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A REVIEW AND COMPILATION OF LP MODELS

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1. INTRODUCTION

Since the development of the Simplex Algorithm by Dantzig in 1947, Linear Programming (LP) has been used to solve optimization problems in various industries. An extensive list of LP applications has been provided by Gass in his bibliography (Gass, 1985). Many other authors have presented real life problems which may be solved with LP techniques. Shapiro, for- example, describes actual business case studies *(Shapiro, 1984)*. See also *(Hillier & Lieberman, 1986), (Winston,1987),(Williams,1990).* Such examples will be formulated in this review.

The formulation of real life problems into LP models has lead to the classification of models into typical prototypes, eg: production; distribution; blending; manpower planning; cutting stock; transportation; network; process-flow. Each of these classes of models has intrinsic characteristics which demand particular modeling techniques. Many real situations will involve a combination of these prototype models and will therefore require a mixture of such techniques. Thus the simple skills used for formulating these prototypes form the basis of LP modelling. A compilation of basic prototype models follows in section two with illustrative examples. Although these examples are formulated in this section in a mathematical form, the models are not the most clearly defined. Thus a discussion of model formulation and a more formal approach to model description follows in section three. The basic prototype models may also be extended to deal with multi-time periods or multi-locations. This is illustrated in section four. Section five examines process flow models in more detail and section six describes network models which due to their special structure may be solved with specialist algorithms.

2. A COMPILATION OF BASIC PROTOTYPE MODELS

2.1 Production (how much of each product to produce)

This type of problem is also commonly known as a **product mix** problem. This is because it usually involves the production of various products competing for the same limited production lines and resources; the task being to determine the optimum production level for each product, which maximizes profit, subject to the production capacity available and the production capacity required for each product. The following example is taken from Hillier and Lieberman *(Hillier & Lieberman, 1986, p30f).*

Example 2.la

A glass manufacturer decides to produce two new products: product 1 is an aluminium framed glass door and product 2 is a wood-framed window. The company has three plants. Aluminium frames and hardware are made in plant 1, wood frames are made in plant 2 and plant 3 is used to produce the glass and assemble the products. The market is such that the company could sell as much of either product as could be produced with the available capacity. However, because both products would be competing for the same production capacity in plant 3, it is not clear which mix between the two products would be most profitable.

The **O.R.** Department have provided the necessary data for this problem, ie:

(1) the percentage of each plant's production capacity that would be available for these products;

(2) the percentages required by each product for each unit produced per minute and

(3) the unit profit for each product.

This information is summarized in the table below:

Table 2.la

Plant	%capacity used user per unit production rate Product 1	product 2	Capacity Available
1	1	0	4
2	0	2	12
3	3	2	18
Unit profit	\$3	\$5	

<u>Formulation of Example 2.la</u>

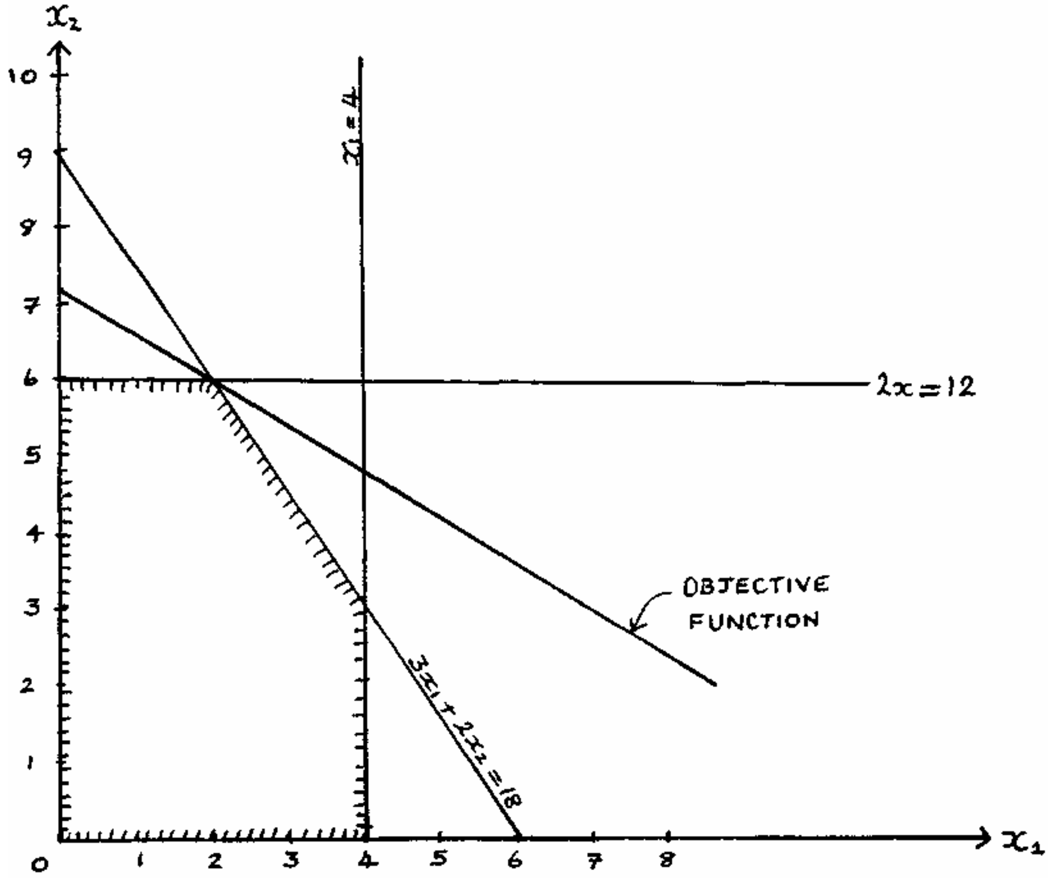
Let x_1 and x_2 represent the number of units of product 1 and product 2 respectively, produced per minute. The objective is to find values for x_1 and x_2 so as to maximize profit, z say. That is, Maximize $z = 3x_1 + 5x_2$, subject to the production capacity restrictions. From the table it is clear that each unit of product 1 produced per minute requires 1% of plant 1 capacity, whereas only 4% is available. This can be expressed mathematically as $x_1 \leq 4$. Similarly plant 2 provides the restriction $2x_2 \leq 12$ and plant 3 gives the constraint $3x_1 + 2x_2 \leq 18$. Finally, production rates cannot be negative, thus $x_1 \geq 0$ and $x_2 \geq 0$.

The **LP** formulation of this problem is therefore:

Maximize $z = 3x_1 + 5x_2$

$$
\begin{array}{llrl}
\text{subject to} & x_1 & & \leq 4 \\
 & & 2x_2 & \leq 12 \\
 & 3x_1 + & 2x_2 & \leq 18 \\
 & & x_1 \geq 0, x_2 \geq 0. &
\end{array}
$$

This is a simple two variable problem which can be solved graphically as shown below.



The inequality restrictions form a **convex** region (shown shaded). Any point within this region will satisfy all the constraints. Thus this region is known as the **feasible region** and any point in this region is a **feasible solution.** The objective function z= $3x_1 + 5x_2$ is a straight line with gradient -3/5. As the formulation states that this function is to be maximized, the optimum solution is a point where the straight line representing the objective function has the largest z value and coincides with a feasible solution. Thus moving the straight line upwards over the feasible region until a point is reached where further movement would, force the line outside the feasible region provides the optimum solution. Thus the optimum solution is $x_1 = 2$, $x_2 = 6$ which gives a profit of $36.

Problems involving more than two decision variables are not solvable by such two- dimensional methods. The following example is another production example taken from Shapiro *(Shapiro,* 1984, *p12f*). It is more complicated than example 2.la since one raw material can be made by using another raw material. In addition, there are four decision variables.

