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ERROR ANALYSIS OF GALERKIN METHODS FOR DIRICHLET PROBLEMS CONTAINING BOUNDARY SINGULARITIES

BY

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1. Introduction and Weak Problem

In this paper we derive Galerkin solutions with error bounds for two dimensional elliptic problems involving boundary singularities for the case in which the governing equation is Poisson's equation and there are Dirichlet boundary conditions. In such problems the function u(x,y) satisfies

$$-\Delta[u(x,y)] = g(x,y)\ ,\ (x,y) \in \Omega, \tag{1.1}$$

$$u(x,y) = f(x,y)\ ,\ (x,y) \in \partial\Omega, \tag{1.2}$$

where for our case $\Omega \subset E^2$ is a simply connected open domain with polygonal boundary $\partial\Omega$, $f \in L_2(\partial\Omega)$ and $g \in L_2(\Omega)$. The Laplaoian operator in (1.1) could be replaced by a general second order linear self-adjoint elliptic operator.

Many finite element solutions with theoretical error bounds have been derived for problems of the type (1.1) -(1.2). However, these bounds demand that specific derivatives of u be bounded throughout $\overline{\Omega}$, the closure of Ω. When the boundary $\partial\Omega$ contains corners with certain interior angles, for example re-entrant corners, the solution u contains a singularity and these conditions are not satisfied. This shortcoming was first overcome by Fix (1969) for self-adjoint linear second order elliptic problems with homogeneous boundary conditions where Ω is a rectangular domain having a re-entrant corner of interior angle $3\pi/2$. Fix uses rectangular elements and constructs approximations from spaces of piecewise polynomials augmented with functions having the form of the singularity. He is thus able for his problem to obtain a bound on the finite element error using the tensor product results of Birkhoff, Schultz and Varga (1968). Adopting a similar approach we here produce an 0(h) error bound using piecewise linear approximation over triangular elements for the problem

(1.1) - (1.2) which has a more general shaped boundary and nonhomogeneous boundary conditions.

The weak problem corresponding to (1.1) - (1.2) is the following: find $u \in \psi + \overset{\circ}{W}{}_2^1(\Omega)$ such that

$$a(u,v) = (g,v) \qquad \forall\, v \in \overset{\circ}{W}{}_2^1(\Omega), \tag{1.3}$$

where $\psi = f$ on $\partial\Omega$ and $\psi \in W_2^1(\overline{\Omega})$. The space $W_2^1(\overline{\Omega})$ is the Sobolev space of functions which together with their first generalised derivatives exist and are in $L_2(\Omega)$, WHEREAS $\overset{\circ}{W}{}_2^1(\Omega)$ is the subspace of these functions which assume homogeneous Dirichlet boundary values. The notation $u \in \psi + W_2^1(\Omega)$ means that $u = \psi + v$ where $v \in \overset{\circ}{W}{}_2^1(\Omega)$. Thus $u \in W_2^1(\Omega)$ and $u = f$ on $\partial\Omega$. The bilinear functional a(u,v) is defined to be

$$a(u,v) = \iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dx\, dy \qquad \forall\, u,v \in W_2^1(\Omega). \tag{1.4}$$

The <u>energy</u> <u>norm</u> (semi norm) for (1.4) is

$$\|v\|_E = (a(v,v))^{\frac{1}{2}}. \tag{1.5}$$

The solution of (1.3) is the <u>generalized solution</u> u of (1.1)-(1.2) and u is approximated by the Galerkin approximation U.

2. <u>Galerkin Techniques</u>

The region Ω is discretized into triangular elements with generic length h so that there are m internal and n boundary nodes. We consider piecewise linear approximations. Following Barnhill and Gregory (1972) we let $\{B_i(x,y)\}_{i=1}^{m}$ and $\{Cj(x,y)\}_{j=1}^{n}$ be two sets of functions that are biorthonormal (see Davis (1963)) with respect to the evaluations at the nodes. That is, the $B_i(x,y)$

have zero boundary values and $B_i(x_k, y_k) = \delta_{ik}$, $1 \leqq i, k \leqq m$, for the internal nodes $\{(x_k, y_k)\}_{k=1}^{m}$, where $C_j(x,y)$ are zero at all the interior nodes and $C_j(x_{m+p}, y_{m+p}) = \delta_{j,p}$, $1 \leqq j, p \leqq n$, for the boundary nodes $\{(x_{m+p}, y_{m+p})\}_{p=1}^{n}$. The B_i and C_j are thus pyramid functions.

We define $S^h \subset f^h + \overset{\circ}{W}{}_2^1(\Omega)$ to be the set of all functions $s(x,y)$ of the form

$$s(x,y) = \sum_{i=1}^{m} A_i B_i(x,y) + \sum_{j=1}^{n} f_j^h \; C_j(x,y)\,, \tag{2.1}$$

where in (2.1) the A_i are constants to be found and the f_j^h are the discretized boundary data. We let S_o^h be the m-dimensional subspace of $\overset{\circ}{W}{}_2^1(\Omega)$ generated by the B_i.

