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INTERPOLATION REMAINDER THEORY FROM TAYLOR EXPANSIONS WITH NON - RECTANGULAR DOMAINS OF INFLUENCE

by

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ABSTRACT

Sobolev norm error bounds are derived for interpolation remainders on triangles using two types of Taylor expansion. These bounds are applied to the finite element analysis of Poisson's equation on a triangulation of a polygonal region.

1. INTRODUCTION

In this paper we consider two methods of calculating Sobolev norm bounds for linear functionals defined on a triangle. These are based on a rectilinear Taylor expansion and a directional derivative Taylor expansion, respectively. The first is a generalisation of the Sard Taylor expansion in $\underset{=}{B}_{1,1}$ for rectangles [8]. The second is a special case of the Frechet derivative expansions of Ciarlet and Wagschal [5] and Ciarlet and Raviart [6].

We apply these methods to the finite element analysis of Poisson's equation on a polygonal region Ω. The energy pseudonorm $\|v\|_{A(\Omega)}$ is defined by

$$\|v\|^2_{A(\Omega)} = \iint_{\Omega} \left[v^2_{1,0} + v^2_{0,1} \right] dx\, dy \tag{1.1}$$

If u is the weak solution of Poisson's equation, then the Galerkin approximation U to u satisfies the beat approximation property

$$\| u - U \|_{A(\Omega)} \leq \| u - \tilde{u} \|_{A(\Omega)} \tag{1.2}$$

for all $\tilde{u}$ in the same subset of the Sobolev space $W^1_2(\Omega)$ as U. (See, for example, Barnhill and Gregory [1].) For a triangulation $\Omega = \bigcup_i T_i$,

$$\| u - U \|_{A(\Omega)} = \{ \sum_i \| u - U \|^2_{A(T_i)} \}^{\frac{1}{2}} \quad . \tag{1.3}$$

If $\tilde{u}$ Is an interpolant, Inequality (1.2) is the relationship between interpolation remainder theory and the Galerkin method. We let $\tilde{u}$ be the piecewise linear interpolant that interpolates to u at the vertices of the sub triangles and, in particular on the triangle T with vertices at (0,0), (1,0) and (0,1). A change of variable to similar triangles, indicating the order of convergence, is given in the Appendix.

The two methods can be generalized to handle higher order elliptic operators, different regions, and other interpolants.

2. RECTILINEAR TAYLOR EXPANSION IN A TRJANGLE.

Let Ω be a rectangle with $(a,b) \in \Omega$. For u in the Sard space $\underset{=}{B}_{1,1}(\Omega)$ and $(x,y) \in \Omega$, the Sard Taylor expansion is the following:

$$u(x,y) = u(a,b) + (x-a)\, u_{1,0}(a,b) + (y-b)\, u_{0,1}(a,b)$$
$$+ \int_a^x (x-\tilde{x}) u_{2,0}(\tilde{x},b)\, d\tilde{x} + \int_b^y \int_a^x u_{1,1}(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y}$$
$$+ \int_b^y (y-\tilde{y}) u_{0,2}(a,\tilde{y})\, d\tilde{y} \quad . \tag{2.1}$$

Let L be a linear operator that involves only point evaluations of a function and its first partial derivatives, that is,

$$Lu = \sum_{k,\ell} \alpha_{k,\ell}(x,y)\, u(x_k, y_\ell) + \sum_{k,\ell} \beta_{k,\ell}(x,y)\, u_{1,0}(x_k, y_\ell)$$

$$+ \sum_{k,\ell} y_{k,\ell}(x,y) \quad u_{0,1}(x_k, y_\ell) \quad . \tag{2.2}$$

The remainder is R = I - L, where I is the identity operator, that is,

$$R\,u\,(x,y) = u\,(x,y) - L\,u\,(x,y)\;. \tag{2.3}$$

Also let

$$R_{i,j}\,u = \frac{\partial^{i+j}}{\partial y^j\, \partial\, x^i}\, R\,u = u_{i,j}(x,y) - L_{i,j}\, u(x,y)\;,\quad 0 \le i + j \le 1, \tag{2.4}$$

where $L_{i,j}\,u = \dfrac{\partial^{i+j}}{\partial y^j\, \partial\, x^i}\, Lu$.