Example 2.1b

A power drill manufacturer produces 4 different types of drills (models 1, 2 ,3 and 4) which consist of an electric motor encased in a plastic and metal housing. The raw material requirements and the profit contribution for each model is provided in the table below. Up to 16 000 pounds of plastic and no more than 5 000 pounds of copper alloy are available for the production quarter. Wire for the motor winding is available from two sources. It can be produced internally at a cost of 14c/yard from the copper alloy, on machines with the capacity to draw 80 000 yards of wire per quarter. Each 100 yards of wire uses 3.6 pounds of copper alloy. Alternatively it can be ordered from outside suppliers in virtually unlimited quantities at 29c/yard. At the start of the quarter, 8 000 yards of wire is expected to be in stock. The other materials required in the production of these drill model;s are in abundant supply and can be obtained easily. Management believes that all the drills they make can be sold. However, for customer satisfaction, at least as many units should be produced of models 1 and 2 as are produced of the newer models 3 and 4. The company needs to plan its product mix for the quarter.

Table 2. 1b

	Model 1	Model 2	Model 3	Model 4
Plastic Required (lb)	0.82	0.62	1.42	2.03
Copper Alloy Required (lb)	0.43	0.69	0.33	0.20
Wire Required (yd)	15	16	9	9
Contribution (excluding wire cost)	\$12.50	\$11.30	\$17.20	\$19.90

Formulation of Example 2.1b

Let x_1, x_2, x_3 and x_4 represent the number of models 1, 2, 3 and 4 produced respectively and let w_p and w_m be the yards of wire purchased and manufactured respectively. The objective is again to maximize profit, z say. That is, the objective may be stated as Maximize $z = 12.50x_1 + 11.30x_2 + 17.30x_3 + 19.90x_4 - 0.14w_m - 0.29w_p$. The restrictions imposed are resource limitations as well as the customer satisfaction production constraint. The resource restriction for plastic may be stated as: $0.82x_1 + 0.62x_2 + 1.42x_3 + 2.03x_4 \leq 16\,000$. The copper alloy restriction may be expressed as: $0.43x_1 + 0.69x_2 + 0.33x_3 + 0.20x_4 + 0.036w_m \leq 5\,000$. Note that w_m had to be included in this expression as wire production uses copper alloy. The wire constraint is a little more complicated since the wire available = wire produced + wire bought + wire on hand = $w_m + w_p + 8000$. Thus the constraint in full is: $15x_1 + 16x_2 + 9x_3 + 9x_4 \leq w_m + w_p + 8000$. As the wire production machines have a limited capacity we need to include the constraint $w_m \leq 80\,000$. Finally the customer satisfaction constraint may be written as: $x_1 + x_2 \geq x_3 + x_4$.

Thus the formulation of this problem (with all the variables on the left hand side) is:

Maximize $z = 12.50x_1 + 11.30x_2 + 17.20x_3 + 19.90x_4 - 0.14w_m - 0.29w_p$

subject to

$$
\begin{array}{l}
0.82x_1 + 0.62\, x_2 + 1.42\, x_3 + 2.03\, x_4 \qquad \le 16\ 000; \\
0.43x_1 + 0.69\, x_2 + 0.33\, x_2 + 0.20\, x_4 + 0.036w_m \quad \le\ 5\ 000; \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad w_m \quad \le 80\ 000; \\
5x_1 + \quad 16\, x_2 + \quad 9x_3 + \quad 9\, x_4 - \quad w_m - w_p \ \le\ \ 8\ 000; \\
x_1 + \qquad x_2 - \quad x_3 - \qquad x_4 \qquad\qquad \ge\ 0; \\
x_1,\ x_2,\ x_3,\ x_4, w_m\ w_p \ge 0.
\end{array}
$$

2.2 Distribution (finding a distribution plan to satisfy demand)

This type of problem is concerned with finding an optimal distibution strategy which will satisfy the demand and at the same time keep within the capacities and limitations that exist. The following example illustrates:

Example 2.2

A company has two factories and four depots. It sells its products to six customers each of whom may be supplied either from a depot or direct from the factory. The company has to pay distribution costs (in pounds per ton) for the deliveries. These are shown in the table below. Dashes indicate that certain deliveries are impossible.

Supplied to	Liverpool Factory	Brighton Factory	Newcastle Depot	Birmingham Depot	London Depot	Exeter Depot
Depots						
Newcastle	0.5	-				
Birmingham	0.5	0.3				
London	1.0	0.5				
Exeter	0.2	0.2				
Customers						
C1	1.0	2.0	-	1.0	-	-
C2	-	-	1.5	0.5	1.5	-
C3	1.5	-	0.5	0.5	2.0	0.2
C4	2.0	-	1.5	1	-	1.5
C5	-	-	-	0.5	0.5	0.5
C6	1.0	-	1.0	-	1.5	1.5

Each factory has a monthly capacity which cannot be exceeded:

Factory	Capacity in tons
Liverpool	150 000
Brighton	200 000

Each depot has a maximum monthly throughput given below which cannot be exceeded:

Depot	Max throughput in tons
Newcastle	70 000
Birmingham	50 000
London	100 000
Exeter	40 000

Each customer has a monthly requirement which must be met:

Customer	Monthly Requirement
C1	50 000
C2	10 000
C3	40 000
C4	35 000
C5	60 000
C6	20 000

The company wishes to find a distribution pattern which minimizes cost *(Williams 1990, p263f)*.

<u>Formulation of Example 2.2</u>

Let x_{ij} the quantity sent from factory i to depot j,

Y_{ik} be the quantity sent from factory i to customer k

and z_{jk} be the quantity sent from depot j to customer k. ($1 = 1,2; j=1,\ldots,4; k=1,\ldots,6$)

As some of the deliveries are impossible, some x_{ij}, y_{ik} and z_{jk} will not be defined (namely x_{21}, y_{12}, y_{22}, y_{23}, y_{24}, y_{15}, y_{25}, y_{26}, z_{11}, z_{15}, z_{26}, z_{31}, z_{34}, z_{41}, z_{42}).

The objective is to minimize cost, z, say. That is:

$$
\begin{aligned}
\text{Minimize } z = {} & 0.5x_{11} + 0.5x_{12} + x_{13} + 0.2x_{14} + 0.3x_{22} + 0.5x_{23} + 0.2x_{24} \\
& + y_{11} + 1.5y_{13} + 2y_{14} + y_{16} + 2y_{21} \\
& + 1.5z_{12} + 0.5z_{13} + 1.5z_{14} + z_{16} + z_{21} + 0.5z_{22} + 0.5z_{23} + z_{24} \\
& + 0.5z_{25} + 1.5z_{32} + 2z_{33} + 0.5z_{35} + 1.5z_{36} + 0.2z_{43} + 1.5z_{44} + 0.5z_{45} \\
& + 1.5z_{46}.
\end{aligned}
$$

Each factory has a capacity, this adds two restrictions that all x_{ij} and y_{ik} added together for each factory must not exceed each factories capacity. For example, for factory 1 (Liverpool):

$$x_{11} + x_{12} + x_{13} + x_{14} + y_{11} + y_{13} + y_{14} + y_{16} \leq 150\ 000.$$

Each depot too has a capacity. For example, for depot 1 (Newcastle):

$x_{11} \leq 70\ 000$. There are three other such constraints (one for each depot).

The customer requirements must also be satisfied. For example, for customer C1:

$$y_{11} + y_{21} + z_{21} \geq 50\,000.$$

There are five more such restrictions (one for each customer).

Finally, like in the blending problem, there are continuity restrictions. The continuity at each depot must be considered. For example, at depot 1:

$$z_{12} + z_{13} + z_{14} + z_{16} = x_{11}$$

Again as there are three more depots, there will be three more of these constraints.

Williams provides a general formulation of this problem *(op. cit. p305f)*. Distribution models like this example often occur together with production models. These are then called **production cum distribution** models. Distribution models may also be viewed as network flow models which is considered in section six.

2.3 Blending (how to blend raw materials)

This type of problem usually involves the blending of raw materials to make an end product (or products) so that the cost of the blend is minimized (or profit from sales of the final product is maximized), subject to various quality restrictions. A typical characteristic of such a problem is that certain balancing constraints (also known as **continuity constraints)** need to be taken into account. These constraints arise as a result of conservation of mass during blending. The following example taken from Williams illustrates this.