The Galerkin method is :

find $U \in S^h$ such that

$$a(U,V) = (g,V) \qquad \forall\ V \in S_o^h\,. \tag{2.2}$$

Lemma The Galerkin approximation U is the best approximation to the generalized solution u of (1.3) from S^h in the energy norm (1.5).

Proof Equation (2.2) holds for $V = B_k$, $k = 1,2,\ldots, m$ and so, after substitution from (2.1) for U, the equations for the calculation of the A_i are

$$\sum_{i=1}^{m} A_i a(B_i, B_k) = (g, B_k) - \sum_{j=1}^{n} f_j^h \; a(C_j, B_k)\,, \tag{2.3}$$

$$k = 1,2,\ldots, m.$$

Substitution of $a(u,B_k) = (g,B_k)$, $k = 1,2,\ldots, m$ from (1.3) into (2.3) produces the normal equations for the best approximation U. That is

$$\sum_{i=1}^{m} A_i \; a(B_i, B_k) = a\Big(u - \sum_{j=1}^{n} f_j^h C_j, B_k\Big) \qquad k=1,2,\ldots,\ m,$$

see Davis (1963). Hence

$$a(u-U,\ u-U) \leqq a(u-w,\ u-w) \qquad \forall\ w \in S^h \qquad // \tag{2.4}$$

If $\tilde{u} \in S^h$ interpolates u at the m+n nodal points, then (2.4) implies that

$$\|u-U\|_E \leqq \|u-\tilde{u}\|_E \tag{2.5}$$

and so an upper bound on the interpolation error yields an upper bound on the Galerkin error.

Bounds on the right hand side of (2.5) can be obtained from Theorem 5 of Ciarlet and Raviart (1972). In order to quote this theorem we denote the interpolant $\tilde{u}$ by πu, where π is the linear operator corresponding to piecewise linear interpolation over the triangulation.

Theorem (Ciarlet and Raviart).

Let Ω and π be defined as above and ρ = sup{diameter of all circles that can be inscribed in the triangles of the triangulations], then $u \in W_2^2(\Omega)$ implies that there exists a constant K such that

$$\| u-\pi u \|_{W_2^1(\Omega)} \leqq K \frac{h^2}{\rho} |u|_2 , \tag{2.6}$$

where

$$|u|_2 = \left\{ \left\|\frac{\partial^2 u}{\partial x^2}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^2 u}{\partial x \partial y}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^2 u}{\partial y^2}\right\|^2_{L_2(\Omega)} \right\}^{\frac{1}{2}} .$$

Proof In the notation of Ciarlet and Raviart we let m = 1 and 1=2, and note that our π is a bounded linear operator from $W_2^2(\Omega)$ to $W_2^1(\Omega)$ such that $\pi p = p$ whenever p is a polynomial of degree 1. The operator π is bounded since by the Sobolev imbedding theorem $W_2^2(\Omega)$ is imbedded in $C(\overline{\Omega})$. //

Since

$$\|u-\tilde{u}\|_E \leqq \|u-\tilde{u}\|_{W_2^1(\Omega)} ,$$

it follows from (2.5) and (2.6) that

$$\|u-U\|_E \le K\frac{h^2}{\rho}|u|_2 \ . \qquad (2.7)$$

We assume that $h_i/\rho_i \leqq \alpha$ for some fixed α, where, as the mesh is refined, $\{h_i\}$ is a sequence of values of h and the $\{\rho_i\}$ are the corresponding values of ρ; i.e. the triangulations form a <u>regular</u> <u>family</u> (see Ciarlet and Raviart p.174). With this assumption (2.7) yields an 0(h) error bound.

3. <u>Boundary Singularities</u>

In order to use the bound (2.7) we need $u \in W_2^2(\Omega)$. If $\partial\Omega$ is sufficiently smooth, and f(x,y) is sufficiently well behaved, this condition is satisfied. However, if the boundary contains a corner at which the internal angle $\phi = \alpha\pi/\beta = \pi/\gamma$ is such that either

