0.56	0.48
	1.5	0.66	0.57	0.49	0.41	0.59	0.54	0.47	0.40
	2.0	0.57	0.49	0.42	0.35	0.53	0.47	0.41	0.34
	2.5	0.51	0.44	0.37	0.31	0:48	0.42	0.37	0.31
g= 10	3.0	0,46	0.40	0.34	0.2S	0.44	0.39	0.33	0.28
	4.0	0.39	0.34	0.29	0.24	0.38	0.33	0.28	0.23
	5.0	0.35	0.30	0.25	0.21	0.34	0.29	0.25	0.21

Table 3

Asymptotic variance efficiencies $E^{(0)}_{1,a}$ and $E^{(0)}_{2,a}$ for b_1 for the case of a single regressor variate with P_1 - p and $P_g = \theta^{-1}$ p.

	θ\p	$E^{(0)}_{1,a}$ 0.8	0.7	0.6	0.5	$E^{(0)}_{2,a}$ 0.8	0.7	0.6	0.5
g=5	1.0	0.80	0.70	0.60	0.50	0.65	0.62	0.56	0.48
	1.5	0.65	0.57	0.49	0.41	0.58	0.53	0.47	0.40
	2.0	0.57	0.49	0.42	0.35	0.51	0.46	0.41	0.34
	2.5	0.51	0.44	0.38	0.31	0.46	0.42	0.36	0.31
	3.0	0.46	0.40	0.34	0.28	0.42	0.38	0.33	0.28
	4.0	0.40	0.35	0,29	0.24	0.36	0.33	0.29	0.24
	5.0	0.35	0.31	0.26	0.22	0.33	0.29	0.25	0.21
g=10	1.0	0.80	0.70	0.60	0.50	0,65	0.62	0.56	0.48
	1.5	0.65	0.57	0.49	0.41	0.58	0.53	0.47	0.40
	2.0	0.57	0.50	0.43	0.35	0.52	0.47	0.41	0.35
	2.5	0.51	0.45	0.38	0.32	0,47	0.42	0.37	0.31
	3.0	0.47	0.41	0.35	0.29	0.43	0.39	0.34	0.28
	4.0	0.41	0.35	0.30	0.25	0.38	0.34	0.29	0.25
	5.0	0.36	0.32	0.27	0.22	0.34	0.30	0.26	0.22

6. SMALL SAMPLE MOMENT PROPERTIES OF THE ESTIMATORS

In this section, we report the results of a Monte Carlo investigation to examine the moment properties of the ML and WLS estimators of the regression coefficients for the case of a single explanatory variable. The small sample variance efficiencies of the estimators relative to the maximum likelihood estimation when no censoring is present (t_i.= ∞, i=1,,.. ,g) are compared with their asymptotic values given in the previous section.

In the simulation investigation, equally spaced values for the regressor variate were taken, the regression model for the group true means being $\mu_i = \exp(\beta_0+\beta_1 x_i)$, with

$$x_i = i - \frac{1}{2}(g+1)\ , i - 1,\ldots,g \quad . \tag{6.1}$$

Equal sample sizes were used with $n_i = n$ = 10,20,50, i - 1,... ,g and g = 5,10. Without loss of generality, the value of β_0 was taken as zero, the probability of failure in the ith group being $P_i = 1 - \exp(-te^{\beta_1 X_1})$. Values of t and β_1 were used such that P_1 = p, P_g = p/θ with p = 0.5,0.6,0.7,0.8 and θ = 1(1)5. The ML and WLS estimates were obtained using GLIM, the run-size being 2000 in each case.

Values of the biases × 10^2 of the estimators of β_0 and β_1 are shown in tables 4 and 5, respectively, from which the following broad conclusions can be reached.

a) The bias patterns for the ML estimators $\hat{\beta}_{10}$ and $\hat{\beta}_{20}$ of B_0 are similar with the bias increasing as the degree of censoring increases with p becoming smaller and θ becoming larger. For θ = 1 and p > 0.5 the biases of $\hat{\beta}_{10}$ and $\hat{\beta}_{20}$ are negative. When there is no censoring, the bias of $\hat{\beta}_{10}$ is - N^{-1} to order N^{-1} (Al-Abood and Young (1985)).

b) The WLS estimator $\hat{\beta}_{30}$ of $\hat{\beta}_0$ has a negative bias in nearly all cases, the absolute value of the bias increasing with increasing values of and decreasing values of p.

c) The ML estimators $\hat{\beta}_{11}$ and $\hat{\beta}_{21}$ of β_2 have similar bias patterns. The biases are positive in nearly all cases and increases with increasing θ (increasing B_i) but the degree of censoring appears to have little effect.

d) The bias of the WLS estimator $\hat{\beta}_{31}$ of β_1 is very small when θ is small but there is a strong negative bias when θ is large.

