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ON THE USE OF CRUDE EIGENVALUE BOUNDS IN THE SOLUTION OF ROBBIN'S PROBLEM *

BY

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ABSTRACT

In this paper we consider what effect the use of crude eigenvalue estimates, for the evaluation of the iterative parameters, has on the convergence of the A.D.I. process which is employed in the numerical integration of the Third Boundary Value Problem.

1, Introduction

In the numerical solution of the Third Boundary Value Problem by means of an Alternating Direction Implicit scheme the extremal eigenvalues of a matrix are required. If the degenerate case of the Second Boundary Value problem is excluded, analytical expressions are not available for these eigenvalues.

Moreover, the study carried out in [3], effectively showed that not only can an algorithm be set up for the efficient evaluation of the extremal eigenvalues, but furthermore this can be done with an accuracy which is at our disposal.

In the same paper, it was also pointed out that crude bounds to the extremal eigenvalues may be found and the question which arises is about the cost-benefit ratio of such a usage. Obviously, it will save the computational effort required in the evaluation of the strict bounds, since the calculation of the crude counterparts is straightforward involving only simple analytical expressions. On the other hand, the use of such crude bounds will yield iterative parameters slightly different from the optimal ones and consequently the convergence of the procedure will not be as rapid as it would be in the case with iterative parameters produced on the basis of strict bound spectra.

In this work, firstly we shall show how such bounds may be obtained and secondly we shall discuss their effect on the convergenœ of the iterative procedure which will be the Extrapolated Alternating Direction Implicit (E.A.D.I.) scheme in the four forms studied in [3] and [4] .

2. General Considerations

The problem we shall deal with consists of Laplace's equation

$$u_{x_1x_1}(x_1, x_2) + u_{x_2x_2}(x_1, x_2) = 0 \qquad (1)$$

in a region R, satisfying the boundary condition

$$u_n + \alpha(x_1, x_2)u = H(X_1, X_2) \qquad (2)$$

on the boundary ∂ R of R, where u_n stands for the normal derivative directed away from the region R, and a, α are given functions properly defined on ∂R.

For the numerical solution of problem (1), (2) a uniform mesh of mesh size h is imposed on the region $R \cup \partial R$.

Now, if we assume that the region R is the unit square and adopt the ordering given in [3], then by using a 5-point difference formula the problem is transformed into the following matrix one:

$$(H + V)\, u = K, \qquad (3)$$

with

$$H = I_{N+1} \otimes U_1, \quad V = U_2 \otimes I_{N+1}, \qquad (4)$$

and

$$U_i = \begin{bmatrix} 2(1+p_i h) & -2 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2(1+q_i h) \end{bmatrix}, \qquad i = 1,2$$

matrices of order (N+l).

The vector u of order $(N+1)^2$ is the approximating solution, and K is a similar constant vector coming from the boundary conditions [see 3],

3. The E.A.D.I. Scheme

For the iterative solution of the matrix equation (3) we shall use the extrapolated form of the A.D.I, procedures [see 1, 2, 6] described by

$$\begin{aligned} (I+rH)u^* &= [(I+rH) - wr(H+V)]\, u^{(n)} + rwK; \\ (I+rV)u^{(n+1)} &= u^* + rVu^{(n)} . \end{aligned} \tag{5}$$

where $u^{(n)}$ is the approximating vector at the nth iteration ($u^{(0)}$ arbitrary), I denote;the unit matrix of order $(N+l)^2$, and w,r signify the extrapolation and the acceleration parameters respectively.

If we assume that the parameters (p_i, q_i), i=l,2 satisfy the conditions:

$$\begin{aligned} Pi + q_i &> 0 , \\ p_i q_{i\cdot} + p_i + q_i &> 0 , \end{aligned} \qquad i = 1,2 , \tag{6}$$

which imply that the operators H and V are positive definite, and accordingly the convergence of the scheme will be secured [see 5]. One can then choose appropriate iterative parameters, namely w and r, for speeding up the convergence of the scheme.

4. The Eigenvalues

The eigenvalues of H and V because of relationships (4) will be identical to those of U_i , i=l,2 and consequently they can be determined as roots of the determinantal equations

$$| U_i - \lambda I | = 0 , i=1,2$$

which can easily be put in the forms

$$\{\lambda^2-2\lambda+4h^2 p_i q_i\} T_{N-1}(\lambda) + 2h(p_i+q_i) [T_N(\lambda)-T_{N-2}(\lambda)] = 0 \quad (7)$$

with $T_N(\lambda)$ as in formula (10) in [3].