Then $R_{i,j}$ can be applied to (2.1) and, if $R_{i,j}$ has linear polynomial precision, upper bounds can be found for $|R_{i,j}[u(x,y)]|$ in terms of norms of the derivatives $u_{2,0}(\tilde{x},b)$, $u_{1,1}(\tilde{x},\tilde{y})$, and $u_{0,2}(a,\tilde{y})$ [1 ,2], for example, in terms of

$\| u_{2,0}(\tilde{x},b) \|_{L_2(\tilde{x})}$, $\| u_{1,1,}(\tilde{x},\tilde{y}) \|_{L_2(\tilde{x},\tilde{y})}$, and

$\| u_{0,2}(a,\tilde{y}) \|_{L_2(\tilde{y})}$. (The notation $L_2(\tilde{x})$ means the L_2 norm with respect to the variable $\tilde{x}$ over its domain of definition etc.) Since the norms of $u_{2,0}(\tilde{x},b)$ and $u_{0,2}(a,\tilde{y})$ are taken only with respect to $\tilde{x}$ and $\tilde{y}$ respectively, this upper bound is not the Sobolev norm $\overset{o}{W}{}_2^2(\Omega)$ defined by

$$\| u \|_{\overset{o}{W}{}_2^2(\Omega)} = \Big\{ \sum_{|\alpha|=2} \| D^{\alpha} u \|^2_{L_2(\Omega)} \Big\}^{\frac{1}{2}} , \tag{2.5}$$

where $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha| = \alpha_1 + \alpha_2$. For some applications, such as finite element analysis, the Sobolev norm is preferable to the Sard norm.

Birkhoff, Schultz and Varga [3] used the Sard kernel theorem instead of the Taylor expansion, and also they evaluated $R_{i,j}[u(x,y)]$ at $(x,y) = (a,b)$. Then they varied (a,b) over the rectangle to produce a Sobolev norm in the variables a and b. However, this method is restricted to rectangles as we now explain. We consider a region Ω that is not necessarily a rectangle. For each point (x_k, y_ℓ) in (2.2), the method of Birkhoff, Schultz and Varga yields error bounds that involve the values of derivatives in the rectangle with opposite corners at (x_k, y_ℓ) and (a,b), where (a,b) varies over Ω For most common bivariate interpolants this amounts to the assumption that 0 be a rectangle.

Thus (a,b) can either be fixed to produce univariate integrals or, for a rectangle, (a,b) can be varied over the rectangle to produce bivariate integrals and a Sobolev norm. We consider the second possibility over a non-rectangular region, namely, the triangle T with vertices at (0,0), (1,0) and (0,1).

<u>Theorem 2.1</u>. Let $(a,b) \in T$ and let A_1, A_2 and A_3 be subsets of T as follows:

$A_2 = \{(x,y) \in T : x > 1 - b\}$, $A_3 = \{(x,y) \in T : y > 1 - a\}$, and $A = T - (A_1 \cup A_2)$. (See Figure 1).

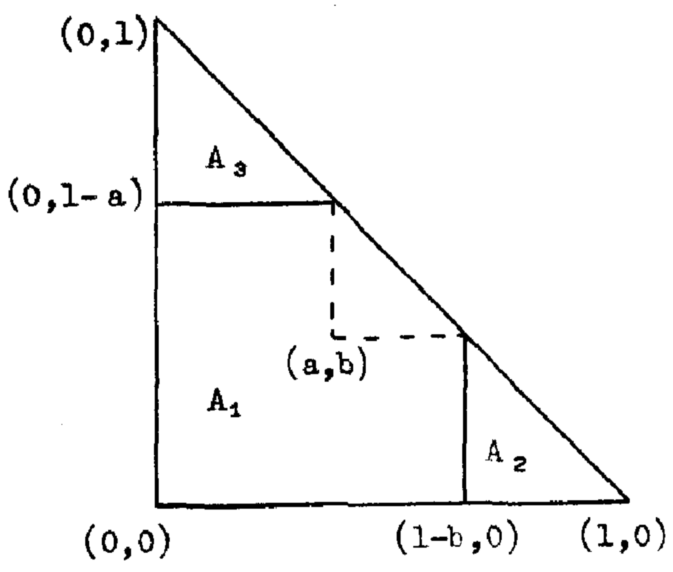

Figure 1.