Table 4

Values of bias × 10^2 for estimators of β_0

n = 10, g = 5

	p = 0,.8			p = 0.7		
θ	$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$	$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
1	-0.91	-2.71	3.96	-0.30	-1,00	-1.69
2	0.12	0.52	-2.21	1.34	0.92	-4.85
3	0.86	1.44	-5,50	1.65	1.27	-7.78
4	2.29	1.61	-6.92	3.48	3.14	-9.30
5	3.70	3.12	- 7.80	5.52	5.29	-10.00

	p = 0.			p = 0.5		
θ	$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$	$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
1	-0,01	-0.56	-4.44	0.95	0.62	-5.53
2	2.26	2.09	-6.17	3.48	3.37	-7.77
3	2-67	2.42	-9.73	3.51	3.44	-12.76
4	4-91	4.67	-11.62	6.95	6.76	-14.42
5	6.88	6.75	-13.28	9.21	9.11	-17.03

(Table 4 cont)

n = 20, g = 5

P = 0.8

θ	$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
1	-0.55	-1.22	-1.44
2	-0.04	-0.37	-2.66
3	0.88	0.59	-2.74
4	1.06	0.93	-3.35
5	1.42	1.15	-4.14

p = 0.7

$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
-0.32	-0.66	-2.39
0.28	0.12	-2.91
1.26	1.08	-3.19
1.42	1.32	-4.11
2.19	2.02	-4.86

P= 0.6

θ	$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
1	0.14	-0.35	-2.69
2	0.45	0.24	-3.57
3	1.82	1.70	-3.48
4	1.66	1.52	-5.20
5	2.78	2.62	-5.62

p = 0.5

$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
0.30	0.18	-2.82
1.12	0.99	-3.81
2.41	2.31	-4.26
2.38	2.28	-6.15
3.78	3.69	-6.80

n = 10, g = 10

P - 0.8

θ	$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
1	-0.55	-1.22	3.83
2	-0.16	-0.53	-4.93
3	0.69	0.37	-7.25
4	1.09	0.98	-9.11
5	1.18	1 .08	-11.01

p = 0.7

$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
-0.32	-0.66	-2.51
0.23	0.03	-7.33
1.29	1.14	-9.56
1 .38	1.26	-12.13
1.73	1.50	-14.53

P = 0.6

θ	$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
1	-0.14	-0.35	-5.62
2	0.44	0.25	-9.65
3	1.93	1.87	-11 .73
4	1.93	1.81	-15.37
5	2.76	2.64	-17.39

p = 0.5

$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
0.29	0.17	-7.64
1.26	1.17	11.84
2.40	2.34	15.07
2.32	2.21	19.54
3.63	3.52	-21.74

n = 20, g = 10

P = 0.8

θ	$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
1.	-0.56	-0.91	-1.57
2	0.02	-0.10	-3.12
3	0.12	-0.06	-4.25
4	0.50	0.32	-4.92
5	0.50	0.34	-5.64

p = 0.7

$\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
-0.42	-0.55	-2.79
0.11	-0.01	-3.85
0.33	0.23	-4.94
0.79	0.67	-5.57
0.92	0.88	-6.66

(Table 4 cont)

θ	P = 0.6 $\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$	p = 0.5 $\hat{\beta}_{10}$	$\hat{\beta}_{20}$	$\hat{\beta}_{30}$
1	-0.34	-0.43	-3.47	0.17	-0.25	-4.16
2	-0.33	0.24	-4.50	0.50	0.41	-5.53
3	0.50	0.38	-5.89	1.05	0.98	-6.87
4	1.17	1.12	-6.64	1.44	1.40	-8.33
5	1.36	1.30	-7.71	1.74	1.67	-9.85