On the other hand, conditions (6) also ensure that the smalles root of (7) will be in the interval (0, 4 $\sin^2 \pi h/2$), while the largest one (which will be greater than 4) is less than

$$\text{u.b.} = \min \left\{ \begin{matrix} 1 + z_i \\ \max(5, z_i) \end{matrix} \right\}, \ i =1,2 \quad (8)$$

according to Geschgörin's theorem, where

$$z_i - 3 + 2h \max (p_i , q_i), \ i = 1,2 \quad (9)$$

The above upper bound, given by (8), takes the following forms along the real axis:

(i) if $z_i \geqq 5$ then u.b. = z_i else

(ii) if $4 \leqq z_i < 5$ then u.b. = 5 else (10)

(iii) u.b. - 1 + z_i .

Moreover, if x represents the $\max(p_i,q_i)$, then for the various number of subdivisions N relationships (10) will appear as follows:

$$\begin{aligned} &\text{(i) u.b.} = 3 + 2x/N && \text{for } N \phi x ; \\ &\text{(ii) u.b.} = 5 && \text{for } N > x \text{ and } N \phi 2x ; \\ &\text{(iii) u.b.} = 2(2+x/N) && \text{for } N > 2x. \end{aligned} \tag{11}$$

To find bounds for the smallest eigenvalue, we consider the polynomial which is produced from (7) by the transformation

$$\lambda = 4 \sin^2 \phi /2 , \phi \varepsilon R - \{0,\pi\}$$

that is

$$F(\phi) - (h^2 p_i q_i - \sin^2 \phi) \sin(N\phi) + h(p_i + q_i) \sin \phi \cos(N \phi). \tag{12}$$

The smallest root of (12) will obviously be within the interval $(0,\pi h)$ and it can easily be checked that $F(\pi h)<0$ holds. Therefore a bound on the smallest eigenvalue may be any point $\mu\pi h$, $\mu\varepsilon (0,1)$

$$F(\mu\pi h) > 0 \tag{13}$$

In the rest of this paragraph we find conditions which ensure that some characteristic points are in fact crude lower bounds.

To do this, we consider the cases:

I. The point $\phi_1 = \pi h/4$.

Condition (13) clearly becomes

$$F(\pi h/4) = \{ h^2 p_i q_i - \sin^2 (\pi h/4)\} \sin\left(\frac{\pi}{4}\right) + h(p_i + q_i) \sin\left(\frac{\pi h}{4}\right) \cos\left(\frac{\pi}{4}\right)$$

or equivalently

$$\text{sign}\,[F(\pi h/4)] = \text{sign}\{-\sin^2(\pi h/4)+h(p_i+q_i)\sin(\pi h/4)+h^2 p_i\, q_i\}.$$

Now, to have the L.H.S. greater than or equal to zero is tantamount to getting the quadratic involved in the R.H.S. non-negative, which in turn implies either

$$p_i q_i > 0\,, \quad \sin(\pi h/4 \leq \frac{h}{2}\,(\sqrt{p_i^{\,2}+q_i^{\,2}+6p_i q_i}+p_i+q_i)\;;$$

or

$$P_i\, q_i < 0\,,\; P_i^{2} + q_i^{2} + 6p_i\, q_i > 0\,, \text{ and}$$

$$\frac{h}{2}(p_i+q_i-\sqrt{p_i^{\,2}+q_i^{\,2}+6p_i q_i}) \leq \sin\left(\frac{\pi h}{4}\right) \leq \frac{h}{2}(p_i+q_i+\sqrt{(p_i^{\,2}+q_i^{\,2}+6p_i\, q_i)}$$

II The point $\phi_2 = \pi h/3$

By direct substitution we get

$$F(\pi h/3) = [h^2 p_i\, q_i - \sin^2(\pi h 3)]\,\sin(\pi/3) + h(p_i+q_i)\,\sin\left(\frac{\pi h}{3}\right)\,\cos\left(\frac{\pi}{3}\right)$$

and consequently

$$\text{sign}\,[F(\pi h/3)] = \text{sign}\,(-\sqrt{3}\,\sin^2(\pi h/3) + h(p_i+q_i)\,\sin(\pi h/3) + \sqrt{3}\,h^2 p_i q_i\,\}$$