Let X_{A_i} (x,y) denote the characteristic function

$$X_{A_i}(x,y) \;=\; \begin{cases} 1, & (x,y) \in A_i \\ 0, & \text{otherwise} \end{cases} . \qquad \text{Then} \quad (x,y) \in T \ \text{ implies}$$

that

$$u(x,y) = u(a,b) + (x-a)\, u_{1,0}(a,b) + (y-b)\, u_{0,1}(a,b)$$

$$+ \, x_{A_1}(x,y) \Big[\int_a^x (x-\tilde{x})\, u_{2,0}(\tilde{x},b)\, d\tilde{x} + \int_b^y \int_a^x u_{1,1}(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y}$$

$$+ \int_b^y (y-\tilde{y})\, u_{0,2}(a,\tilde{y})\, d\tilde{y} \Big]$$

$$+ \, x_{A_2}(x,y) \Big[\int_b^y (y-\tilde{y})\, u_{0,2}(a,\tilde{y})\, d\tilde{y} + \int_a^x \int_a^{\tilde{x}} u_{2,0}(x', 1-\tilde{x})\, dx'\, d\tilde{x}$$

$$+ \int_a^x \int_b^{1-\tilde{x}} u_{1,1}(a,y^*)\, dy^*\, d\tilde{x} + \int_a^x \int_{1-\tilde{x}}^y u_{1,1}(\tilde{x},\tilde{y})\, d\tilde{y}\, d\tilde{x} \Big]$$

$$+ \, x_{A_3}(x,y) \Big[\int_a^x (x,\tilde{x})\, u_{2,0}(\tilde{x},b)\, d\tilde{x} + \int_b^y \int_a^{1-\tilde{y}} u_{1,1}(\tilde{x},b)\, d\tilde{x}\, d\tilde{y}$$

$$+ \int_b^y \int_{1-\tilde{y}}^x u_{1,1}(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} + \int_b^y \int_b^{\tilde{y}} u_{0,2}(1-\tilde{y},y')\, dy'\, d\tilde{y} \Big] \qquad (2.6)$$

We assume that the derivatives in (2.6) exist, in the generalized sense.

Proor: For $(x,y) \in A_1$, (2.6) is the usual Sard Taylor expansion (2.1). For $(x,y) \in A_2$,

$$u(x,y) = u(a,y) + \int_a^x u_{1,0} (\tilde{x},y)\, d\tilde{x} ,$$

$$u(a,y) = u(a,b) + (y-b)\, u_{0,1}(a,b) + \int_b^y (y-\tilde{y})\, u_{0,2}(a,\tilde{y})\, d\tilde{y} .$$

We make the following expansions (compare with Figure 2):

$$u_{1,0}(\tilde{x},y) = u_{1,0}(\tilde{x}, 1-\tilde{x}) + \int_{1-\tilde{x}}^y u_{1,1}(\tilde{x},\tilde{y})\, d\tilde{y}$$

$$u_{1,0}(\tilde{x},1-\tilde{x}) = u_{1,0}(a, 1-\tilde{x}) + \int_a^{\tilde{x}} u_{2,0}(x',1-\tilde{x})\, dx'$$

$$u_{1,0}(a,1-\tilde{x}) = u_{1,0}(a,b) + \int_b^{1-\tilde{x}} u_{1,1}(a,y^*)\, dy^*.$$

Hence

$$u(x,y) = u(a,b) + (x-a)\, u_{1,0}(a,b) + (y-b)\, u_{0,1}(a.b)$$

$$+ \int_b^y (y-\tilde{y})\, u_{0,2}(a,\tilde{y})\, d\tilde{y} + \int_a^x \int_{1-\tilde{x}}^y u_{1,1}(\tilde{x},\tilde{y})\, d\tilde{y}\, d\tilde{x}$$

$$+ \int_a^x \int_a^{\tilde{x}} u_{0,2}(x',1-,\tilde{x})\, dx'\, d\tilde{x} + \int_a^x \int_b^{1-\tilde{x}} u_{1,1}(a,y^*)\, dy^*\, d\tilde{x} .$$

The proof for $(x,y) \in A_3$, is dual. Q.E.D.