Table 5

Values of bias × 10^2 for estimators of β_1

n = 10, g = 5

θ	P = 0.8 $\hat{\beta}_{11}$	$\hat{\beta}_{21}$	$\hat{\beta}_{31}$	p = 0.7 $\hat{\beta}_{11}$	$\hat{\beta}_{21}$	$\hat{\beta}_{31}$
1	0,15	0.27	-0.02	0,03	0.02	-1.03
2	0.96	1.67	-1.91	0.90	1.31	-1.19
3	1.32	2.38	-3.12	1.32	1.89	-2.82
4	2.21	3.49	-4.14	2.43	3.00	-4.01
5	2.79	4.13	-4.91	3.33	4.05	-4.72

θ	P - 0.6 $\hat{\beta}_{11}$	$\hat{\beta}_{21}$	$\hat{\beta}_{31}$	P=0.5 $\hat{\beta}_{11}$	$\hat{\beta}_{21}$	$\hat{\beta}_{31}$
1	0.03	-0.08	-0.09	0.07	-0.04	-0.02
2	0.66	0.74	-1.55	0.99	1.04	-1.95
3	1.38	1.81	-3.29	1.19	1.44	-4.86
4	2.70	2.97	-4.83	3.30	3.49	-6.40
5	3.42	3.81	-6.26	4.19	4.40	-8.41

n = 20, g - 5

θ	P = 0.8 $\hat{\beta}_{11}$	$\hat{\beta}_{21}$	$\hat{\beta}_{31}$	p = 0.7 $\hat{\beta}_{11}$	$\hat{\beta}_{21}$	$\hat{\beta}_{31}$
1	-0.02	0.02	0.10	-0.12	-0.24	-0.24
2	0.88	1.19	0.03	0.86	0.97	0.29
3	1.03	1.43	-0.18	1.04	1.24	-0.40
4	0.91	1.28	-0.89	1.07	1.26	-1.07
5	1.13	1.55	-1.33	1.36	1.59	-1.70

Table 5 (cont)

θ	P = 0.6 $\hat\beta_{11}$	$\hat\beta_{21}$	$\hat\beta_{31}$	P = 0.5 $\hat\beta_{11}$	$\hat\beta_{21}$	$\hat\beta_{31}$
1	0.03	-0.02	-0.02	0.10	0.08	0.07
2	0.87	0.98	-0.16	1.02	1.07	-0.04
3	1.13	1.25	-0.68	1.29	1.34	-1.13
4	1.08	1.24	-1.53	0.98	1.21	-2.52
5	1.43	1.64	-2.18	1.54	1.62	-3.29

n = 10, g = 10

θ	P = 0.8 $\hat\beta_{11}$	$\hat\beta_{21}$	$\hat\beta_{31}$	p = 0, .7 $\hat\beta_{11}$	$\hat\beta_{21}$	$\hat\beta_{31}$
1	-0.01	0.00	0.02	0.05	-0.10	-0.10
2	0.37	0.52	-1.18	0.39	0.46	-0.73
3	0.46	0.60	-1.62	0.50	0.60	-1.45
4	0.45	0.59	-2.44	0.49	0.61	-2.26
5	0.51	0.65	-3.18	0.49	0.59	-3.00

θ	P=0.6 $\hat\beta_{11}$	$\hat\beta_{21}$	$\hat\beta_{31}$	p = 0.5 $\hat\beta_{11}$	$\hat\beta_{21}$	$\hat\beta_{31}$
1	0.02	0.01	0.03	0.02	0.01	0.04
2	0.33	0.36	0.79	0.43	0.44	-0.92
3	0.46	0.52	-1.57	0.40	0.45	-2.06
4	0.56	0.65	-2.62	0.66	0.71	-3.37
5	0.59	0.68	-3.37	0.73	0.78	-4.18

n = 20, g = 10

θ	P=0.8 $\hat\beta_{11}$	$\hat\beta_{21}$	$\hat\beta_{31}$	p = 0.7 $\hat\beta_{11}$	$\hat\beta_{21}$	$\hat\beta_{31}$
1	0.12	0.12	0.09	0.13	0.10	0.10
2	0.14	0.18	-0.27	0.15	0.19	-0.21
3	0.18	0.28	-0.48	0.14	0.18	-0.57
4	0.18	0.32	-0.76	0.14	0.19	-0.94
5	0.20	0.29	-1.03	0.22	0.28	-1.10