Following the same strategy as above we obtain the conditions

$$p_i q_i > 0\,, \quad \sin(\pi h/3) \leq \frac{h}{2\sqrt{3}}\,(p_i+q_i+\sqrt{p_i^{\,2}+q_i^{\,2}+14p_i q_i})$$

or

$$p_i\, q_i < 0\,,\; p_i^{2} + q_i^{\,2} + 14p_i q_i > 0\,, \text{ and}$$

$$\frac{h}{2\sqrt{3}}\,1\;d_i - \sqrt{p_i^{\,2}+q_i^{\,2}+14p_i q_i}) \leq \sin\left(\frac{\pi h}{3}\right) \leq \frac{h}{2\sqrt{3}}(p_i+q_i+\sqrt{p_i^{\,2}+q_i^{\,2}+14p_i q_i}$$

III The point $\phi_3 = \pi h/2$

In this case we can readily find the condition

$$\sin^2(\pi h/2) \leq h^2 p_i q_i .$$

IV The point $f_4 = 2\pi h/3$

This time we have

$$\text{sign}[F(2\pi 2 = \text{sign}\{-\sqrt{3}\ \sin^2\left\{\frac{2\pi h}{3}\right\} - h(p_i + q_i)\ \sin\left\{\frac{2\pi\pi}{3}\right\} + \sqrt{3}\, h^2 p_i q_i\}$$

which yields the conditions

$$p_i > 0,\ q_i > 0,\quad \sin\left\{\frac{2\pi h}{3}\right\} \leq \frac{h}{2\sqrt{3}}\{\sqrt{p_i^2 + q_i^2 + 14 p_i q_i} - p_i - q_i\}.$$

V The point $f_5 = 3\pi h/4$

Adopting the same approach as before we reach the conditions

$$p_i > 0,\ q_i > 0,\quad \sin\left\{\frac{3\pi h}{4}\right\} \leq \frac{h}{2}\{\sqrt{p_i^2 + q_i^2 + 6 p_i q_i} - p_i - q_i\}.$$

5. The Iterative Parameters

The iterative parameter sets we shall consider are the following:

1. The set consisting of a constant acceleration parameter and a varying extrapolation one.

2. The one with a constant extrapolation parameter and a varying acceleration paranietei of Douglas type in the two formulations studied in [4].

3. That in 2, above, but with an acceleration parameter of Samarskii and Andreyev type.

4. Finally, a combination of a constant acceleration and a constant extrapolation parameter.

For the definition of the above sets we shall refer the reader to [3] and [4].

In this paper we shall only propose a third setting of the Douglas parameter cycle incorporated in case 2 above and which appears to result in improved convergence.

The idea behind it is that stricter bounds for a variable can be produced by making the painless assumption that the spectrum on the basis of which the determination of the iterative parameters will be carried out is the one with the greatest upper bound. If this is the case, then because of the symmetry of the function f in formula (13) in [4] , the extremes f^*_m and f^*_M will be given by

$$f^*_M = \max\{f(\mu, v,), f(\mu\frac{L2}{U1}, v)\} \equiv f(\mu\frac{L2}{U1}, v) \quad ;$$

$$f^*_m = \min\{f(\mu\frac{L2}{U1}, \mu), f(v,v)\}.$$

Besides, if we equate the two terms giving the f^*_m we get the extra condition

$$\frac{\mu(U1+L2)}{(1+\mu)(U1+\mu L2)} = \frac{2v}{(1+v)^2}$$

which reduces to the following equation linking up the parameters μ and v

$$v = \frac{(U1+\mu^2 L2) + \sqrt{(1-\mu^2)(U1^2-\mu^2 L2^2)}}{\mu(U1+L2)}$$

whereby the required condition for v, $v \geqq 1$ (see condition 11 in [4]) can immediately be verified.

Now, to study the effects on the convergence of the scheme (when the crude bounds reached in the previous paragraph are used for the determination of the iterative parameters considered above) is equivalent to comparing the measures CTSC, CF, CVS and CVD arrived at in [3] and [4], and which are respectively given by

$$\begin{aligned} &\text{CTSC} \sim - k_o/\ell n\, T_{k_o} \left\{\frac{b+a}{b-a}\right\} \\ &\text{CF} \sim -1/\ell n\, p^{*}\,; \\ &\text{CVS, CVD} \sim - \ell n\left(^{L1}/_{U1}\right)/\ell n\left\{\frac{\mu^{*}}{v^{*}}\right\}/\ell np\,(\mu^{*}, v^{*})\,; \end{aligned} \tag{14}$$

where

(U1,L1), (U2,L2) stand for either the strict or the crude eigenvalue spectra of the operators H and V;

k_0 , a and b respectively denote the length of the extrapolation parameter sequence to be used (in our case it will be assumed to be 2) and the lower and upper pounds to the function f [see 4];

μ^{*} ,v^{*} ,p^{*} signify the optimal values for the parameters μ and v, and the maximum amplification factor.