Example 2.3

"A food is manufactured by refining raw oils and blending them together. The raw oils come in two categories:

Vegetable Oils	VEG1	and	Non-Vegetable Oils	OIL1
	VEG2			OIL2
				OIL3

Vegetable oils and non-vegetable oils require different production lines for refining. In any month it is not possible to refine more than 200 tons of vegetable oil and more than 250 non-vegetable oils. There is no loss of weight in the refining process and the cost of refining may be ignored.

There is a technological restriction of hardness in the final product. In the units in which hardness is measured this must lie between 3 and 6. It is assumed that hardness blends linearly. The cost (per ton) and the hardness of the raw oils are:

	VEG1	VEG2	OIL1	OIL2	OIL3
Cost	£110	£120	£130	£110	£115
Hardness	8.8	6.1	2.0	4.2	5.0

The final product sells for £150 per ton.

How should the food manufacturer make his product in order to maximize his net profit? " *(op. cit. p8)*

Formulation of Example 2.3

Let v_1 and v_2 represent the quantity of vegetable oils, veg1 and veg2 respectively, (in tons) and w_1,w_2, and w_3 represent the non-vegetable oils, oil1, oil2 and oil3 respectively, (in tons) which are blended together. Let y represent the quantity of food (also in tons) that is manufactured by blending these raw oils. The objective is to maximize the total profit, z say. This profit consists of the revenue from the final product minus the cost of the oils which are blended. Thus the objective may be represented as Maximize $z = 150y - (110v_1 + 120v_2 + 130w_1 + 110w_2 + 115w_3)$.

As with the production problem there are restrictions on the available resources: the sum of the vegetable oils that are blended cannot be greater than 200 tons (the amount available). Thus, $v_1 + v_2 \leq 200$. Similarly for the non-vegetable oils, $w_1 + w_2 + w_3 < 250$. The technical restrictions on the hardness implies that $3y \leq 8.8v_1 + 6.1v_2 + 2w_1 + 4.2w_2 + 5w_3 \leq 6y$. Finally, although it is not stated implicitly in the problem description, it is necessary to model the balance of materials in the blending process. In other words, the mass of the final product y must equal the total mass of all the ingredients. This can be represented as $v_1 + v_2 + w_1 + w_2 + w_3 = y$.

Thus the full formulation may be written as:

Maximize $z = 150y - (110v_1 + 120v_2 + 130w_1 + 110w_2 + 1.15w_3)$

subject to:

$$
\begin{array}{lcl}
v_1 + v_2 & \leq & 200; \\
w_1 + w_2 + w_3 & \leq & 250; \\
8.8v_1 + 6.1v_2 + 2w_1 + 4.2w_2 + 5w_3 - 6y & \leq & 0; \\
8.8v_1 + 6.1v_2 + 2w_1 + 4.2w_2 + 5w_3 - 3y & \geq & 0; \\
v_1 + v_2 + w_1 + w_2 + w_3 - y & = & 0; \\
v_1, v_2, w_1, w_2, w_3 \geq 0. & &
\end{array}
$$

Often in blending problems involving more than one final product there will be several continuity restrictions.

2.4 Manpower Planning (how many people to allocate to each shift)

This type of problem is concerned with deciding how many people should be employed in each shift so as to minimize the overall cost but still satisfy the requirements for the scheduling period. Hence this class of problems is also known as **work-scheduling problems.** The following example is taken from Winston *(Winston, 1987, p70f).*

Example 2.4

A post office requires different numbers of full-time employees on different days of the week. The number of full-time employees required on each day is given in the table below. Union rules state that each full-time employee must work 5 consecutive days and then receive 2 days off. The post office wants to meet daily requirements with only full-time employees. Formulate an LP that the post office can use to minimize the number of full-time employees that must be hired.

			Number of Full-time Employees Required
Day	1	Mon	17
	2	Tue	13
	3	Wed	15
	4	Thu	19
	5	Fri	14
	6	Sat	16
	7	Sun	11

Formulation of Example 2.4

Let x_i represent the number of people who start work on day i, (i = 1,2,3,...7). The objective is to minimize the total number of full-time employees, z say, thus this can be expressed as

$$\text{Minimize } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7.$$

As the union rule restricts the number of consecutive days a person can work, any person who starts work on a Tuesday or Wednesday will not be able to work on Monday. Similarly, any person who starts work on a Wednesday or Thursday will not be able to work on Tuesday. Thus two X_i's will be omitted from each days constraint. For example to satisfy the staffing requirement for Monday,

$$x_1 + x_4 + x_5 + x_6 + x_7 \geq 17.$$

The full formulation of the problem is:

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$.

$$
\begin{aligned}
\text{subject to:}\quad & x_1 + x_4 + x_5 + x_6 + x_7 \geq 17;\\
& x_1 + x_2 + x_5 + x_6 + x_7 \geq 13;\\
& x_1 + x_2 + x_3 + x_6 + x_7 \geq 15;\\
& x_1 + x_2 + x_3 + x_4 + x_7 \geq 19;\\
& x_1 + x_2 + x_3 + x_4 + x_5 \geq 14;\\
& x_2 + x_3 + x_4 + x_5 + x_6 \geq 16;\\
& x_3 + x_4 + x_5 + x_6 + x_7 \geq 11;\\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7. \geq 0.
\end{aligned}
$$

Strictly speaking there should be the further restriction that all variables must take integer values as it is not meaningful to have fractional values for variables representing numbers of people. This brings us into the realm of Integer Programming (or IP). However, as the first step of IP is to solve the model using

L.P. techniques, for the purposes of illustration we shall ignore this condition. In addition, many of the fractional decision variable values in such problems often turn out to be close to integer values and taking rounded values can provide insight into the problem.

In the above example, we were told the daily personnel requirements and the shift pattern (five consecutive days on and two days off) were already known. Frequently, the personnel requirements for a scheduling period have to be forecasted and the possible shift patterns need to be identified. There may well be more than one possible shift pattern. The next type of problem, the cutting stock problem, deals with this. Here, possible cutting patterns need to be constructed before the optimum cutting schedule can be selected.

2.5 Cutting Stock (how much of each cutting pattern to use)

This type of problem usually occurs in industries such as the paper industry where a product needs to be cut into a variety of smaller size pieces. The requirements for the final sizes of the product are forecasted and possible cutting patterns are established. The objective is to determine how much of each pattern should be run so as to satisfy the final size requirements at minimum cost. The following example is taken from Schrage (*Schrage, 1981, p66-69*).

Example 2.5

A company which produces household appliances purchases sheet steel in coils of widths 72 inches, 48 inches and 38 inches. In the manufacturing process eight different widths of this sheet steel are required. Namely, 60, 56, 42, 38, 34, 24, 15 and 10 inches. In cutting the sheet steel there is trim waste. The prices per foot of the sheet steel are given as follows:

Width in (inches)	Price per foot (in cents)
72	28
48	19
36	15

The coils may be cut in any feasible way. Possible cutting patterns for the sheet steel are tabulated as follows:

PATTERNS FOR THE 72" RAW MATERIAL

Pattern Designation	NO. TO CUT OF THE REQUIRED WIDTH								Waste in inches
	60”	56”	42”	38”	34”	24”	15”	10”	
Al	1	0	0	0	0	0	0	1.	2
A2	0	1	0	0	0	0	1	0	1
A3	0	1	0	0	0	0	0	1	6
A4	0	0	1	0	0	1	0	0	6
A5	0	0	1	0	0	0	2	0	0
A6	0	0	1	0	0	0	1	1	5
A7	0	0	1	0	0	0	0	3	0
A8	0	0	0	1	1	0	0	0	0
A9	0	0	0	1	0	1	0	1	0
B0	0	0	0	1	0	0	2	0	4
B1	0	0	0	1	0	0	1	1	9
B2	0	0	0	1	0	0	0	3	4
B3	0	0	0	0	2	0	0	0	4
B4	0	0	0	0	1	1	0	1	4
B5	0	0	0	0	1	0	2	0	8
B6	0	0	0	0	1	0	1	2	3
B7	0	0	0	0	1	0	0	3	8
B8	0	0	0	0	0	3	0	0	0
B9	0	0	0	0	0	2	1	0	9
C0	0	0	0	0	0	2	0	2	4
C1	0	0	0	0	0	1	3	0	3
C2	0	0	0	0	0	1	2	1.	8
C3	0	0	0	0	0	1	1	3	3
C4	0	0	0	0	0	1	0	4	8
C5	0	0	0	0	0	0	4	1	2
C6	0	0	0	0	0	0	3	2	7
C7	0	0	0	0	0	0	2	4	2
C8	0	0	0	0	0	0	1	5	7
C9	0	0	0	0	0	0	0	7	2

