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A NUMERICAL CONFORMAL TRANSFORMATION METHOD FOR HARMONIC MIXED BOUNDARY VALUE PROBLEMS IN POLYGONAL DOMAINS

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ABSTRACT

A method is given for solving two dimensional harmonic mixed boundary value problems in simply-connected polygonal domains with re-entrant boundaries. The method consists of a numerical conformal mapping together with three other conformal transformations. The numerical mapping transforms the original domain onto the unit circle, which in turn is mapped onto a rectangle by means of two bilinear and one Schwarz-Christoffel transformations. The transformed problem in the rectangle is solved by inspection.

1. Introduction

Let $R \in E^2$ be a simply-connected domain with boundary S, where $S = \bigcup s_i, i = 1,2,3,4,$, and S_{i+1} being adjacent subarcs of S, and consider the two dimensional mixed boundary value problem in which the function u(x,y) satisfies

$$\left.\begin{array}{lll} \Delta[u(x,y)] & = 0, & (x,y) \in R, \\ u(x,y) & = u_o, & (x,y) \in S_1, \\ \dfrac{\partial u(x,y)}{\partial \nu} & = 0, & (x,y) \in S_2, \\ u(x,y) & = u_1, & (x,y) \in s_3, \\ \dfrac{\partial u(x,y)}{\partial \nu} & = 0, & (x,y) \in S_4. \end{array}\right] \qquad (1)$$

In (1) Δ is the Laplacian operator and $\frac{\partial}{\partial \nu}$ is the derivative in the direction of the outward normal to the boundary.

A conformal transformation method (CTM) has recently been proposed [6] for the numerical solution of problems of type (l). In particular, this method has been developed to deal with problems containing boundary singularities for which standard numerical techniques, such as finite differences and finite elements, fail to produce accurate solutions.

The CTM consists of three conformal transformations whose combined effect is to map $G \equiv R \cup S$ in the w = x+iy plane onto the rectangle

$$G' \equiv \{(\xi, n) : 0 \leq \xi \leqq 1, \quad 0 \leqq \eta \leqq H\}, \qquad (2)$$

in the $w' = \xi + i\eta$ plane and thus to transform the original problem in G into the problem,

$$\left.\begin{array}{l} \Delta[v(\xi,n)] = 0, \quad 0 < \xi < 1, \quad 0 < \eta < H, \\ \dfrac{\partial v(\xi,0)}{\partial \eta} = \dfrac{\partial v(\xi,H)}{\partial \eta} = 0, \quad 0 < \xi < 1, \\ v(0,\eta) = u_o, \quad v(1,\eta) = u_1, \quad 0 \leqq \eta \leqq H \end{array}\right] \tag{3}$$

in G'. The solution of (3) is

$$v(\xi, n) = (u_1 - u_o)\left(\xi + \frac{u_o}{u_1 - u_o}\right), \tag{4}$$

and hence if $P \equiv (x,y) \in G$ is mapped into $P' \equiv (\in,\eta) \in G'$, it follows that $u(P) = v(P')$ and so from (4) the solution of (1) at P is known immediately if the real co-ordinate of the point P' is found.

If $A \equiv w_1, B \equiv w_2, c \equiv w_3$ and $D = w_4$ are respectively the four end points of the subarcs S_1, S_2, S_3, and S_4 (with $\overline{AB} = S_1, \overline{BC} = S_2$ etc), then the three transformations of the CTM are:

(i) The transformation,

$$z = T(w) \quad , \tag{5}$$

where T(w) is a function of w, analytic in R, so that (5) is a conformal mapping of $G \in$ w-plane onto $G_1 \equiv$ the upper half z - plane.

(ii) The bilinear transformation,

$$t = \left\{\frac{T(w_4) - T(w_2)}{T(w_4) - T(w_1)}\right\} \cdot \left\{\frac{z - T(w_1)}{z - T(w_2)}\right\}, \tag{6}$$

mapping $G_1 \in$ z-plane onto $G_2 \equiv$ the upper half t-plane.

(iii)The Schwarz-Christoffel transformation,

$$w' = \frac{1}{K(m)} sn^{-1}(t^{\frac{1}{2}}, m), \tag{7}$$

where an denotes the Jacobian elliptic sine and K(m) is the complete elliptic integral of the first kind with modulus,

$$m = \left\{\left(\frac{T(w_4) - T(w_2)}{T(w_4) - T(w_1)}\right), \left(\frac{T(w_3) - T(w_1)}{T(w_3) - T(w_2)}\right)\right\}^{-\frac{1}{2}}.$$

The effect of (7) is to map $G_2 \in$ t- plane onto the rectangle G' given by (2) with $H = K\{(1 - m^2)^{\frac{1}{2}}\}/K(m)$, in the $w' = \xi + i\eta$ plane.