θ	P =0.6 $\hat\beta_{11}$	$\hat\beta_{21}$	$\hat\beta_{31}$	p = 0.5 $\hat\beta_{11}$	$\hat\beta_{21}$	$\hat\beta_{31}$
1	0.14	0.12	0.12	0.16	0.14	0.15
2	0.17	0.20	-0.27	0.13	0.14	-0.43
3	0.17	0.21	-0.66	0.13	0.15	-0.84
4	0.18	0.24	-1.05	0.23	0.25	-1.45
5	0.27	0.31	-1.34	0.32	0.35	-1.77

Values of the variances $\times 10^2$ of the estimators of βo and β_1 are shown in tables 6 and 7, respectively. It is seen that for both estimation of β_0 and β_1 that the WLS estimators β_{30} and β_{31} have the smallest variances, their variance advantage becoming more pronounced as p decreases and θ increases, particularly for the smaller sample size. The approximating variances for the ML estimators $\hat{\beta}_{10}$, $\hat{\beta}_{2o}$, $\hat{\beta}_{11}$, $\hat{\beta}_{21}$ are consistently smaller than the simulated variances when n = 10 but provide satisfactory approximations for the larger sample size.

Table 6

Values of variances $\times$ 10^2 for estimators of So. Approximate variances given by (5.6) and (5.7) are shown in parentheses.

θ	var($\hat{\beta}_{10}$)	var($\hat{\beta}_{20}$)	var($\hat{\beta}_{30}$)	var($\hat{\beta}_{10}$)	var($\hat{\beta}_{20}$)	var($\hat{\beta}_{30}$)
			(i) n = 10, g = 5			
	p = 0 .8			P = 0.7		
1	2.601(2.500)	3.789(3.088)	2.820	2.889(2.857)	3.409(3.219)	2.862
2	3.778(3.544)	4.233(3.831)	3.741	4.376(4.084)	4.690(4.280)	4. 148
3	5.026(4.426)	5.497(4.657)	4.789	5.684(5.128)	5.976(5.288)	5.111
4	6.150(5.223)	6.581(5.426)	5.384	7.439(6.075)	7.648(6.215)	6.100
5	6.546(5.960)	6.784(6.144)	5.621	8.360(6.951)	8.446(7.079)	6.689
	p = 0 .6			P = 0.5		
1	3.353(3.333)	3.673(3.573)	3.336	4.202(4.000)	4.453(4.163)	4. 116
2	5.330(4.790)	5.535(4.930)	5.054	6.957(5.770)	7.086(5.870)	6.106
3	6.764(6.039)	6.925(6.154)	5.873	8.460(7.297)	8.601(7.380)	7.063
4	9.044(7.174)	9.187(7.275)	7.142	12.116(8.687)	12.284(8.761)	8.732
5	11.051(8.226)	11.192(8.319)	8.400	14.956(9.979)	15.070(10.047)	10.443
			(ii) n = 20, g - 5			
	p = 0 .8			P = 0.7		
1	1.253(1.250)	1.625(1.544)	1 .468	1.409(1.429)	1.625(1.610)	1.547
2	1.829(1.772)	2.006(1.916)	1 .911	2.073(2.042)	2.165(2.140)	2.066
3	2.249(2.213)	2.359(2.329)	2.274	2.696(2.564)	2.758(2.644)	2.610
4	2.670(2.612)	2.792(2.713)	2.633	3.162(3.037)	3.261(3.107)	3.078
5	3.076(2.980)	3.212(3.072)	3.010	3.708(3.476)	3.808(3.540)	3.498
	p = 0 .6			P = 0.5		
1	1.647(1.667)	1.774(1.787)	1.732	2.032(2.000)	2.158(2.081)	2.092
2	2.507(2.395)	2.566(2.465)	2.441	3.042(2.885)	3.066(2.935)	2.901
3	3.309(3.019)	3.368(3.077)	3.123	4.011(3.649)	4.047(3.690)	3.797
4	3.672(3.587)	3.728(3.637)	3.454	4.474(4.344)	4.510(4.380)	4.124
5	4.399(4.113)	4.463(4.159)	4.008	5.430(4.990)	5.476(5.023)	4.882

table 6 (cont)