6. Applications

From the various examples we treated, we shall demonstrate here the following two paradigms

Par 1. $P_1 = 4.0$, $q_1 = 4.0$ $P_2 = 1.0$, $q_2 = 0.5$;

Par 2. $P_1 = 4.0$, $q_2 = 4.0$, $p_2 = 2.0$, $q_2 = 1.25$.

The crude bounds have been calculated for N in the range N=10(10)100 and set out in Table 1. Furthermore, for comparison reasons we also found the strict bounds correct to five decimal digits.

Now, using these strict and crude bounds we evaluated the optimum parameters involved and the measures (14) in three different combinations. That is,

(i) the case where all bounds were strict and for which Table 2 displays the measures;

(ii) all bounds were crude and the similar results are shown in Table 3.

(iii) because of the characteristic of the dominant eigenvalue to be easily traced by several methods, we included the combination of strict upper bounds with crude lower ones.

The corresponding measures were calculated again and shown for N in the range N=10(10)50 in Table 4.

An inspection of Tables 2, 3 and 4 reveals the clear superiority of the third variant of the Douglas parameter cycle with respect to the other two. The improvement effected varied, and roughly speaking was something round 15% and 30% against the first and the second variant respectively. It was rather remarkable, since sometimes it even reached the level of 50% (with respect to the second variant, see e.g. Table 2, problem 1, N=10).

As regards the extra calculation which is inflicted on the iterative procedure by the usage of the crude bounds, a comparative study of Tables 2, 3 and 4 leads us to the following conclusions:

First, the two sets with the fixed acceleration parameter suffered most; in addition, the one with fixed and the extrapolation parameter (Case 1) is the worst off set. Moreover, the sets with constant extrapolation parameter and with varying acceleration parameter, are the better off sets, with the Samarskii and Andreyev one (Case 3) being the rather less affected set.

Second, the magnitude of the penalty imposed is higher in the small numbers of subdivisions and keeps diminishing as we move on to greater numbers of subdivisions. Its size for our two problems approximately varied within the intervals : (7.9,2.1) for the first set; (4.6,0.4) for the second one (variant 3); (3.6,0.4) for the Samarskii and Andreyev set; and (9.4,2.1) for the last one.

The extremes were attained at the marginal subdivisions, namely 10 and 100 respectively.

Third, the combination of strict upper and crude lower bounds ameliorated the situation quite appreciably as it is apparent from Table 4.

Finally, in view of the fine grids and hence the large numbers of subdivisions the applications require and consequently the possible small amount of extra calculation which may be involved, the use of crude bounds becomes almost attractive and worth considering at any rate despite some untoward cases which may crop up.

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TABLE 1

CRUDE EIGENVALUE BOUNDS

S/D	P_1=4.0,	q_1=4.0	P_2 =2.0, q_2 = l.25		P_3= 1.0,	q_3=0.5
	ℓ .b.	u. b .	ℓ..b.	u. b .	ℓ . b .	u . b .
10	0.04370	4.80000	0.02462	4.40000	0.01096	4.2 0000
20	0.01096	4.40000	0.00617	4.20000	0.00274	4.10000
30	0.00487	4.26667	0.00274	4.13333	0.00122	4.06667
40	0.00274	4.20000	0.00154	4.I0000	0.00069	4.05000
50	0.00l75	4.16000	0.00099	4.08000	0.00044	4.04000
60	0.0012 2	4.13000	0.00069	4.06667	0.00030	4.03333
70	0.00090	4.11429	0.00050	4.05714	0.00022	4.02857
80	0.00069	4.l0000	0.00039	4.0 5000	0.00017	4.02500
90	0.00054	4.08989	0.00030	4.04444	0.00014	4.02200
100	0.00044	4.08000	0.00025	4.04000	0.00011	4.02000