The form of the mapping function T(w) clearly depends on the geometry of the domain R. However, once transformation (5) is constructed, the solution of (1) is determined from (6), (7) and (4) by a standard procedure which is described with full computational details in [6].

In [5] and [6] the CTM is applied to problems defined in rectangular domains which contain boundary singularities. In these oases for bounded domains the mapping function T(w) is a Jacobian elliptic sine and for a semi-infinite strip it is a trigonometric cosine. In the present paper we apply the CTM to problems of type (1) defined in non-rectangular polygonal regions containing re-entrant corners at which boundary singularities occur. We consider in particular L-shaped domains and a domain with two re-entrant corners. For these regions the mapping function T(w) is not known in a closed form and thus the transformation (5) is performed numerically, The numerical mapping technique used is due to Symm [3], and is described briefly in Section 2 below. The CTM solutions to the problems considered in Section 3 are in good agreement with those recently obtained by Symm [4], who uses an integral equation method modified to deal with the singularities at the re-entrant corners.

Results that we have obtained by the CTM, for problems of the type considered in the present paper, have been used by Bell and Crank [1] in a study of diffusion in a continuum containing non-permeable rectangular prisms.

2. Numerical Conformal Mapping Technique.

Following Symm [3], we consider the simply-connected domain R of (1). We take 0 to be the origin of co-ordinates in the w-plane and assume, without loss of generality, that $O \in R$. It is well-known that the function F(w) which maps

G $\equiv$ R$\cup$S conformally onto the unit disc $|\zeta| \leqq 1$ in the ζ- plane, so that the point O G$\in$ R is transformed into the centre $\zeta \equiv (0,0)$ of the disc, is given uniquely, apart from an arbitrary rotation, by

$$\zeta = F(W) = \exp\{\log W + g(x,y) + ih(x,y)\} \tag{8}$$

In (8) the functions g(x,y) aad h(x,y) are real valued conjugate harmonic in R, and g(x,y) satisfies the boundary value problem

$$\begin{aligned} &\Delta[g(x,y)] = 0, && (x,y) \in R, \\ &g(x,y) = -\tfrac{1}{2}\log(x^2+y^2), && (x,y) \in S. \end{aligned} \tag{9}$$

When the harmonic function g(x,y) is represented as a single-layer logarithmic potential

$$g(x,y) = \int_S \log|w - w|\sigma(\omega)|d\omega|, \quad w \equiv (x,y) \in G, \tag{10}$$

where $\sigma(\omega)$ is a certain source density on S, its conjugate h(x,y) is given by

$$h(x,y) = \int_S \arg(w-\omega)\sigma(\omega)\,|d\omega|, \quad w \equiv (x,y) \in G \tag{11}$$

and the boundary condition of (9) becomes

$$\int_S \log|w - \omega|\sigma(\omega)|d\omega| = -\frac{1}{2}\log(x^2+y^2), \quad w \equiv (x,y) \in S. \tag{12}$$

Equation (12) is a Fredholm integral equation of the first kind in $\sigma(\omega)$ and its solution always exists, subject only to a possible rescaling of R; see Jaswon [2]. The method of Symm consists of solving (12) numerically for $\sigma(\omega)$. The functions g(x,y) and h(x,y) are then calculated, for any $w \in G$, from approximations to the relations (10) and (11) and finally an approximation to the mapping function F(w) is determined, for any $w \in G$, from (8).

The numerical solution of (12) is obtained as follows: the boundary S is divided into the N intervals $I_1, I_2, \ldots,$ I_N, not necessarily of the same length, and $\sigma(\omega)$ is approximated by $\sigma(\omega) = \sigma_r$, where σ_r is constant for any point $\omega \in I_r$, e1,2,... ,N. Equation (12) is thus replaced by

$$\sum_{r=1}^{N} \left\{ \int_{Ir} \log \left| w - \omega \right| \left| d\omega \right| \right\} \sigma r = -\tfrac{1}{2} \log(x^2 + y^2), \quad w \equiv (x,y) \in S,$$

and is then applied to N "nodal" points $w = (x_3, y_3) \in I_5$, s=1,2,...., N. For the polygonal regions considered in the present paper w_s is taken to be the mid-point of I_S. There results the system of N linear equations

$$\sum_{r=1}^{N} \left\{ \int_{I_r} \log \left| W_s - \omega \right| \left| d\omega \right| \right\} \sigma_r = -\tfrac{1}{2} \log\left(x_s^2 + y_s^2 \right), \qquad s = 1,2, \ldots .N, \tag{13}$$

for the N unknowns $\sigma_1, \sigma_2, \ldots, \sigma_N$. Denoting the end points of I_r by $w_{r-\frac{1}{2}}$ and $w_{r+\frac{1}{2}}$, the coefficients of σ_r in (13) are approximated by,