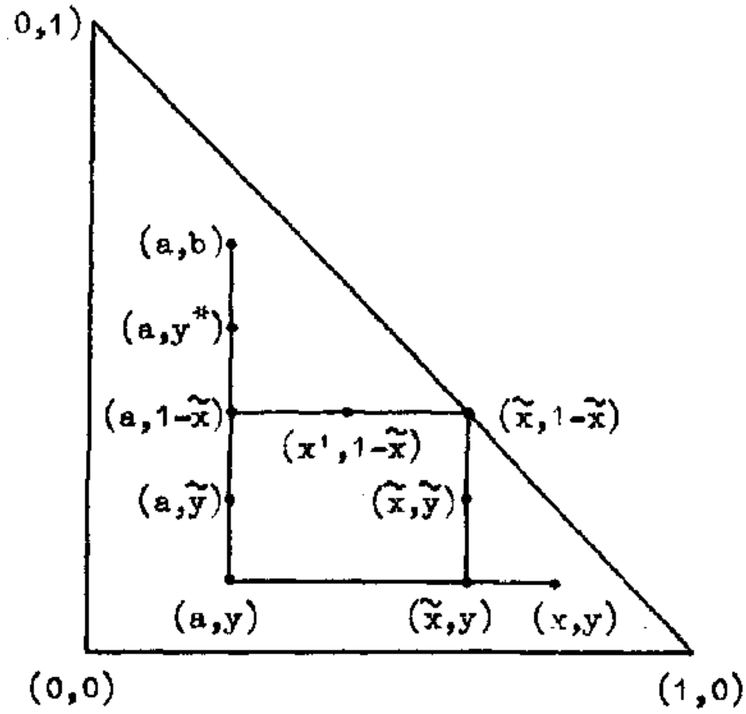

Figure 2.

LINEAR INTERPOLATION ON THE TRIANGLE

In this section, the Taylor expansion (2.6) is used to derive a bound on the linear interpolant over the triangle T, in the energy pseudonorm (1.1).

The remainder for linear interpolation on T evaluated at (a,b) is

$$R\, u(a,b) = u(a,b) - [u(0,0)(1-a-b) - u(1,0)a - u\,(0,1)b] \tag{2.7}$$

Hence

$$R_{1,0}\, u(a,b) = u_{1,0}\,(a,b) + u(0,0) - u(1,0) \tag{2.8}$$

$$R_{0,1}\, u(a,b) = u_{0,1}\,(a,b) + u(0,0) - u(0,1) \tag{2.9}$$

$R_{1,0}$ and $R_{0,1}$ are of finite element interest because of (1.2) and (1.1) and with $\tilde{u}$ as the above linear interpolant we have

$$||u - \tilde{u}||_{A(T)} = \left\{ ||R_{1,0}\, u\, ||^2_{L_2(T)} + ||R_{0,1}\, u\, ||^2_{L_2(T)} \right\}^{\frac{1}{2}} \quad . \tag{2.10}$$

The application of $R_{1,0}$ to (2.6) yields

$$\begin{aligned} R_{1,0}\, u(a,b) = {} & \int_a^0 \tilde{x}\, u_{2,0}(\tilde{x},b)\,d\tilde{x} + \int_a^0 \int_b^0 u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x} \\ & - \int_a^1 \int_a^{\tilde{x}} u_{2,0}(x', 1-\tilde{x})\,dx' - \int_a^1 \int_b^{1-\tilde{x}} u_{1,1}(a,\tilde{y})\,d\tilde{y}\,d\tilde{x} \\ & - \int_a^1 \int_{1-\tilde{x}}^0 u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x} \quad . \end{aligned} \tag{2.11}$$

Thus

$$\begin{aligned} ||R_{1,0}\, u(a,b)\,||_{L_2(a,b)} \leq {} & ||\int_0^a \tilde{x}\, u_{2,0}(\tilde{x},b)\; d\tilde{x}\; ||_{L_2(a,b)} \\ & + ||\int_0^a \int_0^b u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{y}\;d\tilde{x}\;||_{L_2(a,b)} + ||\int_a^1 \int_a^{\tilde{x}} u_{2,0}(x', 1-\tilde{x})\,dx'\,d\tilde{x}\,||_{L_2(a,b)} \\ & + ||\int_a^1 \int_b^{1-\tilde{x}} u_{1,1}(a,\tilde{y})\,d\tilde{y}\;d\tilde{x}\;||_{L_2(a,b)} + ||\int_a^1 \int_{1-\tilde{x}}^0 u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{y}\;d\tilde{x}\,||_{L_2(a,b)} \end{aligned} \tag{2.12}$$

The notation $L_2(a,b)$ means the L_2 norm with respect to the variables (a,b) ever the triangle T. We next compute upper bounds for each of the five norms on the right hand side of (2.11).