(i) $1/\gamma < 1$ and the number y is non-integer,

or

(ii) $1/\gamma > 1$ in which case the corner is <u>re-entrant</u>,

then $u \in W_2^{[\gamma]+1} - W_2^{[\gamma]+2}$, where $[\gamma]$ is the greatest integer $\leqq \gamma$. Interesting cases occur when $\phi > \pi$, so that $u \in W_2^1(\Omega) - W_2^2(\Omega)$, and we consider only these. Suppose there is a re-entrant corner at a point 0 on $\partial\Omega$ and that f(x,y) = 0 on the arms of the corner. Then in terms of local polar co-ordinates (r,θ) with origin at 0 and zero angle along one of the arms of the corner, the asymptotic form of u may be written, see Lehman (1959), as

$$u(r,\theta) = \sum_i a_i\phi_i(r,\theta) \ , \qquad (3.1)$$

where the a_i are unspecified constants. The series expansion contains powers of r which are non-integer; e.g. the leading term is $a_1 r^\gamma \sin\gamma\theta$. These cause a <u>boundary</u> <u>singularity</u> in u at 0 in that $\partial u/\partial r$ is unbounded at r = 0 although $u \in W_2^1(\Omega) - W_2^2(\Omega)$.

Following Fix (1969) and Barnhill and Whiteman (1973) the dominant part of the singularity in u is subtracted off in a neighbourhood $N(r_1) \subset \overline{\Omega}$ of 0. Here $0 < r_\circ < r_1$, and we define

$$N(r_j) \equiv \{(r,\theta)\,; \quad 0 \le r \le r_j, \quad 0 \le \theta \le \phi\}, \qquad j = 0,1.$$

We form the functions

$$w_i(r,\theta) = \begin{cases} \phi_i(r,\theta) & , \ (r,\theta) \in N(r_0) \\ g_i(r)\, h_i(\theta), & (r,\theta) \in N(r_1) - N(r_0), \\ 0 & , \ (r,\theta) \in \overline{\Omega} - N(r_1), \end{cases} \tag{3.2}$$

i = 1,2,...,N, where N is discussed below and the ϕ_i are as in (3.1). The $g_i(r)$ are cubic Hermite polynomials chosen so that each function $w_i(r,\theta)$ is in $W_2^2(\overline{\Omega} - N(r_\circ))$. The $h_i(\theta)$ are appropriate functions so that the w_i all satisfy the homogeneous boundary conditions on the arms of the corner. Using (3.2) we form the function

$$w = u - \sum_{i=1}^{N} a_i\, w_i\,(r,\theta)\,, \tag{3.3}$$

and choose N so that w would be in $W_2^2(\Omega)$ if the a_1, were known exactly. It is the function w that is approximated throughout Ω by the Galerkin solution U, and clearly if the a_i are known, making $w \in W_2^2(\Omega)$ the error bound (2.7) will then apply. However, the a_i cannot be calculated exactly. In practice approximations are calculated by the method of augmenting with singular functions the trial function space in the Galerkin procedure. This is denoted by Aug S^h. In each element the trial functions now have the form

$$a + bx + cy + \sum_{i=1}^{N} c_i\, w_i(r,\theta)\,.$$

By (3.2) these are the usual trial functions of S^h for elements in $\overline{\Omega}-N(r_1)$. Extra equations are added to the linear system which when solved give the Galerkin solution $\sum_{i=1}^{N} \hat{a}_i\, w_i(r,\theta) + \hat{U} \in$ Aug S^h and so only approximations $\hat{a}_i$ to the a_i in (3.3) are obtained from the same numerical calculation as that which gives the values of $\hat{U}$ at the nodal points.

The Lemma of Section 2 holds for Galerkin solutions from Aug S^h, and so we have the following theorem.

Theorem 3.

Under the above conditions on u and N

$$\|u-(\sum_{i=1}^{N} \hat{a}_i\, w_i + \hat{U})\|_E \leqq Kh\, |u-\sum_{i=1}^{N} a_i\, w_i|_2 ,$$

where $\sum_{i=1}^{N} \hat{a}_i\, w_i + \hat{U}$ is the Galerkin approximation to u from Aug S^h and the a_i, are the coefficients in the asymptotic expansion (3.1).

Proof The Best Approximation Lemma applied to Aug S^h implies that

$$\|u-(\sum_{i=1}^{N} \hat{a}_i\, w_i + \hat{U})\|_E \leqq \|u-\sum_{i=1}^{N} a_i w_i = \tilde{u}\|_E ,$$

where $\tilde{u} \in S^h$ interpolates $u-\sum_{i=1}^{N} a_i w_i$ at the m+n nodal points.

By the Ciarlet-Raviart theorem it follows that

$$\|(u-\sum_{i=1}^{N} a_i w_i) - \tilde{u}\|_E \leqq Kh|u-\sum_{i=1}^{N} a_i w_i|_2 ,$$

so that we thus have an 0(h) error bound. //

As was stated earlier Fix uses rectangular elements and bilinear trial functions. An advantage of our use of triangular rather than rectangular elements with the corresponding trial functions is that the computation is much simpler whilst the accuracy of the numerical solutions is comparable. This saving is even more valuable for higher order problems. The use of triangular elements also enables the method to be used for problems in more general polygonal regions.

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