$$a_{sr} = \frac{hr}{3}\left\{\log|w_s - w_{r-\frac{1}{2}}| + 4\log|w_e - w_r| + \log|w_s - w_{r+\frac{1}{2}}|\right\}, s \neq r, \tag{14}$$

and

$$a_{rr} = 2h_r\left\{\log w_r - |w_{r-\frac{1}{2}}| - 1\right\} = 2h_r\{\log h_r - 1\}, \tag{15}$$

where $2h_r$ represents the length of I_r . Thus, the system of equations actually solved is,

$$A\underline{\sigma} = \underline{r}\ , \tag{16}$$

where A = $[a_{sr}]$ is the NxN matrix whose coefficients are given by (14) and (15), and $\underline{\sigma} = \{\sigma_s\}_{s=1}^{N}$, $\underline{r} = \{r_s\}_{s=1}^{N}$ are N dimensional column vectors with ' $r_s = -\frac{1}{2}\log\left(x_s^2 + y_S^2\right)$. Equations (16) are solved by Gaussian elimination and then approximations $\tilde{g}(x,y)$ and $\tilde{h}(x, y)$ to g(x,y) and h(x,y) respectively, are computed by replacing (10) and (11) by

$$\tilde{g}(x,y) = \sum_{r=1}^{N} \ell_r(w)\sigma_r, w \equiv (x,y) \in R, \tag{17}$$

$$\tilde{h}(x, y) = \sum_{r=1}^{N} \alpha_r(w)\sigma_r, w \equiv (x, y) \in G, \tag{18}$$

For any point w ≡ (x,y) ϵ S the function g(x,y) is of course computed from the boundary condition of (9). In (17) and (18) $\ell_r(w)$ and $\alpha_r(w)$ denote respectively suitable approximations to the integrals $\int_{Ir} \log|w-\omega|\,|d\omega|$ and $\int_{I_r} \arg(w-\omega)|d\omega|$.

For the regions considered in the present paper we take in particular,

$$\ell_r(w)=\frac{hr}{3}\left\{\log|w-w_{r-\frac{1}{2}}|+4\log|w-w_r|+\log|w-w_{r+\frac{1}{2}}|\right\}, w\in R, \tag{19}$$

$$\alpha_r(W)=\begin{cases}\frac{hr}{3}\left\{\arg(w-w_{r-\frac{1}{2}})+4\arg(w-w_r)+\arg(w-w_{r+\frac{1}{2}})\right\}, & w\in G-I_r,\\ hr\left\{\arg(w-w_{r-\frac{1}{2}})+\arg(w-w_{r+\frac{1}{2}})\right\}, & w=w_r,\\ 2h_r\arg(w-w_r), & w-w_{r\pm\frac{1}{2}}.\end{cases} \tag{20}$$

Once $\tilde{g}(x, y)$ and $\tilde{h}(x, y)$ are computed the approximation $\tilde{F}(w)$ to the mapping function F(w) is obtained by replacing (8) by

$$\tilde{F}(w) = \exp\{\log w + \tilde{g}(x, y) + i\tilde{h}(x, y)\}.$$

Finally, the bilinear transformation

$$z = i\left\{\frac{1 + \zeta}{1 - \zeta}\right\},$$

which maps the disc $|\zeta| \leqq 1$ onto the upper half z-plane, is used to approximate the mapping function T(w) of (5) by,

$$\tilde{T}(w) = i\left\{\frac{1 + \tilde{F}(w)}{1 - \tilde{F}(w)}\right\}.$$

We point out that for straight line intervals I_r there exist formulae for the exact evaluation of the integrals

$$\int_{I_r} \log|w-\omega||d\omega|, \quad w \in G.$$

However, most of the error in $\tilde{F}(w)$ arises from the replacement of the continuous function $\sigma(w)$ by the step function σ_r,r =1,2,...,N and thus the use of the exact formulae, in place of the approximations (14),(15) and (19), does not appreciably improve the accuracy of the numerical mapping. The approximations are comoutionally more convenient and for this reason are used in the present paper. Furthermore, formulae of type (14),(15) and (19) are also suitable for curved intervals of S and their use increases the flexibility of our procedure.