1st term:

$$\| \int_0^a \tilde{x}\, u_{2,0}(\tilde{x},b)\, d\tilde{x} \|^2_{L_2(a,b)} = \int_0^1 \int_0^{1-a} \{\int_0^a \tilde{x}\, u_{2,0}(\tilde{x},b)\, d\tilde{x}\}^2 \, db \, da$$

$$\leq \int_0^1 \int_0^{1-a} \left\{\int_0^a \tilde{x}^p d\tilde{x}\right\}^{2/p} \left\{\int_0^a u_{2,0}(\tilde{x},b)^{p'} d\tilde{x}\right\}^{2/p'} db \, da$$

$$\leq \int_0^1 \left(\frac{a^{p-1}}{p+1}\right)^{2/p} (1-a)^{1/r} \left[\left\{\int_0^a |u_{2,0}(\tilde{x},b)^{p'} d\tilde{x}\right\}^{2r'/p'} db\right]^{1/r'} da,$$

where $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{r} + \frac{1}{r'}$.

Let $r' = p'/2$, which implies $r = p/(2-p)$ and hence $1 \leq p \leq 2$ and $p' \geq 2$.

Then the above is bounded by

$$\| u_{2,0}(\tilde{x},b) \|^2_{Lp'(\tilde{x},b)} \frac{1}{(p+1)^{2/p}} \int_0^1 a^{2+\frac{2}{p}} (1-a)^{\frac{2}{p}-1} da \, . \tag{2.13}$$

The univariate integral in (2.13) equals $B(3+\frac{2}{p},\frac{2}{p})$, where $B(m,n)$, m, n > 0, is the beta function, which has the properties that if m and n are integers, then

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

and

$$B(m+\tfrac{1}{2},\ n+\tfrac{1}{2}) = \frac{\pi(m-\frac{1}{2})\ldots(\frac{1}{2})(n-\frac{1}{2})\ldots(\frac{1}{2})}{(m+n)!}$$

<u>2nd term :</u>

$$\left\| \int_0^a \int_0^n u_{1,1}(\tilde{x},\tilde{y})\, d\tilde{y}\, d\tilde{x} \right\|^2_{L_2(a,b)}$$

$$\leq \| u_{1,1}(\tilde{x},\tilde{y}) \|^2_{L_p.(\tilde{x},\tilde{y})} \left(\frac{p}{2+p}\right) B\left(\frac{2}{p}+1,\ \frac{2}{p}+2\right). \tag{2.14}$$

<u>3rd term :</u>

$$\left\| \int_a^1 \int_a^{\tilde{x}} u_{2,0}(x',1-\tilde{x})\, dx'\, d\tilde{x} \right\|^2_{L_2(a,b)}$$

$$\leq \| u_{2,0}(x',1-\tilde{x}) \|^2_{L_{p'},(x',1-\tilde{x})} \frac{1}{(2+\frac{4}{p})2^{2/p}} \quad . \tag{2.15}$$

4th term:

$$\| \int_a^1 \int_b^{1-\tilde{x}} u_{1,1}(a,\tilde{y})\, d\tilde{y}\, d\tilde{x} \|^2_{L_2(a,b)}$$

$$\leq \int_0^1 \int_0^{1-b} \left\{ \int_0^1 \int_0^{1-a} | u_{1,1}(a,\tilde{y}) |\, d\tilde{y}\, d\tilde{x} \right\}^2 da\, db$$

$$\leq \int_0^1 \int_0^{1-b} (1-a)^{2+\frac{2}{p}} \left\{ \int_0^{1-a} | u_{1,1}(a,\tilde{y}) |^{p'} d\tilde{y} \right\}^{\frac{2}{p'}} da\, db$$

$$\leq \| u_{1,1}(a,\tilde{y}) \|^2_{L_{p'}(a,\tilde{y})} \left(\frac{2-p}{4+p}\right)^{\frac{2-p}{p}} \int_0^1 \left(1 - b^{\frac{4+p}{2-p}}\right)^{\frac{2-p}{p}} db$$