θ	$var(\hat{\beta}_{10})$	$var(\hat{\beta}_{20})$	$var(\hat{\beta}_{3o})$	$var(\hat{\beta}_{10})$	$var(\hat{\beta}_{2o})$	$var(\hat{\beta}_{30})$
			(iii) n = 10, g - 10			
	p = 0,8			p = 0.7		
1	1.253(1.250)	1.623(1.544)	1.170	1.409(1.429)	1.625(1.610)	1.314
2	1.777(1.755)	1.905(1.893)	1.710	2.057(2.025)	2.109(2.120)	1.941
3	2.191(2.169)	2.283(2.278)	1.981	2.668(2.518)	2.750(2.595)	2.357
4	2.612(2.540)	2.729(2.634)	2.388	3.084(2.961)	3.189(3.027)	2.729
5	2.943(2.881)	3.057(2.965)	2.732	3.502(3.368)	3.591(3.428)	3.106
	p = 0.6			p = 0.5		
1	1.648(1.667)	1.773(1.787)	1.572	2.051(2.000)	2.157(2.081)	1.964
2	2.494(2.377)	2.560(2.446)	2.267	3.091(2.865)	3.126(2.914)	2.700
3	3.288(2.970)	3.359(3.025)	2.819	3.940(3.592)	3.982(3.633)	3.283
4	3.639(3.502)	3.712(3.551)	3.086	4.435(4.247)	4.481(4.282)	3.592
5	4.216(3.993)	4.276(4.037)	3.479	5.296(4.850)	5.341(4.883)	4.061
			(iv) n = 20, g = 10			
	p = 0.8			p = 0.7		
1	0.647(0.625)	0.802(0.772)	0.738	0.742(0.714)	0.834(0.805)	0.797
2	0.887(0.877)	0.953(0.947)	0.886	0.995(1.012)	1.056(1.060)	0.991
3	1.109(1.084)	1.158(1.139)	1.115	1.307(1.259)	1.343(1.297)	1.288
4	1.257(1.270)	1.322(1.317)	1.234	1.506(1.480)	1.549(1.514)	1.446
5	1.436(1.440)	1.474(1.483)	1.370	1.749(1.684)	1.780(1.714)	1.653
	p = 0.6			p = 0.5		
1	0.869(0.833)	0.933(0.893)	0.913	1.039(1.000)	1.074(1.041)	1.040
2	1.196(1.188)	1.241(1.223)	1.185	1.147(1.432)	1.459(1.457)	1.390
3	1.576(1.485)	1.623(1.513)	1.537	1.874(1.796)	1.901(1.816)	1.770
4	1.754(1.751)	1.774(1.775)	1.632	2.191(2.123)	1.198(2.141)	1.928
5	2.118(1.996)	2.135(2.018)	1.950	2.651(2.425)	2.663(2.441)	2.314

<u>Table 7</u>

Values of variances × 10^2 for estimators of $_1$. Approximate variances given by (5.6) and (5.7) are shown in parentheses.

	$var(\hat{\beta}_{11})$	$var(\hat{\beta}_{21})$	$var(\hat{\beta}_{31})$	$var(\hat{\beta}_{11})$	$var(\hat{\beta}_{21})$	$var((\hat{\beta}_{31})$
			(i) n = 10, g = 5			
	P = 0.8				p = 0.7	
1	1.304(1.250)	1.863(1.544)	1.445	1.550(1.429)	1.866(1.610)	1.556
2	1.917(1.768)	2.328(1.964)	1.996	2.190(2.026)	2.406(2.154)	2.169
3	2.389(2.175)	2.905(2.385)	2.327	2.789(2.498)	3.082(2.633)	2.577
4	2.950(2.523)	3.808(2.752)	2.829	3.443(2.902)	3.860(3.049)	3.058
5	3.253(2.833)	4.123(3.080)	3.032	3.977(3.263)	4.386(3.421)	3.492
	P = 0.6				p = 0.5	
1	1.757(1.667)	1.958(1.787)	1.759	2.148(2,000)	2,301(2.081)	2.116
2	2.669(2.368)	2.793(2.456)	2.495	3.343(2.845)	3.426(2.907)	2.877
3	3.332(2.923)	3.577(3.016)	2.968	4.040(3.516)	4.233(3.581)	3.431
4	4.115(3.401)	4.303(3.501)	3.413	5.275(4.095)	5.375(4.164)	4.091
5	4.899(3.828)	5.237(3.935)	4.146	6.092(4.612)	6.255(4.687)	4.856
			(ii) n = 20, g = 5			
	P = 0.8				p = 0.7	
1	0.633(0.625)	0,826(0.772)	0.765	0.731(0.714)	0.861(0.805)	0.835
2	0.892(0.884)	1.046(0.982)	0.972	1.015(1.013)	1.104(1.077)	1.047
3	1.095(1.088)	1.260(1.193)	1.158	1.291(1.249)	1.396(1.317)	1.313
4	1.315(1.262)	1.494(1.376)	1.386	1.514(1.451)	1.630(1.525)	1.519
5	1.444(1.416)	1.618(1.540)	1.500	1.759(1.631)	1.876(1.711)	1.709
	P = 0.6				p = 0.5	
1	0.835(0.833)	0.912(0.893)	0.889	1.029(1.000)	1.081(1.041)	1.045
2	1.182(1.184)	1.242(1.228)	1.172	1.453(1.422)	1.476(1.453)	1.353
3	1.538(1.462)	1.623(1.508)	1.506	1.819(1.758)	1.875(1.790)	1.708
4	1.763(1.701)	1.864(1.751)	1.709	2.163(2.048)	2.242(2.082)	2.023
5	2.097(1.914)	2.202(1.968)	1.971	2.529(2.306)	2.585(2.343)	2.285