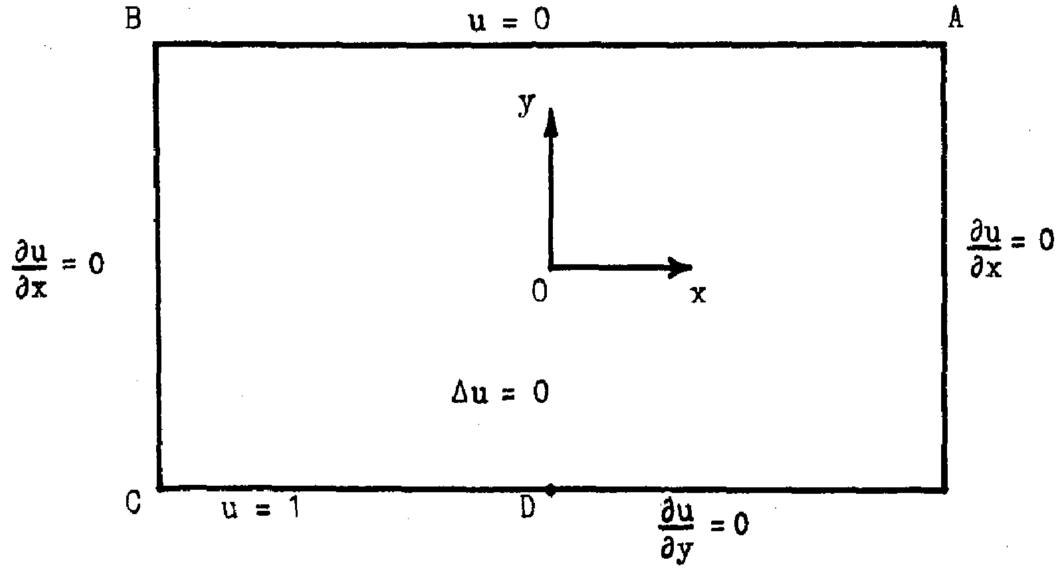
An attractive feature of Symm's mapping technique is that the intervals I_r, need not be of the same length. It is thus possible to refine the mesh locally, by increasing the density of the intervals, in any subarc S' of S where inaccuracies in the numerical mapping are expected (e.g. in the neighbourhood of a re-entrant corner). However, from our experiments it appears that a uniform refinement of the mesh over the whole boundary S gives better overall accuracy than a local refinement over S', when the same number of intervals is used. For the regions considered here, we take $h_r = h$ for all r We do however recognise the need for further investigation on the possibility of increasing the accuracy of the mapping by other distributions of the intervals on S.

3. Numerical Results

In this Section the CTM is applied to problems of type (1) defined in various L-shaped regions and also to a problem defined in an octagonal region containing two re-entrant corners. However, in order that the accuracy of the CTM may be investigated when the transformation (5) is performed numerically, the method is first applied to a problem defined in a rectangular domain.

In all the problems considered, the numerical conformal mapping is performed by dividing the boundary S of G into N straight line intervals of equal length. When G is symmetric about an axis then, by assuming the function $\sigma'(\omega)$ to have the same symmetry and distributing the intervals appropriately about the axis, the number of equations (16) to be solved is reduced from N to N/2.

Problem 1, Consider the harmonic mixed boundary value problem illustrated in Figure 1 for which G is the rectangle $\left\{(x, y) : |x| \leq 1, |y| \leq 0.5\right\}$. The problem is of type (1) with $w_1 \equiv A \equiv (1,0.5)$, $W_2 \equiv B \equiv (-1,0.5)$, $W_3 \equiv C \equiv (-1,-0.5)$ and $W_4 \equiv D \equiv (0, -0.5)$.



w -plane.

Figure 1.

The "analytic" CTM solution of this problem has been obtained in [5] by taking the origin of coordinates in the w-plane to be the point D and using the exact mapping function

$$T(w) = \mathrm{sn}\left(Kw,\ 1\sqrt{2}\right),\quad K = K\left(1/\sqrt{2}\right),$$