PATTERNS FOR 0THE 48" RAW MATERIAL

Pattern Designation	NO TO CUT OF THE REQUIRED WIDTH								waste in inches
	60”	56”	42”	38”	34”	24”	15”	10”	
D0	0	0	1	0	0	0	0	0	6
Dl	0	0	0	1	0	0	0	1	0
D2	0	0	0	0	1	0	0	1	4
D3	0	0	0	0	0	2	0	0	0
D4	0	0	0	0	0	1	1	0	9
D5	0	0	0	0	0	1	0	2	4
D6	0	0	0	0	0	0	3	0	3
D7	0	0	0	0	0	0	2	1	8
D8	0	0	0	0	0	0	1	3	3
D9	0	0	0	0	0	0	0	4	8

PATTERNS FOR THE 36" RAW MATERIAL

Pattern Designation	NO TO CUT OF THE REQUIRED WIDTH 60"	56"	42 "	38"	34"	24"	15"	10"	waste in inches
E0	0	0	0	0	1	0	0	0	2
El	0	0	0	0	0	1	0	1	2
E2	0	0	0	0	0	0	2	0	6
E3	0	0	0	0	0	0	1	2	1
E4	0	0	0	0	0	0	0	3	6

The lengths of various widths required for the planning period are:

Width (in inches)	60	56	42	38	34	24	15	10
Feet required	500	400	300	450	350	100	800	1000

The amount of sheet steel available this period is as follows:

Width	Feet available
72	1 600
48	10 000
36	10 000

The company wishes to determine the number of feet of each pattern which should be cut so as to satisfy the width requirements at minimum cost.

Formulation of Example 2.5

Let A_1, A_2,...E_4 (as in the previous table) denote the number of feet to cut of the corresponding pattern. In addition, let

T_1 = No. feet cut of 72 inch patterns

T_2 = No. feet cut of 48 inch patterns

T_3 = No. feet cut of 32 inch patterns

X_1 = No feet of excess cut of 60 inch width

X_2= No feet of excess cut of 56 inch Width

.

.

.

X_8 = No feet of excess cut of 10 inch width

The objective is to minimize the total cost, z say. That is,

Minimize $z = 28T_1 + 19T_2 + 15T_3$

The availability of raw material imposes 3 constraints one for each width. For example for the 72 inch width, $T_1 \leq 1\,600$.

The definition of T_1, T_2 and T_3 give three more constraints. For example, for T_2:

$T_2 - D_0 - D_1 - D_2 - D_3 - D_4 - D_5 - D_6 - D_7 - D_8 - D_9 = 0.$

In order to satisfy the demand for the 8 widths, 8 more constraints are required. For example, for the 60 inch width:

$$A_1 - X_1 = 500$$

and for the 42 inch width:

$$A_4 + A_5 + A_6 + A_7 + D_0 - X_3 = 300.$$

Schrage provides the full formulation of this problem (*op., cit. p68*).

In larger problems it may be impractical if not impossible to generate all possible patterns. This will be discussed in a later review of software tools.

There may also be additional cost considerations ***(op. cit. p69-70).*** For example, there may be a fixed cost of setting up a particular pattern. Thus this encourages solutions with fewer patterns.

2.6 Transportation (how to transport goods)

Transportation problems are concerned with the transportation of goods from a set of supply points (for example warehouses) to a set of demand points (for example, retail outlets). Each unit of the goods transported has an associated cost. The objective is thus to minimize the total transportation cost whilst simultaneously satisfying the demand. If the total quantity of the goods in supply equals the total quantity of the goods in demand, the problem is known as a **balanced transportation problem.** It is desirable to formulate transportation problems as balanced transportation problems as the solution of balanced problems are easier. Unbalanced transportation problems, with the total supply greater than the total demand, can be balanced by creating a dummy demand point with a demand equal to the excess supply. This is detailed further in section five. Obviously unbalanced problems with the total supply less than the total demand are infeasible, though it may be possible to permit some demand to be unmet with a penalty associated with this *(Winston, 1987, p265-267).* The following example is a balanced transportation problem.

Example 2.6

A certain product is produced at 3 plants (plants 1, 2 and 3). Plants 1, 2, and 3 have 100, 120 and 120 tons (respectively) of this product which is to be delivered to 5 warehouses (warehouses 1, 2, 3,...5). Each warehouse requires its quota of this product, ie. 40, 50, 70, 90 and 90 tons respectively. The transportation cost for a given unit of the product is shown in the table below.

warehouses Plants	1	2	3	4	5
1	4	1	2	6	9
2	6	4	3	5	7
3	5	2	6	4	8

The problem is to work out a transportation plan so as the total cost of transporting the product is minimized *(Kaufman, 1963, p51f).*

Formulation of Example 2.6

Let x_{ij} = the quantity of product produced at plant i. (i=1,2,3) and shipped to warehouse j (j= 1,2,3,….5). The objective is to minimize the total transportation cost, z say. Each x_{ij} has an associated cost c_{ij} as given in the table above. The availability of the product at each plant leads to three constraints. For example for plant 1,

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 100$$

The demand for each warehouse imposes 5 restrictions. For example, for warehouse 3, $x_{13} + x_{23} + x_{33} = 70.$

Thus the LP formulation of this problem is:

$$\begin{aligned} \text{Minimize } z = {} & 4x_{11} + x_{12} + 2x_{13} + 6x_{14} + 9x_{15} \\ & + 6x_{12} + 4x_{22} + 3x_{23} + 5x_{24} + 7x_{25} \\ & + 5x_{31} + 2x_{32} + 6x_{33} + 4x_{34} + 8x_{35} \end{aligned}$$

subject to

$$\begin{array}{lllr} x_{11}+ x_{12}+ x_{13}+ x_{14}+ x_{15} & & & =100; \\ & x_{21}+x_{22}+ x_{23}+ x_{24}+ x_{25} & & =120; \\ & & x_{31}+ x_{32}+ x_{33}+ x_{34}+ x_{35} & =120; \\ x_{11} & +x_{21} & +x_{31} & = 40; \\ \quad x_{12} & \quad +x_{22} & \quad +x_{32} & = 50; \\ \qquad x_{13} & \qquad +x_{23} & \qquad +x_{33} & = 70; \\ \qquad\quad x_{14} & \qquad\quad +x_{24} & \qquad\quad +x_{34} & = 90; \\ \qquad\qquad x_{15} & \qquad\qquad +x_{25} & \qquad\qquad +x_{35} & = 90; \end{array}$$

$$X_{ij} \geq 0 \qquad i= 1,2,3 \;\; j= 1,2,3,4,5.$$

Notice the special structure of this model. Transportation models have a particular structure which enables them to be solved more easily than other types of models of comparative size *(Williams, 1990, p84)*. The transportation model is a special case of what is called a **generalized network flow model** which is concerned with finding the minimal cost flow through a network. Network models are considered in more detail in section six.

2.7 Process Flow

Process flow models are concerned with a production process, for example, where the output from one stage of the process is either used directly or is used as an input to later stage of the process. The following example illustrates.