table 7 (cont)

θ	var($\hat{\beta}_{11}$)	var($\hat{\beta}_{21}$)	var($\hat{\beta}_{31}$)	var($\hat{\beta}_{11}$)	var($\hat{\beta}_{21}$)	var($\hat{\beta}_{31}$)
			(iii) n = 10, g = 10			
	P = 0.8			P = 0.7		
1	0.155(0.152)	0.199(0.187)	0.143	0.176(0.173)	0.208(0.195)	0.174
2	0.217(0.212)	0.258(0.235)	0.215	0.247(0.244)	0.268(0.258)	0.236
3	0.262(0.259)	0.303(0.282)	0.244	0.310(0.298)	0.336(0.313)	0.286
4	0.312(0.298)	0.346(0.323)	0.281	0.362(0.344)	0.390(0.360)	0.326
5	0.338(0.333)	0.375(0.359)	0.301	0.400(0.385)	0.423(0.402)	0.354
	P= 0.6			P = 0.5		
1	0.201(0.202)	0.220(0.217)	0.201	0.246(0.242)	0.259(0.252)	0.235
2	0.289(0.285)	0.302(0.295)	0.270	0.356(0.343)	0.362(0.350)	0.314
3	0.372(0.349)	0.396(0.360)	0.340	0.428(0.421)	0.441(0.428)	0.372
4	0.430(0.404)	0.454(0.415)	0.367	0.517(0.487)	0.535(0.495)	0.423
5	0.487(0.453)	0.514(0.464)	0.419	0.589(0.546)	0.604(0.555)	0.477
			(iv) n = 20 g = 10			
	P = 0.8			P = 0.7		
1	0.072(0,076)	0.095(0.094)	0.086	0.080(0.087)	0.093(0.098)	0.083
2	0.105(0,106)	0.116(0.117)	0.109	0.122(0.122)	0.130(0.129)	0.123
3	0.138(0,129)	0.156(0.141)	0,145	0.157(0.149)	0.168(0.157)	0.159
4	0.153(0,149)	0.164(0.161)	0.149	0.185(0,172)	0.193(0.180)	0.178
5	0.171(0.166)	0.189(0.180)	0.169	0.197(0.192)	0.205 (0.201)	0,187
	P = 0.6			P = 0.5		
1	0.093(0.101)	0.101(0.108)	0.097	0.117(0.121)	0.123(0.126)	0.117
2	0.144(0.143)	0.149(0.148)	0,142	0.175(0.171)	0.179(0.175)	0.168
3	0.183(0.175)	0.190 (0.180)	0.181	0.228 (0.210)	0.234(0.214)	0.215
4	0.219(0.202)	0.224 (0.208)	0.206	0.277 (0.244)	0.281(0.248)	0.247
5	0.243(0.226)	0.248 (0.232)	0.224	0.298 (0.273)	0.301(0.277)	0.266