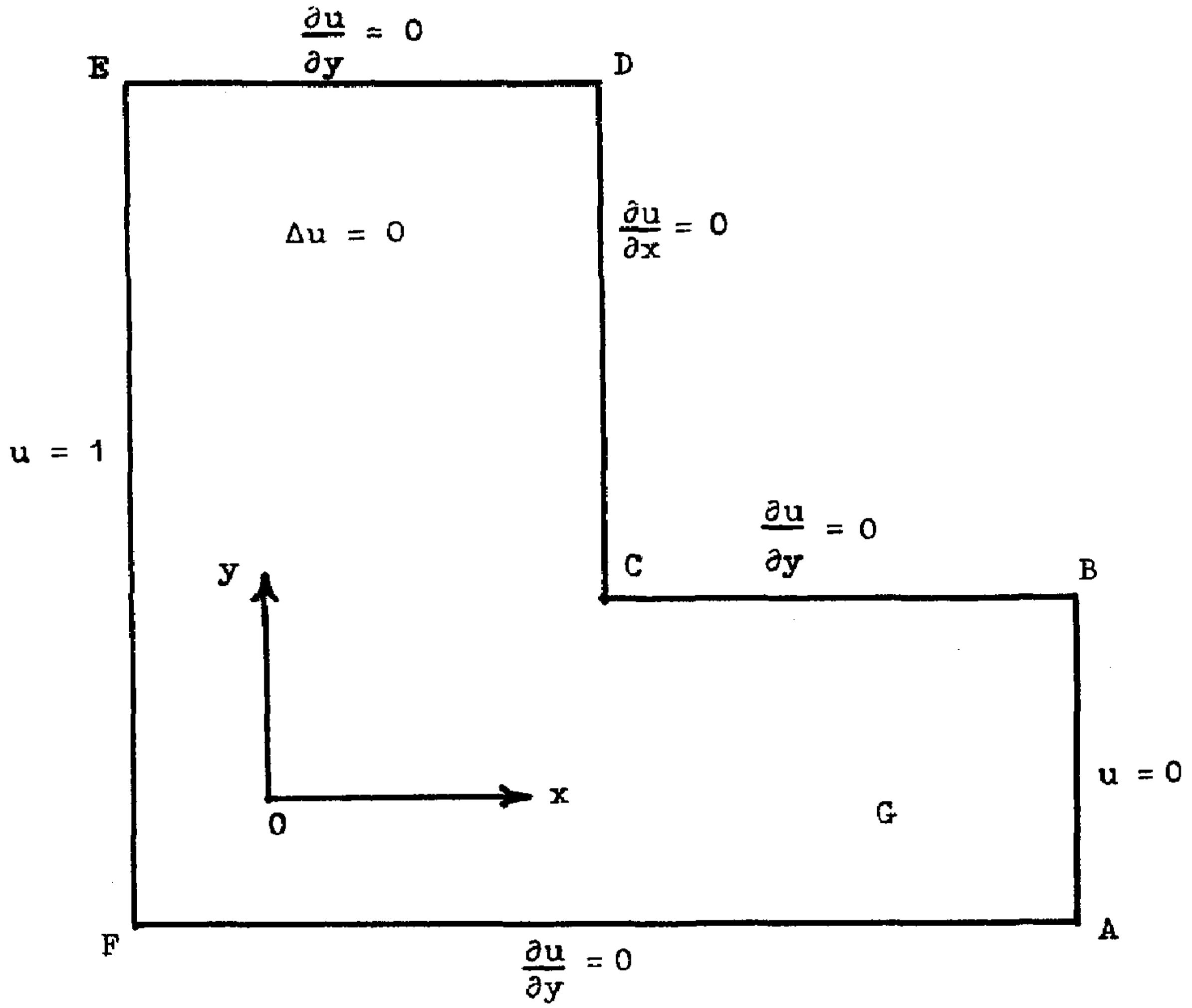
in the transformation (5). In the present paper the CTM. solution of this problem is again obtained but the transformation (5) is now performed numerically using the technique of Section 2 with N=120. The symmetry of G about the y-axis is used in the numerical conformal mapping and thus the number of equations (16) actually solved is 60. The "numerical" CTM results are given, on a square mash of length 0.2, in Table 1 and are compared with the "analytic" CTM results of [51.

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1928	0.1915	0.1871	0.1788	0.1656	0;1477	0.1270	0.1073	0.0917	0.0819	0.0786
0.1924	0.1915	0.1871	0.1788	0.1656	0.1476	0.1270	0.1073	0.0917	0.0819	0.0784
0.3882	0.3858	0.3781	0.3626	0.3364	0.2982	0.2531	0.2104	0.1774	0.1572	0.1505
0.3880	0.3858	0.3781	0.3626	0.3363	0.2982	0.2530	0.2103	0.1773	0.1572	0.1504
0.5879	0.5854	0.5766	0.5575	0.5203	0.4568	0.3761	0.3028	0.2499	0.2190	0.2089
0.5880	0.5853	0.5765	0.5575	0.5202	0.4567	0.3760	0.3027	0.2498	0.2189	0.2089
0.7924	0.7907	0.7846	0.7700	0.7335	0.6370	0.4880	0.3730	0.3002	0.2602	0.2474
0.7928	0.7907	0.7845	0.7699	0.7334	0.6368	0.4778	0.3728	0.3001	0.2601	0.2474
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.5464	0.4008	0.3185	0.2747	0.2609
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.5460	0.4005	0.3183	0.2745	0.2608

TABLE 1

At each mesh point the numbers represent: "Analytic" CTM [5] / "Numerical" CTM

Problem 2 The harmonic mixed boundary value problem illustrated in Figure 2 is of **type** (1) with $W_1 \equiv A$, $W_2 \equiv B$, $W_3 \equiv E$ and $W_4 \equiv F$.



w – plane

Figure 2.

CTM solutions to this problem are obtained for the following geometries of the L-shaped domain G.

(i) AB = BC = CD = DE = 0.5, EF = FA = 1.

(ii) AB = 0.6, BC = CD = 0.4, DE = 0.6, EF = FA = 1 .

(iii) AB = BC = 0.4, CD = DE = 0.6, EF = FA = 1.

(iv) AB = 0.8, BC = CD = 0.2, DE = 0.3, EF = 1, FA = 0.5.