Example 2.7

"Furnco manufactures tables and chairs. A table requires 40 board ft of wood and a chair requires 30 board ft of wood. Wood may be purchased at a cost of \$1 per board ft and 40 000 board ft of wood are available for purchase. It takes two hours of skilled labor to manufacture an unfinished table or an unfinished chair. Three more hours of skilled labour will turn an unfinished table into a finished table and two more hours of skilled labor will turn an unfinished chair into a finished chair. A total of 6000 hours of skilled labor are available (and have already been paid for). All furniture produced can be sold at the following unit prices: unfinished table, \$70; finished table \$140; unfinished chair \$60; finished chair \$110. Formulate an LP that will maximize the contribution to profit from manufacturing tables and chairs." *(Winston, 1987, p90)*

Formulation of Example 2.7

Let uc be the unfinished chairs made,
ut be the unfinished tables made,
sc be the unfinished chairs sold,
st be the unfinished tables sold,
fc be the finished chairs made and sold
ft be the finished tables made and sold.

The objective is to maximize profit, z say. That is

Maximize $z = 70ut + 60uc + 140ft + 110fc$.

All unfinished furniture can be either sold as unfinished or finished and then sold. Thus there are two restrictions, one for chairs and one for tables:

$uc = sc + fc$ and
$ut = st + ft$.

There is a restriction on the availability of wood and each item of furniture requires different amounts of this resource. Thus,

$40ut + 30uc \leq 40\,000$.

There is also a limited amount of labour. Thus, $2uc + 2ut + 3ft + 2fc \leq 6000$.

Therefore the full formulation for this example is:

Maximize $z = 70ut + 60uc + 140ft + 110fc$.

subject to:

$$uc - sc - fc = 0;$$

$$ut - st - ft = 0;$$

$$40ut + 30uc \leq 40\,000;$$

$$2uc + 2ut + 3ft + 2fc \leq 6000;$$

$$uc, ut, sc, st, fc, ft \geq 0.$$

This is obviously a very simple example in which two different processes (manufacturing and finishing) are interrelated. A more complicated form of this type of problem occurs in input-output models which can represent interrelationships between different parts of the economy.

3. MODEL FORMULATION AND LP MODELS IN DECLARATIVE FORM

In section two basic prototype models were reviewed together with illustrative examples. The formulation of these examples was presented in a mathematical form which was somewhat incomplete. For much larger problems, as found in real applications, this incomplete format may be difficult to read and comprehend. It is clearly not the easiest form with which to communicate to other modellers. Representation of models, indeed representation of knowledge, can take two forms: **declarative** and **procedural**. "Knowledge items of the first type have the advantages of being easy to read and to modify and of not requiring anything to be said in advance of how they are to be used. The disadvantage however, is that processing such knowledge items can take a relatively long time. Items of the second type, procedural items, have precisely the opposite advantages and disadvantages" *(Bonnet, 1985, p83)*. For the purpose of representing LP models declarative form is preferable. (However, for the purposes of solving these models this form is not suitable and must be changed or converted into a more appropriate form. This will be taken up in chapter two where LP modelling tools will be reviewed.). Models may be represented in a declarative form as follows:

- **Subscripts, Ranges:**

  $i = 1,\ldots,m$; $j = 1,\ldots,n$

- **Variables and coefficients**

  x: x_j, $j = 1,\ldots,n$, r: r_i, $i = 1,\ldots,m$,

  c: cj, $j = 1,\ldots,n$ b: b_i, $i = 1,..,m$,

  A: a_{ij}, $i = 1,\ldots m$, $j = 1,\ldots,n$.

- **Linear Objective function and constraints**

Max

$$\sum_{j=1}^{n} c_j x_j$$

subject to r_i: $\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1,\ldots,m$

$x_j \geq 0, \qquad j = 1,\ldots,n.$

Thus a strategy for model formulation may be defined as follows:

Strategy for Model Formulation

Step 1 *Define the subscripts and their ranges. These emerge from the categories of the problem. For example the number of planning periods, number of different products, number of different locations etc.*

Step2 *Define the model variables and the coefficients in terms of the subscripts defined in step 1. In defining the decision variables of the model some new subscripts which may have been overlooked in step 1 may become apparent The model coefficients provide the data for the problem being considered.*

Step3 *Specify the linear relationships which connect the items defined in step 2. These include an objective function and constraints which arise from the physical restrictions of the problem. For example capacity restrictions, continuity constraints, blending requirements etc.*

These three steps lead to a declarative formulation of the model which is clear, precise and easy to comprehend. To illustrate this, reconsider example 2.3.

Example 3.1 (example 2.3 in declarative form)

Subscripts, Ranges $i = 1,2,3,4,5$ denotes the raw oils (veg1, veg2, oil1, oil2 oil3)

Variables x_i the quantity of raw oil i used in the blend (in tons)

y the quantity of the final food produced by blending (in tons)

Coefficients c_i the cost per ton of raw oils

h_i the hardness of raw oils

p price charged per ton of final food sold

L_V refining limit for vegetable oils

L_N refining limit for non-vegetable oils

Linear Constraint Relations: A Mathematical Statement

Maximize profit = $py - \sum_{i=1}^{5} c_i x_i$

subject to:

continuity of mass $\sum_{i=1}^{5} x_i - y = 0;$

refining limits $\sum_{i=1}^{2} x_i \leq Lv;$

$$\sum_{i=3}^{5} x_i \leq L_N;$$

upper limit on hardness

$$\sum_{i=1}^{5} h_i x_i - 6y \leq 0;$$

lower limit on hardness

$$\sum_{i=1}^{5} h_i x_i - 3y \geq 0;$$

and $x_i\ y \geq 0$ $\quad$ $i= 1,2,...,5.$

Strictly, the hardness limits of 6 and 3 should be defined in the coefficients section. Thus should any of the coefficients change, the model can be updated with just one alteration. This is a small model so a change in the hardness limits could be easily accommodated by changing the values in the constraints. In larger problems this would not be such an easy task as it is likely that the same coefficients will be used in many constraints and the size of the model itself would inhibit the operation. Thus wherever possible it is preferable to keep data separate from the linear constraint section of the model. This declarative formulation provides a generalized description of the problem and the data in the coefficients section instantiates the model. In practice the coefficients are stored separately in a data file. This maintains security of data and facilitates maintenance of the model.

4. MULTI-TIME PERIOD AND MULTI-LOCATION EXTENSIONS

In section two, the various classes of basic prototype models were compiled. These basic models may be extended to deal with multi-time periods or multi-locations. The following two examples illustrate this. Both examples have been stated in declarative form as described in section three.

Example 4.1 (Multi-time period problem)

This example is taken from Williams. The problem data is provided for six months so that the model needs to reflect this. It is no longer just a single planning period, but has many planning periods.

"A food is manufactured by refining raw oils and blending them together. The raw oils come in two categories:

vegetable oils	VEG 1	non-vegetable oils	OIL1
	VEG 2		OIL2
			OIL3

Each oil may be purchased for immediate delivery (January) or bought on the future's market for delivery in a subsequent month. Prices now and in the future's market are given below (in £s/ton):

	VEG 1	VEG 2	OIL 1	OIL 2	OIL 3
January	110	120	130	110	115
February	130	130	110	90	115
March	110	140	130	100	95
April	120	110	120	120	125
May	100	120	150	110	105
June	90	100	140	80	135

The final product sells at £150 ton.

Vegetable oils and non-vegetable oils require different production lines for refining. In any month it is not possible to refine more than 200 tons of vegetable oils and more than 250 tons of non-vegetable oils, there is no loss of weight in the refining process and the cost of refining can be ignored.

It is possible to store up to 1000 tons of each raw oil for use later. The cost of storage for vegetable and non-vegetable oil is £5 per ton per month. The final product cannot be stored. Nor can refined oils be stored.

There is a technological restriction of hardness on the final product. In the units in which hardness is measured this must lie between 3 and 6. It is assumed that hardness blends linearly and that hardness of the raw oils are

VEG 1	8.8	OIL 1	2.0
VEG 2	6.1	OIL 2	4.2
		OIL 3	5.0

What buying and manufacturing policy should the company pursue in order to maximize profit?