The integral $\int_0^1 (1-x^{\alpha})^{\delta} dx$, with the change of variable

$x^{\alpha} = u$, equals $\frac{1}{\alpha} B(\delta+1, \frac{1}{\alpha})$. Hence the above is bounded by

$$\begin{cases} \left(\frac{2-p}{4+p}\right)^{\frac{2}{p}} \left[B\left(1+\frac{2}{p}, \frac{2-p}{4+p}\right) \right] \||\, u_{1,1}(a,\tilde{y}) \|^2_{L_2}(a,\tilde{y}), p' > 2 \\ \| u_{1,1}(a,\tilde{y}) \|_{L_2}(a,\tilde{y}), p' = 2. \end{cases} \qquad (2.16)$$

<u>5th term</u> :

$$\| \int_a^1 \int_{1-\tilde{x}}^0 u_{1,1}(\tilde{x},\tilde{y})\, d\tilde{y}\, d\tilde{x} \|^2_{L_2(a,b)}$$

$$\leq \quad \| u_{1,1}(\tilde{x},\tilde{y}) \|^2_{L_{p'}(\tilde{x},\tilde{y})} \frac{p}{p+2} \frac{1}{2^{\frac{2}{p}+1}} \quad . \tag{2.17}$$

$\| R_{0,1} u(a,b) \|_{L_2(a,b)}$ is dual to $\| R_{1,0} u(a,b) \|_{L_2(a,b)}$.

Therefore, from (2.12) - (2.17) with p = 2, we obtain the following norms over the triangle T :

$$\left\{ \|R_{1,0}\ u\|^2_{L_2(T)} + \|R_{0,1},\ u\|^2_{L_2(T)} \right\}^{\frac{1}{2}}$$

$$\leq \|R_{1,0}\ u\|_{L_2(T)} + \|R_{0,1}\ u\|^2_{L_2(T)}$$

$$\leq \frac{1}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right)\left[\|u_{2,0}\|_{L_2(T)} + \|u_{0,2}\|_{L_2(T)}\right]$$

$$+ 2\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + 1\right)\|u_{1,1}\|_{L_2(T)} \quad . \tag{2.18}$$

3. <u>DIRECTIONAL DERIVATIVE TAYLOR EXPANSION</u>

We consider Taylor expansions at the point (x,y) about the point (a,b) along the line between (x,y) and (a,b).

For example

$$u(x,y) = u(a,b) + (x-a)u_{1,0}(a,b) + (y-b)u_{0,1}(a,b)$$

$$+ \int_0^1 \Big[(x-a)^2 u_{2,0}[a+\theta(x-a), b+\theta(y-b)]$$

$$+ 2(x-a)(y-b)u_{1,1}\big[a+\theta(x-a), b+\theta(y-b)\big]$$

$$+ (y-b)^2 u_{0,2}\big[a+\theta(x-a), b+\theta(y-b)\big](1-\theta)\,d\theta. \qquad (3.1)$$

For remainders $R_{i,j}$ of the form (2.4) evaluated at (a,b), each point function evaluation and derivative evaluation is expanded in a. line Taylor expansion about (a,b). This is the same as applying $R_{i,j}$ to the Taylor expansion if $R_{i,j}$ does not contain derivatives at points other than (a,b). If $R_{i,j}$ has derivatives at points other than (a,b), the above is necessary since the direct application of $R_{i,j}$ to (3.1), for example, would involve derivatives of $u_{2,0}$, $u_{1,1}$ and $u_{0,2}$. This occurs because the variables x and y are not covered by integrals. This is in contrast to the rectilinear expansion in which the variables are

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* An example that involves derivative data functionals is the piecewise quintic interpolant, see for example Mitchell [7], which is useful for fourth order elliptic equations.

covered. The use of higher derivatives of u is undesirable because it increases the smoothness assumption.

The above method is applicable to regions which are star shaped with respect to the data functionals in the operators $L_{i,j}$ (2.4). That is, for each (a,b) in Ω the line from (a,b) to (x_k, y_ℓ) is in Ω .