In (i) and (ii) the numerical conformal mapping is performed with N = 100. Since, for both these geometries, G is symmetric about the line CF the actual number of equations (16) solved is 50. In (iii) and (iv) the numerical conformal mapping is performed with N = 100 and N = 75 respectively. The CTM solutions for the oases (i), (ii), (iii) and (iv) are given, on a square mesh of length 0.1, in Tables 2, 3, 4 and 5 respectively. In Tables 2 and 5 the solutions, for the cases (i) and (iv), obtained by Symm [4] are included for comparison.

TABLE 2

At each mesh point the numbers represent:

CTM
Integral Eqn. Method, [4]

1.0000	0.9698	0.9425	0.9202	0.9055	0.9005					
0.9999	0.9700	0.9427	0.9205	0.9060	0.9009					
1.0000	0.9686	0.9398	0.9163	0.9007	0.8953					
1.0000	0.9687	0.9400	0.9166	0.9012	0.8957					
1.0000	0.9647	0.9319	0.9044	0.8856	0.8787					
1.0000	0.9648	0.9322	0.9048	0.8860	0.8793					
1.0000	0.9584	0.9188	0.8839	0.8581	0.8482					
1.0000	0.9585	0.9191	0.8843	0.8587	0.8487					
1.0000	0.9502	0.9013	0.8548	0.8146	0.7948					
1.0000	0.9503	0.9015	0.8553	0.8154	0.7961					
1.0000	0.9411	0.8816	0.8207	0.7560	0.6663	0.4884	0.3580	0.2371	0.1170	0.0000
1.0000	0.9412	0.8818	0.8210	0.7565	0.6667	0.4869	0.3579	0.2364	0.1177	0.0000
1.0000	0.9324	0.8632	0.7897	0.7066	0.6026	0.4788	0.3555	0.2355	0.1174	0.0000
1.0000	0.9325	0.8633	0.7898	0.7066	0.6019	0.4780	0.3549	0.2352	0.1172	0.0000
1.0000	0.9254	0.8487	0.7672	0.6774	0.5760	0.4646	0.3490	0.2326	0.1162	0.0000
1.0000	0.9254	0.8487	0.7671	0.6772	0.5756	0.4642	0.3486	0.2323	0.1161	0.0000
1.0000	0.9204	0.8388	0.7528	0.6605	0.5606	0.4539	0.3428	0.2294	0.1149	0.0000
1.0000	0.9204	0.8387	0.7527	0.6603	0.5604	0.4536	0.3425	0.2291	0.1147	0.0000
1.0000	0.9175	0.8331	0.7450	0.6516	0.5524	0.4477	0.3387	0.2271	0.1139	0.0000
1.0000	0.9175	0.8331	0.7449	0.6515	0.5521	0.4474	0.3385	0.2269	0.1138	0.0000
1.0000	0.9163	0.8315	0.7426	0.6486	0.5500	0.4452	0.3376	0.2257	0.1140	0.0000
1.0001	0.9166	0.8313	0.7424	0.6487	0.5495	0.4453	0.3371	0.2261	0.1134	0.0000

TABLE 3

1.0000	0.9494	0.9017	0.8586	0.8238	0.8009	0.7928				
1.0000	0.9484	0.8991	0.8545	0.8179	0.7933	0.7844				
1.0000	0.9451	0.8919	0.8426	0.8002	0.7694	0.7579				
1.0000	0.9400	0.8810	0.8241	0.7712	0.7265	0.7049				
1.0000	0.9340	0.8680	0.8020	0.7354	0.6658	0.5726	0.3856	0.2499	0.1244	0.0000
1.0000	0.9279	0.8550	0.7804	0.7017	0.6136	0.5045	0.3754	0.2473	0.1228	0.0000
1.0000	0.9224	0.8437	0.7623	0.6761	0.5816	0.4757	0.3599	0.2402	0.1201	0.0000
1.0000	0.9181	0.8350	0.7490	0.6586	0.5619	0.4579	0.3476	0.2332	0.1170	0.0000
1.0000	0.9152	0.8289	0.7401	0.6474	0.5498	0.4469	0.3392	0.2279	0.1145	0.0000
1.0000	0.9137	0.8255	0.7350	0.6412	0.5432	0.4408	0.3344	0.2247	0.1129	0.0000
1.0000	0.9133	0.8247	0.7334	0.6391	0.5412	0.4385	0.3330	0.2232	0.1128	0.0000