At present there are 500 tons of each type of raw oil in storage. It is required that these stocks will also exist at the end of *June"(Williams, 1990, p245f).*

Formulation

Subscripts, Ranges

i = 1,2,3,4,5 denotes raw oils (VEG1 - OILS)
t = 1,2,3,4,5,6 denotes months (Jan - Jun)

Variables

b_{it} quantity of oils bought in month t

S_{it} quantity of oils stored in month t

U_{it} quantity of oils used in month t

y_{it} quantity of final product made in month t

Coefficients

c_{it} cost of oils i (per ton) per month
h_i hardness of oils i
p price per ton of final product
q storage cost per month per ton

Linear Constraint Relations: A Mathematical Statement

Maximize profit = $$p\sum_{t=1}^{6} y_t - \sum_{t=1}^{6} \sum_{i=1}^{5} c_{it} b_{it} - q\sum_{t=1}^{6} \sum_{i=1}^{5} s_{it}$$

subject to:

continuity of mass $\sum_{i=1}^{5} u_{it} - y_t = 0;$ t=1,2, ...,6

refining limits $\sum_{i=1}^{2} u_{it} \leq 200;$ t=1,2, ...,6

$\sum_{i=3}^{5} u_{it} \leq 250;$ t=1,2, ...,6

upper limit on hardness $\sum_{i=1}^{5} h_i u_{it} - 6y_t \leq 0;$ t=1,2,...,6

lower limit on hardness $\sum_{i=1}^{5} h_i u_{it} - 3y_t \leq 0;$ t=1,2,...,6

initial storage $b_{i1} - u_{i1} - s_{i1} + 500 = 0;$ i=1,2,...,5 (t =1)

linking constraints $s_{it-1} + b_{it} - u_{it} - s_{it} = 0;$ i=1,2 ..... 5 ; t= 2,...,6

final storage $s_{i6} - 500 = 0;$ i=1,2,...,5

and $b_{it}, s_{it}, u_{it}, y_t \geq 0$ i=1,2 .....5 ; t=1,2,...,6

Having multi-time periods has multiplied the number of constraints by six (as there are six time periods). In addition, initial and final storage conditions have to be satisfied and linking constraints (which ensure that the quantity bought in the previous month t-1 + quantity bought in month t equals the quantity used in month t + quantity stored in month t) are required. Such linking constraints are always required for multi-time period problems.

Example 4.2 (Multi-location problem)

This example is a production cum distribution problem involving more than one plant and more than one product.

Two plants, A and B, situated in different locations both produce products P1 and P2. At A there are three machines and at B there are two machines. All machines manufacture both P1 and P2. After manufacture, products may be transported between plants to satisfy demand. The number of units produced per day of each product, the production and transportation costs, the demand for the products, the number of days that each machine has available per month and other numerical information are provided in the tables below.

Plant	A						B			
Machine	M1		M2		M3		M1		M2	
Product	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2
Prodn/day	40	35	42	38	40	37	41	37	42	40
Cost/day	100	102	104	106	98	104	102	105	103	106
Availability (in days)	30		28		24		26		28	

Products	P1		P2	
Plants	A	B	A	B
Demands	1200	800	1500	1100
Transport cost/unit	From A to B = 4		From A to B = 3	
	From B to A = 4		From B to A = 4	

The task is to find the best operating schedule of the machines in each plant, and also the interplant distribution of all the products, all at minimum cost *(Mitra, 1976, p104f)*.

Subscripts and Ranges

$i = 1, 2$ denotes plants (A and B)

$j = 1, 2, 3$ denotes machines (Ml, M2 and M3)

$k = 1, 2$ denotes products (P1, P2)

Variables

X_{ijk} production at plant i, on machine j of product k.

(Note: throughout this model if plant i=2 then machine j=3 does not exist)

t_{ik} transportation of product k from plant i

u_{ik} transportation of product k to plant i

Coefficients

c_{ijk} cost per day of production at plant i, on machine j of product k

p_{ijk} production per day at plant i, on machine j of product k

m_{ij} machine availability (in days) at plant i of machine j

d_{ik} demand at plant i for product k

S_{ilk} transportation cost of product k from plant i

v_{ik} transportation cost of product k to plant i

Linear Constraint Relations: A Mathematical Statement

Minimize cost= $\sum_{i=1}^{2} \sum_{j=1}^{3} \sum_{k=1}^{2} c_{ijk} \; x_{ijk} + \sum_{i=1}^{2} \sum_{k=1}^{2} s_{ik} \; t_{ik} + \sum_{i=1}^{2} \sum_{k=1}^{2} v_{ik} \, u_{ik}$

subject to:

demand $\sum_{j=1}^{3} p_{ijk} \; x_{ijk} - s_{ik} \; t_{ik} + v_{ik} \; u_{ik} = d_{ik}$ i=1,2; k=1, 2

availability of machines $\sum_{k=1}^{2} x_{ijk} \le m_{ij}$ i=1,2; j=1,2,3

and $x_{ijk}, t_{ik}, U_{ik} \ge 0.$

Just as the presence of multi-time periods increased the number of constraints, the occurence of multi-locations has doubled the number of constraints as there were two plants. In addition, as there are two products the number of constraints was doubled yet again.

The most interesting feature of theis model is the undefined variables $x_{32}k$ (k=l,2). This is sometimes dealt with by assigning a very large value to the associated costs $^{C}23k$ (k $^{=}$ 1-2). Thus as this is a minimization problem, when solved x_32k will have zero values as it will not be cost effective to use this production, it is also possible to introduce constraints, i.e. x_{32k} = 0 (k = 1,2) can be added to the list of restrictions. However both these methods increase the size of the model and are therefore not preferable. This problem shall be discussed further in a later review of software tools. Undefined variables often occur in network models.

5. NETWORK AND GENERALIZED FLOW MODELS

A network is a system of arcs connecting different points. Network flow problems are concerned with sending some commodity from certain supply points (sources) to some demand points (sinks). An example of a network flow problem is the transportation problem such as example 2.6. Many network flow problems can be formulated as linear programs. Three special cases which are considered here are the transportation, transshipment and assignment problems.

5.1 The Transportation Problem

Transportation problems usually involve the transportation of a product between several sources and sinks at minimum cost. A transportation problem may be described as follows.

Suppose that there are m sources and n sinks for a particular product. Let the m sources have supplies a_1, $a_2 a_m$ available and the n sinks have demands b_1, b_2,..., b_n for the product. Let the unit cost of transportation from source i to sink j be c_{ij}. the objective is to find an optimal transportation schedule which maximizes the total transportation cost.

In declarative form the model may be stated as follows:

Subscripts, ranges

i = 1,2,..., m denotes sources
j = 1,2,..., n denotes sinks

Variables

x_{ij} quantity of product transported from source i to sink j;

Coefficients

c_{ij} unit cost of transportation from source i to sink j;

a_i the quantity of product available for transportation from source i;

b_j the demand for product at sink j;

Linear Constraint Relations

Minimize cost = $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

subject to:

(available supply at source i) $\sum_{j=1}^{n} x_{ij} \le a_i; \quad i = 1,2,\ldots, m$

(demand at sink j) $\sum_{i=1}^{n} x_{ij} \ge b_j; \quad j = 1,2,\ldots, n$

$x_{ij} \ge 0; \; i = 1,2 \ldots\ldots m; \; j=1,2,\ldots,n$

In the above model, the demand for the product can only be satisfied if the total available supply is greater than or equal to the total demand. If the total available supply equals the total demand, ie. $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ then the inequality relations become strict equalities and the model is balanced. Example 2.6 in section two is an example of a balanced transportation problem. Transportation models like this may arise in practice in a variety of contexts. For example, multi-time period production models may be formulated as transportation models (eg *Hillier & Lieberman, 1986, p!89f* and *Beale, 1986, p34f).*

Transportation models have a special structure: all the coefficients of the variables in the constraint rows are either zero or one and each column contains at most two non-zero coefficients. This means that the multiplications (or divisions) in the simplex algorithm are reduced to additions (or subtractions). Also this special structure guarantees that if all the coefficients a_i and b_j are integer then all the variables in the optimum solution will also be integer. This property can be exploited when integer valuws are required in the solution without the need for Intger Programming (IP).