LINEAR INTERPOLATION ON A TRIANGLE

A bound is now derived on the linear interpolant over the triangle T in the energy paeudonorm (1.1), c.f. Section 2.

A Taylor expansion of each term of the functional $R_{1,0}$ u(a,b), equation (2.8), gives the following :

$$R_{1,0}\ u(a,b) = \int_0^1 \{a^2\ u_{2,0}[a(1-\theta), b(1-\theta)] + 2ab\ u_{1,1}[a(1-\theta), b(1-\theta)]$$

$$+ b^2\ u_{0,2}[a(1-\theta), b(1-\theta)]\}\ (1-\theta)\ d\theta$$

$$-\int_0^1 \{(1-a)^2\ u_{2,0}[a+\theta(1-a), b(1-\theta)] - 2(1-a)b\ u_{1,1}[a(1-\theta), b(1-\theta)]$$

$$+ b^2\ u_{0,2}[a+\theta(1-a), b(1-\theta)]\}\ (1-\theta)\ d\theta \qquad (3.2)$$

(We note that the linear precision of the functional implies that all but the integral remainder terms to zero .)

A bound on $\|R_{1,0}\ u(a,b)\|_{L_2(a,b)}$ is obtained from (3.2) by the application of the triangle inequality to the right hand side.

The first term in the result is the following:

$$\left\| \int_0^1 a^2 \; u_{2,0}\,[a(1-\theta),\, b(1-\theta)](1-\theta)\, d\theta \right\|_{L_2(a,b)}$$

$$= \left[\int_0^1 \int_0^{1-b} \left| \int_0^1 a^2 u_{2,0}\,[a(1-\theta),\, b(1-\theta)](1-\theta)\, d\theta \right|^2 da \;\; db \right]^{\frac{1}{2}}$$

$$\leq \left[\int_0^1 \int_0^{1-b} a^4 \left\{ \int_0^1 \left| u_{2,0}\,[a(1-0),\, b(1-\theta)]\,(1-\theta) \right|^{p'} d\theta \right\}^{\frac{2}{p'}} da\; db \right]^{\frac{1}{2}}$$

$$= \left[\| a^4 \|_{L_{\frac{p}{2-p}}(a,b)} \left\{ \int_0^1 \int_0^1 \int_0^{1-b} \left| u_{2,0}\,[a(1-\theta), b(1-\theta)]\,(1-\theta) \right|^{p'} da \;\; db \;\; d\theta \right\}^{\frac{2}{p'}} \right]^{\frac{1}{2}}$$

$$\text{where} \quad \frac{1}{p} + \frac{1}{p'} = 1\,, \quad p' \geq 2.$$

$$\leq \| a^4 \|^{\frac{1}{2}}_{L_{\frac{p}{2-p}}(a,b)} \left\{ \int_0^1 \int_0^{1-\theta} \int_0^{(1-\theta)(1-\tilde{b})} \left| u_{2,0}(\tilde{a},\tilde{b}) \right|^{p'} d\tilde{a}\; d\tilde{b} \;\; (1-\theta)^{p'-2} \;\; d\theta \right\}^{1/p'}$$

$$\leq \| a^4 \|^{\frac{1}{2}}_{L_{\frac{p}{2-p}}(a,b)} \left(\frac{1}{p'-1} \right)^{\frac{1}{p'}} \| u_{2,0}(\tilde{a},\tilde{b}) \|_{L_{p'}(\tilde{a},\tilde{b})} \qquad (3.3)$$

Equation (3.3) together with a similar consideration of the other terms require the following results :

$$\alpha(r) = \| a^4 \|^{\frac{1}{2}}_{L_r(a,b)} = \| b^4 \|^{\frac{1}{2}}_{L_r(a,b)}$$

$$= \begin{cases} \left[\frac{1}{(4r+2)\ (4r+1)}\right]^{\frac{1}{2r}} & , \ r < \infty , \\ 1 \ , & r = \infty , \end{cases} \tag{3.4}$$