TABLE 4

1.0000	0.9699	0.9419	0.9173	0.8980	0.8858	0.8815				
1.0000	0.9691	0.9401	0.9146	0.8945	0.8817	0.8772				
1.0000	0.9665	0.9347	0.9064	0.8838	0.8690	0.8639				
1.0000	0.9621	0.9258	0.8927	0.8653	0.8465	0.8397				
1.0000	0.9563	0.9137	0.8735	0.8381	0.8117	0.8017				
1.0000	0.9495	0.8993	0.8499	0.8024	0.7608	0.7400				
1.0000	0.9424	0.8842	0.8246	0.7623	0.6946	0.5983	0.4054	0.2634	0.1315	0.0000
1.0000	0.9359	0.8705	0.8017	0.7268	0.6400	0.5290	0.3953	0.2612	0.1300	0.0000
1.0000	0.9309	0.8598	0.7845	0.7021	0.6088	0.5012	0.3811	0.2553	0.1279	0.0000
1.0000	0.9280	0.8533	0.7742	0.6882	0.5929	0.4871	0.3719	0.2506	0.1260	0.0000
1.0000	0.9270	0.8515	0.7706	0.6840	0.5880	0.4823	0.3691	0.2481	0.1257	0.0000

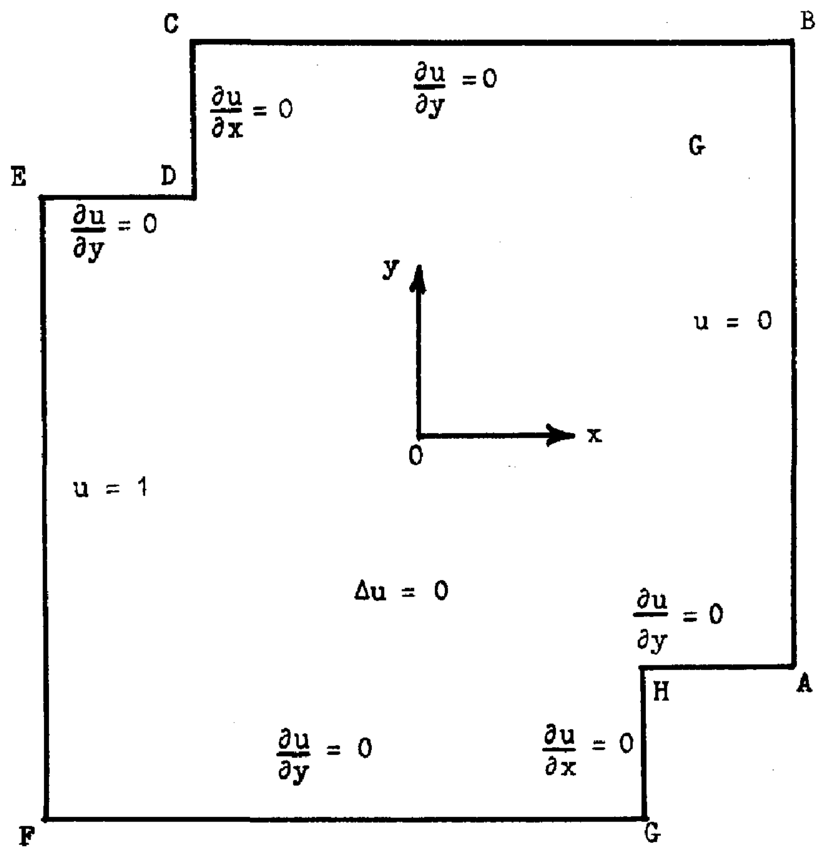
TABLE 5

At each mesh point the numbers represent:

CTM
Integral Eqn. Method [4]

1.0000	0.9015	0.8243	0.7940		
0.9996	0.9005	0.8223	0.7910		
1.0000	0.8917	0.8007	0.7604		
1.0000	0.8906	0.7982	0.7556		
1.0000	0.8667	0.7339	0.5656	0.2432	0.0000
1.0000	0.8660	0.7321	0.5676	0.2469	-0.0002
1.0000	0.8403	0.6702	0.4681	0.2349	0.0000
1.0000	0.8401	0.6700	0.4692	0.2360	0.0000
1.0000	0.8222	0.6365	0.4357	0.2208	0.0000
1.0000	0.8222	0.6365	0.4360	0.2212	0.0000
1.0000	0.8119	0.6193	0.4191	0.2116	0.0000
1.0000	0.8119	0.6193	0.4192	0.2116	0.0000
1.0000	0.8064	0.6103	0.4103	0.2063	0.0000
1.0000	0.8064	0.6103	0.4103	0.2063	0.0000
1.0000	0.8035	0.6057	0.4057	0.2035	0.0000
1.0000	0.8034	0.6056	0.4056	0.2034	0.0000
1.0000	0.8022	0.6035	0.4034	0.2021	0.0000
1.0000	0.8019	0.6031	0.4031	0.2019	0.0000
1.0000	0.8020	0.6026	0.4023	0.2014	0.0000
1.0000	0.8012	0.6020	0.4020	0.2013	0.0000
1.0000	0.8022	0.6027	0.4022	0.2001	0.0000
1.0000	0.8009	0.6016	0.4017	0.2011	0.0009

Problem 3 . Consider the harmonic mixed boundary value problem illustrated in Figure 3. The domain G, an octagon with dimensions AB = BC = EF= FG = 0.8, GH = HA = CD = DE = 0.2, contains two re-entrant corners at D and H. The problem is of type (1) with $w_1 \equiv A$, $w_2 = B$, $w_3 = E$ and $w_4 = F$.



w - plane.