Finally, we examine the small sample variance efficiencies of the estimators under censoring relative to the ML estimators $\hat{\beta}_0$, $\hat{\beta}_1$ when no censoring occurs and all failure times are observed. The variances of β_0 and β_1 are independent of β_0 and β_1 for all sample sizes. Al-Abood and Young (1985) give the following values obtained by simulation using a run-size of 4000

$n = 10,\ g = 5;\quad 10^2\mathrm{var}(\hat{\beta}_0) = 2.048,\quad 10\ \mathrm{var}(\hat{\beta}_1) = 1.033$

$n = 20,\ g = 5;\quad 10^2\mathrm{var}(\hat{\beta}_0) = 0.992,\quad 10^2\mathrm{var}(\hat{\beta}_1) = 0.502$

$n = 10,\ g = 10;\quad 10^2\mathrm{var}(\hat{\beta}_0) = 1.030,\quad 10\ \mathrm{var}(\hat{\beta}_1) = 0.099$

$n = 20,\ g = 10;\quad 10\ \mathrm{var}(\hat{\beta}_0) = 0.508,\quad 10\ \mathrm{var}(\hat{\beta}_1) = 0.060$

Let

$$E_1^{(j)} = \mathrm{var}(\hat{\beta}_j)/\mathrm{var}(\hat{\beta}_{1j}),\ E_2^{(j)} = \mathrm{var}(\hat{\beta}_j)/\mathrm{var}(\hat{\beta}_{2j}),\ E_3^{(j)} = \mathrm{var}(\hat{\beta}_j)/\mathrm{var}(\hat{\beta}_{3j}) \quad (6.2)$$

$j = 0,1$ denote the small sample variance efficiences. Values of $E_t^{(0)}$ and $Et_t^{(1)}$, $t = 1,2,3$ are shown in tables 8 and 9 respectively for $n = 10, 20$, $g = 5, 10$, $p = 0.5(0.1)0.8$ and $\theta = 1(1)5$.

The results show that for estimation of both Bo and B_1 the efficiency values $E_1^{(0)}$ and $E_2^{(0)}$ approach their asymptotic values rapidly. The efficiency of the WLS estimator is appreciably higher than that of the ML estimator based on numbers of failures, when the degree of censoring is small, but the efficiency advantage quickly reduces as the degree of censoring increases.

Table 8

Small, sample variance efficiencies of the estimators for β_0 under censoring relative to the ML estimator β_0 without censoring.

(i) g = 5

	P = 0.8						P = 0.7					
	n = 10			n =20			n = 10			n = 20		
θ	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$	.$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$	$E_1^{(0)}$	.$E_2^{(0)}$	$E_3^{(0)}$	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$
1	0.79	0.54	0.73	0,79	0.61	0.68	0.71	0.60	0.72	0.70	0.61	0.64
2	0.54	0.48	0.55	0.54	0.50	0.52	0.47	0.44	0.49	0.48	0.46	0.48
3	0.41	0.37	0.43	0.44	0.42	0.44	0.36	0.34	0.40	0.37	0.36	0.38
4	0.33	0.31	0.38	0.37	0.36	0.38	0.28	0.27	0.34	0.31	0.30	0.32
5	0.31	0.30	0.36	0.32	0.31	0.33	0.25	0.24	0.31	0.27	0.26	0.28

	p = 0.6						p = 0.5					
	n = 10			n = 20			n = 10			n = 20		
θ	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$
1	0.61	0.56	0.61	0.60	0.56	0.57	0.49	0.46	0.50	0.49	0.46	0.47
2	0.38	0.37	0.41	0.40	0.39	0.41	0.29	0.29	0.34	0.33	0.32	0.34
3	0.30	0.30	0.35	0.30	0.30	0.32	0.24	0.24	0.29	0.25	0.25	0.26
4	0.23	0.22	0.29	0.27	0.27	0.29	0.17	0.17	0.23	0.22	0.22	0.24
5	0.19	0.18	0.24	0.23	0.22	0.25	0.14	0.14	0.20	0.18	0.18	0.20

(ii) g = 10

	P = 0.8						P = 0.7					
	n = 10			n = 20			n = 10			n = 20		
θ	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$
1	0.82	0.63	0.88	0.79	0.63	0.69	0.73	0.63	0.78	0.69	0.61	0.64
2	0.58	0.54	0.60	0.57	0.53	0.57	0.50	0.49	0.53	0.51	0.48	0.51
3	0.47	0.45	0.52	0.46	0.44	0.46	0.39	0.37	0.44	0.39	0.38	0.39
4	0.39	0.38	0.43	0.40	0.38	0.41	0.33	0.32	0.38	0.34	0.33	0.35
5	0.35	0.34	0.38	0.35	0.34	0.37	0.29	0.29	0.33	0.29	0.29	0.31