$$\beta(r) = \left\| \ 4a^2 \ b^2 \ \right\|^{\frac{1}{2}}_{L_r(a,b)} = \begin{cases} 2\left[\frac{B\,(2r + 2, 2r+1)}{2r + 1}\right]^{\frac{1}{2r}} & , r < \infty , \\ \frac{1}{2}, \quad r = \infty , \end{cases} \tag{3.5}$$

where B (m,n) is the beta function defined in Section 2,

$$y\,(r) = \left\| \ (1-a)^4 \ \right\|^{\frac{1}{2}}_{L_r(a,b)} = \begin{cases} \left[\frac{1}{4r + 2}\right]^{\frac{1}{2r}} & , \ r < \infty , \\ 1, \ r = \infty , \end{cases} \tag{3.6}$$

$$\delta\ (r) = \left\| \ 4(1-a)^2 \ b^2 \ \right\|^{\frac{1}{2}}_{L_r(a,b)} = \begin{cases} 2\left[\frac{1}{(4r + 2)\ (2r + 1)}\right]^{\frac{1}{2}} & , \ r < \infty \\ 2 \ , \ r = \infty . \end{cases} \tag{3.7}$$

We then obtain the following bound :

$$\left\| R_{1,0} \quad u\ (a,b) \right\|_{L_2\ (a,b)} \leq$$

$$\left(\frac{1}{p'-1}\right)^{\frac{1}{p'}} \left[\left\{ \alpha\left(\frac{p}{2-p}\right) + \gamma\left(\frac{p}{1-p}\right) \right\} \| u_{2,0}\ (\tilde{a},\tilde{b}) \|_{L_{p'}(\tilde{a},\tilde{b})} \right.$$

$$+ \left\{ \beta\left(\frac{p}{2-p}\right) + \delta\left(\frac{p}{2-p}\right) \right\} \| u_{1,1}\ (\tilde{a},\tilde{b}) \|_{L_{p'}\ (\tilde{a},\tilde{b})}$$

$$\left. + \quad 2\alpha \quad \left(\frac{p}{2-p}\right) \left\| \quad u_{0,2}\ (\tilde{a},\tilde{b}) \right\|_{L_{p'}\ (\tilde{a},\tilde{b})} \right] \tag{3.8}$$

The norm $\| R_{0,1}\ \ u\ (a,b) \|_{L_2}\ (a,b)$ is dual to $\| R_{0,1}\ \ u\ (a,b) \|_{L_2}\ (a,b)$

With p' = p = 2 we obtain the following bound :

$$\left\{ \| R_{0,1}\ \ u \|^2_{L_2(T)} + \| R_{0,1}\ \ u \|^2_{L_2\ (T)} \right\}^{\frac{1}{2}}$$

$$\leq \| R_{0,1}\ \ u \|_{L_2(T)} + \| R_{0,1}\ \ u \|_{L_2(T)}$$

$$\leq 4 \| u_{2,0} \|_{L_2(T)} + 5 \| u_{1,1} \|_{L_2(T)} + 4 \| u_{0,2} \|_{L_2(T)} \tag{3.9}$$

Appendix : Change of Variable Formula

Let $x^* = hx$, $y^* = hy$ be a change of variable from the triangle T with vertices at (0,0), (1,0) and (0,1) to the triangle T* with vertices at (0,0), (h,0) and (0,h).

Then

$$u^*(x^*, y^*) \equiv u\left(\frac{x^*}{h}, \frac{y^*}{h}\right)$$

defines the transformation of a function u(x,y) defined on T, to a function $u^*(x^*, y^*)$ define on T*.

Thus

$$u^*_{i,j}(x^*, y^*) \equiv \frac{1}{h^{i+j}} u_{i,j}\left(\frac{x^*}{h}, \frac{y^*}{h}\right)$$

and hence

$$\int_0^1 \int_0^{1-y} |u_{i,j}(x,y)|^p \, dx \, dy = \int_0^h \int_0^{h(1-y^*)} |h^{i+j} u^*_{i,j}(x^*,y^*)|^p \frac{1}{h^2} \, dx^* \, dy^*$$

We thus obtain the following change of variable formula :

$$\| u_{i,j}(x,y) \|_{L_p(T)} = h^{i+j-2/p} \| u^*_{i,j}(x^*, y^*) \|_{L_p(T^*)}$$

We note the dependence of the order of h on p.

A general change of variable for triangles is given in Bramble and Zlamal [4].

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