In addition, specialist algorithms which are designed to take advantage of this structure can be used to solve balanced transportation problems. These specialist algorithms are more efficient than the simplex method. Thus it is often desirable to formulate unbalanced transportation problems as balanced problems. This is easily performed as follows.

Consider the unbalanced transportation problem where

$$\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$$

To balance the problem formulation, a dummy sink j = n+1 is created with a demand b_{n+1} equal to the excess supply,

ie
$$b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j..$$

The associated unit transportation costs ci,n+l (i = 1,2,...,m) are assumed to be zero. The unbalanced transportation problem may now be stated as:

Minimize cost = $\sum_{i=1}^{m} \sum_{j=1}^{n+1} c_{ij}$

Subject to:
$$\sum_{j=1}^{n+1} x_{ij} = a_i; \qquad i=1,2,\ldots,m$$

$$\sum_{i=1}^{m} x_{ij} = b_i; \qquad i=1,2,\ldots,n+1$$

$$x_{ij} \geq 0 \qquad i=1,2,\ldots m;\ j=1,2,\ldots,n+1$$

where j=n+l is the dummy sink with demand

$$b_{n+1} = \sum_{i=1}^{m} a_j - \sum_{j=1}^{n} b_j$$

and $c_{i,n+1}$ = 0 (i=l,2,...,m) and the variable $x_{i,n+1}$ denotes the excess supply at source i.

An unbalanced transportation problem where the total demand is greater than the total supply is obviously infeasible. However, a transportation schedule may be still required which will supply as much as possible to the sinks. This can be dealt with by creating a dummy source i=m+l to supply the shortage. Thus $x_{m+1,j}$ denotes the shortage at sink j and i = m+1 denotes the dummy source with supply

$$a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i \quad \text{and}\, c_{m+1,j} = 0 \quad (j=1,2,\ldots n).$$

Transportation models arise frequently in practice in a variety of contexts. Sometimes it is desirable to reformulate other models (when appropriate) as transportation models to exploit this special structure.

5.2 The Transshipment Problem

The transshipment problem is a generalization of the transportation problem. In the transportation problem there are sources and sinks and it is assumed that the product concerned is sent directly from the sources to the sinks. For example, a product being sent to sink 2 does not pass through sink 1 first. The transshipment problem relaxes this and permits transportation through intermediate transshipment points (which may be sources or sinks). Thus solving the transshipment problem not only includes finding the optimum amount to ship from each source to each sink but also includes deciding the route for each shipment. However, the transshipment problem may be reformulated as a transportation problem *(Orden, 1956)* and may thus be solved by the same specialist algorithms. This conversion is also of interest as it ensures no difficulty in finding an optimal solution with integer-valued variables.

There are sveral ways to convert a transshipment problem into a transportation problem *(Dantzig, 1972, p336f; Winston, 1987, p300f; Hadley, 1980, p369f)*. The method illustrated here shows the transshipment quantities explicitly and does not involve finding least cost routes between sources and sinks. It is also suitable for handling problems which have fixed capacities on certain flows.

Conversion is performed by considering each transshipment point as firstly a source and then as a sink. When considered as a source, the supply of a transshipment point is set equal to the total quantities available (as it certainly cannot exceed this) plus any supply in the original data. Thus in order to maintain the balance of flow at each transshipment node, this quantitiy is also added to the original demand.. Fictitious shipments xii are included to enable the conversion and in solution these values are ignored *(Wagner, 1975, p178f)*.

Example 5.2

A company wishes to redistribute stock between its eight stores. Some stores have excess stock and some have a demand. Figure 6.2a below shows the possible routes for the transportation of stock. The numbered nodes represent the eight stores and the values next to each node represent the stock available for redistribution. Obviously, a negative value indicates a requirement.

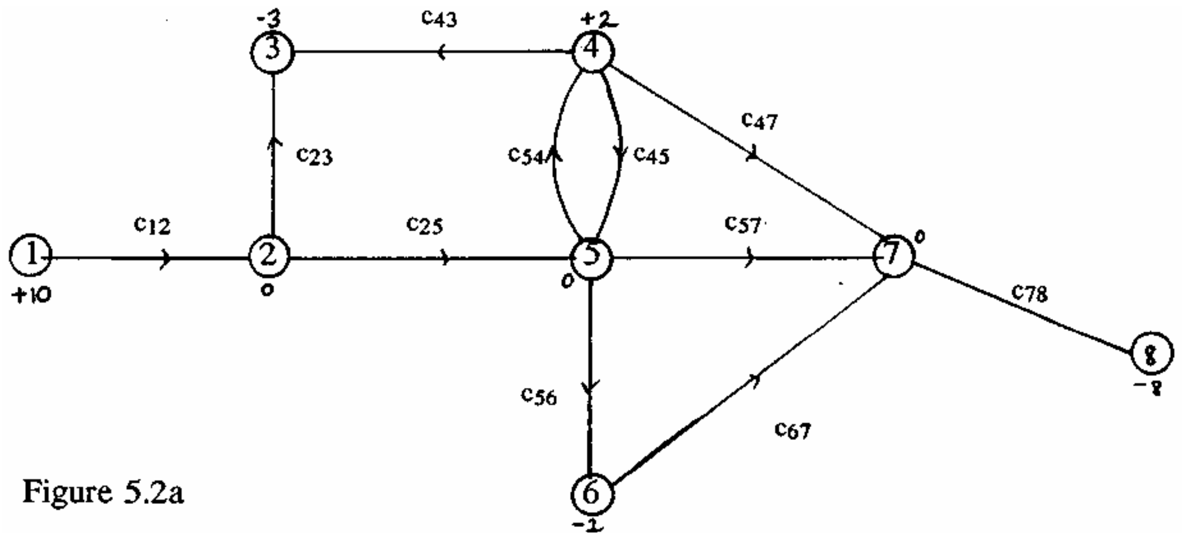


Figure 5.2a

Stores 2, 4, 5, 6 and 7 are transshipment points as stock may be shipped through them to other stores. The other stores are either sources and sinks: store 1 is a source and stores 3 and 8 are sinks.

Each possible shipping route xij will have an associated unit cost cij $\geq$ 0. The company's objective is to redistribute stock at minimum cost.

The formulation for this transshipment problem is as follows:

$$\text{Minimize } c_{12}x_{12} + c_{23}x_{23} + c_{25}x_{25} + c_{43}x_{43} + c_{45}x_{45} + c_{47}x_{47} + c_{54}x_{54} + c_{56}x_{56} + c_{67}x_{67} + c_{78}x_{78}$$

subject to:

$$
\begin{aligned}
x_{12} &= 10 \\
-x_{12} + x_{23} + x_{25} &= 0 \\
-x_{23} - x_{43} &= -3 \\
x_{43} + x_{45} + x_{47} - x_{54} &= 2 \\
-x_{25} - x_{45} + x_{54} + x_{56} &= 0 \\
-x_{56} + x_{67} &= -1 \\
-x_{47} - x_{67} + x_{78} &= 0 \\
x_{78} &= 8
\end{aligned}
$$

Conversion to a transportation model is facilitated by the construction of the following table:

	2	(sink) 3	4	5	6	7	(sink) 8	SUPPLY
(source) 1	x_{12}							10
2	x_{22}	x_{23}		x_{25}				12
4		x_{43}	x_{44}	x_{45}		x_{47}		14
5			x_{54}	x_{55}	x_{56}			12
6					x_{66}	x_{67}		11
7						x_{77}	x_{78}	12
DEMAND	12	3	12	12	12	12	8	

The table shows the possible shipment quantities xij which each have an associated cost c_{ij}. Shipments from a point to itself (x_{ij}) have zero transportation costs and are only included to facilitate the conversion.

There is a row for each node that ships out items (nodes 3 and 8 are sinks so these are excluded from the rows) and there is a column for each node that receives items (node 1 is a source and therefore is excluded from the columns).

The supply for source 1 is ten and the demand for sinks 3 and 8 are three and eight respectively. This follows from the original data. The supply and demand quantities for the transshipment points have been increased by twelve. This number being the total quantity of stock available for redistribution (ten items at store 1 and two items at store 4. Thus for each transshipment point the difference between the supply and demand equals the quantity available for redistribution in the original data *(op. cit. p!76f)*.