	P = 0.6						P = 0.5					
	n = 10			n = 20			n = 10			n = 20		
θ	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$	$E_1^{(0)}$	$E_2^{(0)}$	$E_3^{(0)}$
1	0.63	0.58	0.66	0.59	0.55	0.56	0.50	0.48	0.52	0.49	0.47	0.49
2	0.41	0.40	0.45	0.42	0.41	0.43	0.33	0.33	0.38	0.36	0.35	0.37
3	0.31	0.31	0.37	0.32	0.31	0.33	0.26	0.26	0.31	0.27	0.27	0.29
4	0.28	0.28	0.33	0.29	0.29	0.31	0.23	0.23	0.29	0.23	0.23	0.26
5	0.24	0.24	0.30	0.24	0.24	0.26	0.19	0.19	0.25	0.19	0.19	0.22

Table 9

Small sample variance efficiencies ofthe estimatorsfor β_1 under censoring relative to the ML estimatorβ_1 without censoring.

(i) g = 5

	p =0.8						p = 0.7					
	n = 10			n = 20			n = 10			n = 20		
θ	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$
1	0.78	0.56	0.72	0.79	0.61	0.66	0.67	0.55	0.66	0.69	0.58	0.60
2	0.54	0.44	0.52	0.56	0.48	0.52	0.47	0.43	0.48	0.49	0.46	0.48
3	0.43	0.36	0.44	0.46	0.40	0.43	0.37	0.34	0.40	0.39	0.36	0.38
4	0.35	0.27	0.37	0.38	0,34	0.36	0.30	0.27	0.34	0.33	0.31	0.33
5	0.32	0.25	0.34	0.35	0.31	0.34	0.26	0.24	0.30	0.29	0.27	0.29

	p = 0.6						p = 0.5					
	n = 10			n = 20			n = 10			n = 20		
θ	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$
1	0.59	0.53	0.59	0.60	0.55	0.56	0.48	0.45	0.49	0.49	0.47	0.48
2	0.39	0.37	0.42	0.42	0.40	0.43	0.31	0.30	0.36	0.35	0.34	0.37
3	0.31	0.29	0.35	0.33	0.31	0.33	0.26	0.24	0.30	0.28	0.27	0.29
4	0.25	0.24	0.30	0.29	0.27	0.29	0.20	0.19	0.25	0.23	0.22	0.25
5	0.21	0.20	0.25	0.24	0.23	0.25	0.17	0.17	0.21	0.20	0.19	0.22

(ii) g = 10

	p = 0.8						p = 0.7					
	n = 10			n = 20			a = 10			n = 20		
θ	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$
1	0.64	0.50	0.69	0.83	0.63	0.70	0.56	0.48	0.57	0.75	0.64	0.68
2	0.46	0.38	0.46	0.57	0.52	0.55	0.40	0.37	0.42	0.49	0.46	0.49
3	0.38	0.33	0.41	0.44	0.39	0.41	0.32	0.30	0.35	0.38	0.36	0.38
4	0.32	0.29	0.35	0.39	0.37	0.40	0.27	0.25	0.30	0.32	0.31	0.34
5	0.29	0.27	0.33	0.35	0.32	0.36	0.25	0.23	0.28	0.30	0.29	0.32

	p = 0.6						p = 0.5					
	n = 10			n = 20			n = 10			n = 20		
θ	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$
1	0.49	0.45	0.49	0.65	0.59	0.62	0.40	0.38	0.42	0.51	0.49	0.51
2	0.34	0.33	0.37	0.42	0.40	0.42	0.28	0.27	0.32	0.34	0.34	0.36
3	0.27	0.25	0.29	0.33	0.32	0.33	0.23	0.22	0.27	0.26	0.26	0.28
4	0.23	0.22	0.27	0.27	0.27	0.29	0.19	0.19	0.23	0.22	0.21	0.24
5	0.20	0.19	0.24	0.25	0.24	0.27	0.17	0.16	0.21	0.20	0.20	0.23

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