TABLE 2

STRICT BOUNDS - COMPARISON TABLE

Problems	S/D	CF	CTSC	CV1	CV2	CV3	CVS
P_1=4.0, q_1=4.0, p_2= 1.0, q_2 = 0.5	10	3.05509	1.91033	2.84862	3.64996	2.47340	2.35138
	20	5.95220	3.40676	3.75298	4.57067	3.22529	3.00509
	30	8.86133	4.88136	4.27929	5.10186	3.67078	3.39140
	40	11.75206	6.33769	4.15054	5.47035	3.98666	3.66434
	50	14.66690	7.80212	4.94076	5.75696	4.23418	3.87798
	60	17.55016	9.24855	5.17334	5.99440	4.43279	4.05067
	70	20.55552	10.75487	5.37743	6.20436	4.60721	4.20257
	80	23.45526	12.20742	5.54929	6.37355	4.75415	4.32933
	90	26.31279	13.63828	5.69981	6.51740	4.88290	4.43970
	100	2 9.25214.	15.10969	5.83796	6.651 22	5.00109	4.54129
p_1 =4.0, q_1=4.0, p_2=2.0 , q_2=1.25	10	2.74382	1.74517	2.84292	3.22883	2.46796	2.26585
	20	5.38452	3.11692	3.75095	4.14991	3.22346	2.91840
	30	8.03173	4.46213	4.27824	4.17792	3.66984	3.30373
	40	10.68002	5.79822	4.64990	5.05216	3.98609	3.57764
	50	13.34778	7.13974	4.94032	5.34218	4.23380	3.79170
	60	15.96968	8.45588	5.17302	5.57537	4.43251	3.96369
	70	18.64627	9.79806	5.37718	5.77047	4.60699	4.11229
	80	21.30048	11.12811	5.54909	5.94350	4.75398	4.23975
	90	23.91255	12.43644	5.69965	6.09153	4.88276	4.35053
	100	26.61321	13.78868	5.83783	6.23018	5.00097	4.45295

TABLE 3

CRUDE BOUNDS - COMPARISON TABLE

Problems	S/D	CF	CTSC	CV1	CV2	CV3	CVS
P_1 =4.0, q_1=4.0, P_2=1.0, q_2=0.5	10	3.34167	2.06118	2.98262	3.78953	2.58618	2.43593
	20	6.36873	3.61884	3.84525	4.69308	3.30374	3.06907
	30	9.39200	5.14915	4.35433	5.12499	3.73477	3.44656
	40	12.40594	6.66646	4.71709	5.58170	4.04349	3.71576
	50	15.44374	8.19200	5.00088	5.87098	4.28557	3.92699
	60	18.46949	9.70945	5.22923	6.11716	4.48058	4.09915
	70	21.46593	11.21100	5.42308	6.31649	4.64626	4.24370
	80	24.44874	12.70494	5.59220	6.48218	4.79087	4.36873
	90	27.45042	14.52877	5.74864	6.60690	4.92468	4.48004
	100	30.48723	15.72787	5.87908	6.76197	5.03628	4.58060
P_1 =4.0, q_1 = 4.0, p_2=2.0, q_2=1.25	10	2.96073	1.86042	2.97806	3.29380	2.58051	2.33995
	20	5.66627	3.26090	3.84330	4.18390	3.30175	2.96801
	30	8.37239	4.63437	4.35327	4.70342	3.73374	3.34402
	40	11.07669	5.99790	4.71642	5.07203	4.04286	3.61298
	50	13.77696	7.35530	5.00041	5.35477	4.28513	3.82238
	60	16.44486	8.69425	5.22887	5.58590	4.48025	3.99209
	70	19.16652	10.06033	5.42281	5.79226	4.64602	4.13887
	80	21.78548	11.37107	5.59199	5.95142	4.79067	4.26148
	90	24.67173	12.81660	5.74847	6.11970	4.92453	4.38051
	100	27.18803	14.07645	5.87893	6.23653	5.03615	4.47355

TABLE 4

ROUNDS STRICT FROM ABOVE AND CRUDE FROM BELOW - COMPARISON TABLE

Problems	S/D	CF	CTSC	CV1	CV2	CV3	CVS
p_1=4.0, q_1=4.0, P_2=1.0, q_2=0.5	10	3.18660	1.97968	2.89006	3.75872	2.50734	2.39051
	20	6.19651	3.53121	3.79065	4.67724	3.25706	3.04292
	30	9.21420	5.05945	4.31548	5.20541	3.70153	3.42834
	40	12.22554	6.57578	4.68714	5.57820	4.01786	3.70179
	50	15.26155	8.10058	4.97512	5.86668	4.26352	3.91567
p_1=4.0, q_1=4.0, p_2=2.0, q_2=1.25	10	2.79563	1.77277	2.88334	3.23564	2.50101	2.28576
	20	5.48221	3.16687	3.78827	4.15404	3.25493	2.93671
	30	8.18200	4.53813	4.31426	4.68264	3.70044	3.32218
	40	10.88323	5.90053	4.68639	5.05568	4.01719	3.59623
	50	13.58178	7.25728	4.97461	5.34431	4.26306	3.80881

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