The transshipment problem has now been converted to the following transportation problem:

Minimize $c_{12}x_{12} + c_{23}x_{23} + c_{25}x_{25} + c_{43}x_{43} + c_{45}x_{45} + c_{47}x_{47} + c_{54}x_{54} + c_{56}x_{56} + c_{67}x_{67}$

$+ c_{78}x_{78}$

subject to:

$$
\begin{aligned}
x_{12} &= 10 \\
x_{22} + x_{23} + x_{25} &= 12 \\
x_{43} + x_{44} + x_{45} + x_{47} &= 14 \\
x_{54} \quad x_{55} + x_{56} &= 12 \\
x_{66} + x_{67} &= 11 \\
x_{77} + x_{78} &= 12 \\
x_{12} + x_{22} &= 12 \\
x_{23} + x_{43} &= 14 \\
x_{44} + x_{54} &= 12 \\
x_{25} + x_{45} + x_{55} &= 12 \\
x_{56} + x_{66} &= 12 \\
x_{47} + x_{67} + x_{77} &= 12 \\
x_{78} &= 8
\end{aligned}
$$

$x_{ij} > 0$ for all defined x_{ij}.

In the optimum solution all X_{II} values are ignored.

5.3 The Assignment Problem

The assignment problem is a special case of the transportation problem. It is concerned with assigning n tasks to n agents at minimum cost. In declarative form, the assignment problem may be expressed as:

Subscripts and Ranges

$i=1,2 \ldots\ldots n$ denotes tasks

$j=1,2 \ldots\ldots n$ denotes agents

Variables

$$\mathbf{x}_{ij} = \begin{cases} 1 & \text{if task i is performed by agent j} \\ 0 & \text{otherwise} \end{cases}$$

Coefficients

C_{ij} cost of task i being accomplished by agent j

Linear Constraint Relations

Minimize cost = $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

subject to:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1,2,\ldots,n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1,2,\ldots,n$$

$$x_{ij} = 0 \text{ or } 1 \qquad i = 1,2,\ldots,n;\ j = 1,2,\ldots,n.$$

The first set of linear constraints ($\sum_{j=1}^{n} x_{ij} = 1$ for all i) ensures that only one agent is assigned to task i and the second set of constraints ($\sum_{i=1}^{n} x_{ij} = 1$ for all j) ensures that only one task is undertaken by each agent. The decision variable x_{ij} are restricted to be zero or one. Therefore, this problem is an integer program. However, by comparing this model with the transportation model it is clear that the assignment problem is a special case of the transportation model where a_i and b_j are both equal to one, m=n and $x_{ij} \geq 0$. Thus, as a_i and b_j are integer, the solution of the assignment problem is guaranteed to have integer valued variables and the constraints ensure that these values are either zero or one. The specialist algorithms for the transportation models may be employed to solve assignment problems. However, as $a_i = b_i = 1$, these algorithms may be streamlined further for still more efficient solution procedures.

5.4 Generalized Network Flow Model

In the network models previously reviewed, it was assumed that the total flow out of a point is equal to the total flow into that point. In other words there is no gain or loss. The generalized network flow problem does not have this assumption and may be expressed as

Minimize $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

subject to

$$\sum^{n} x_{ij} \leq a_i \qquad i = 1,2,\ldots,m$$

$$J = 1$$

$$\sum_{i=1}^{n} d_{ij}\, x_{ij} \;\geq\; b_j \qquad j = 1,2,\ldots,n$$

$$x_{ij} \;\geq\; 0 \qquad 1 = 1,2,...m;\; j = 1,2,...,n.$$

A unit of flow from point i becomes $d_{ij} > 0$ on arrivl at point j. Thus if $d_{ij} < 1$ there is a loss. Hence the generalized network flow model is sometimes referred to a **network model with gains.** The LP solution to a generalized network flow model is not guaranteed to have integer valued variables and if it is required that flows are integer then **IP** techniques are necessary. Nevertheless the special structure of these models can be exploited in the algorithms used.

Network models with gains occur in many practical applications. For example, where ther is wastage, interest rates or any situation which does not assume a conservation of flow.

The network models reviewed so far in this section may be summarized by the following diagram.

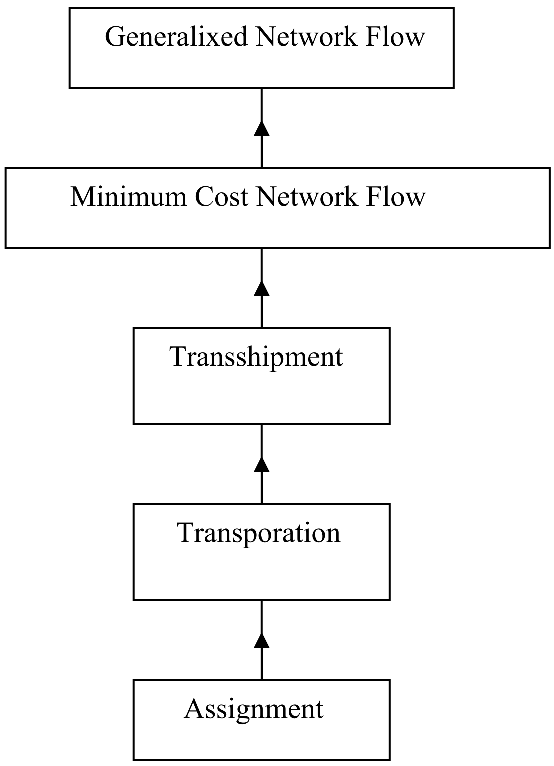


Figure 5.4

Other variations of network problems include **shortest path problems** in which each arc has an associated length. Longest path problems occur in **critical path analysis.** Both these types of network models can be considered as minimum cost flow models by assigning the source a supply of one unit and the sink a demand of one unit. Another important network problem is the **maximal flow problem** which involves finding the maximal flow from a source to a sink. In this problem, the arcs may have upper bounds on the amount of flow able to pass through.

As stated in this section, network flow models have a special structure which, if the right hand side coefficients are integer, brings about integer variable values in the optimum solution. This property can save much time an effort, if an integer solution is required, as it avoids difficulties and computational costs encountered in IP.

In addition, this special structure has been exploited to obtain specialist algorithms which are more efficient for solving such problems. TOherefore it is often desirable to remodel other suitable problems into this form so as to utilize these specialized solution techniques. Not all LP models can be converted to network flow models. Williams describes a method for conversion and a way of showing such conversion to be impossible *(Williams, 1990, p96f).*

A particularly difficult network problem which cannot be solved with **LP** techniques is the **travelling salesman problem** which involves finding the minimum cost route around a set of locations.

6. FINAL COMMENTS

In this review basic prototype models have been briefly introduced with illustrative examples. As stated previously, in reality, most of these models form only part of a problem. It is usual for real life problems to incorporate several of these models.

A proper method for model formulation has also been described. In order to obtain a declarative mathematical statement of an LP problem, preliminary analysis of the problems often required. A diagram is often useful in assisting the modeller to visualize the problem and to consider the relevant information. The modeller has to then identify model entities, variables and constraints. It is then possible to obtain a mathematical statement of the problem which defines subscripts and ranges, variables, coefficients and constraints as described previously. It is useful if not essential to use mnemonic names for variables wherever possible, especially for larger models.

Having modelled the problem the next step is obviously to solve it obtaining the optimum solution which satisfies all the constraints. For most **LP** problems the Simplex algorithm, or a streamlined version of it, is employed. In order to utilize the computer optimizers the models have to be in a certain format. The standard format is known as **MPS** (Mathematical Programming System) format. Unfortunately this format is not easy to read as it consists of a rows section which lists the rows of the model and a columns section which lists the non-zero coefficients of the model column by column. Once the model has been converted to **MPS** format it can be presented to the optimizer as a file. There are many software tools available which assist with the modelling and solution of **LP** problems. Some of these will be discussed in a later review of software tools.

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