Figure 3.

The numerical conformal mapping is performed with N = 100 and, by using the symmetry of G about the line BF, the number of equations (16) actually solved is 50. The CTM results are given, on a square mesh of length 0.1, in Table 6 and are compared with the results of Symm [4].

		0.5655	0.5462	0.4969	0.4295	0.3521	0.2685	0.1803	0.0901	0.0000
		0.5651	0.5452	0.4963	0.4294	0.3521	0.2686	0.1810	0.0908	0.0000
		0.5870	0.5615	0.5052	0.4343	0.3551	0.2704	0.1820	0.0911	0.0000
		0.5858	0.5606	0.5047	0.4341	0.3551	0.2706	0.1823	0.0916	0.0000
0.9993	0.8673	0.6940	0.6030	0.5266	0.4471	0.3635	0.2761	0.1858	0.0934	0.0000
1.0015	0.8682	0.6935	0.6025	0.5263	0.4469	0.3634	0.2762	0.1859	0.0935	0.0000
1.0000	0.8751	0.7511	0.6439	0.5520	0.4641	0.3756	0.2849	0.1917	0.0964	0.0000
1.0000	0.8753	0.7513	0.6438	0.5518	0.4640	0.3755	0.2849	0.1917	0.0964	0.0000
1.0000	0.8857	0.7746	0.6708	0.5746	0.4821	0.3900	0.2962	0.1997	0.1006	0.0000
1.0000	0.8857	0.7746	0.6708	0.5745	0.4821	0.3900	0.2962	0.1997	0.1006	0.0000
1.0000	0.8937	0.7896	0.6895	0.5935	0.5000	0.4065	0.3105	0.2104	0.1063	0.0000
1.0000	0.8937	0.7896	0.6895	0.5935	0.5000	0.4065	0.3105	0.2104	0.1063	0.0000
1.0000	0.8994	0.8003	0.7038	0.6100	0.5179	0.4254	0.3292	0.2254	0.1143	0.0000
1.0000	0.8994	0.8003	0.7038	0.6100	0.5179	0.4255	0.3292	0.2254	0.1143	0.0000
1.0000	0.9036	0.8083	0.7151	0.6244	0.5359	0.4480	0.3561	0.2489	0.1249	0.0000
1.0000	0.9036	0.8083	0.7151	0.6245	0.5360	0.4482	0.3562	0.2487	0.1247	0.0000
1.0000	0.9067	0.8142	0.7239	0.6365	0.5529	0.4734	0.3970	0.3055	0.1327	0.0000
1.0000	0.9065	0.8141	0.7238	0.6366	0.5531	0.4737	0.3975	0.3065	0.1318	-0.0015
1.0000	0.9089	0.8180	0.7296	0.6449	0.5657	0.4948	0.4385	0.4130		
1.0000	0.9084	0.8177	0.7294	0.6449	0.5659	0.4953	0.4394	0.4142		
1.0000	0.9099	0.8197	0.7315	0.6479	0.5705	0.5031	0.4538	0.4345		
1.0000	0.9092	0.8190	0.7314	0.6479	0.5706	0.5037	0.4548	0.4349		

TABLE 6

At each mesh point the numbers represent:

CTM

Integral Eqn. Method, [4]

4. Discussion

The results in [5] and [6] suggest strongly that the "analytic" CTM solution of Problem 1 represents the true solution of the problem to the number of figures quoted. Thus, the good agreement between the two sets of results displayed in Table 1 indicates that the "numerical" CTM solution of Problem 1 is extremely accurate.

We recognise that, since the numerical mapping of an irregular domain is less accurate than that of a rectangle, the comparison of the results in Table 1 can only give some indication of the accuracy of the "numerical" CTM for Problems 2 and 3. However, the CTM results in Tables 2,5 and 6 are in good agreement with those obtained by Symm [4], where an integral equation method modified to deal with the singularities at the re-entrant corners is used.

In the present paper the "numerical" CTM has been applied to problems defined in polygonal domains. However, it is clear that the method, with some simple modifications in the numerical mapping technique, may be used for the solution of problems of type (1) defined in general two dimensional simply-connected domains.

In [5] and [6] the CTM, with the exact mapping function used in (5) has been shown to be extremely efficient for problems defined in rectangular domains. On the strength of the present results we feel that the method deserves strong consideration as a practical technique for solving problems of type (1) even when, due to the geometry of the domain, the transformation (5) has to be performed numerically.

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