TR/101

A CUBIC SPLINE METHOD FOR THE SOLUTION OF A LINEAR FOURTH-ORDER TWO-POINT BOUNDARY VALUE PROBLEM

By

N. Papamichael and A. J. Worsey

W9260234

W9260234

ABSTRACT

A cubic spline method is described for the numerical solution of a two-point boundary value problem, involving a fourth order linear differential equation. This spline method is shown to be closely related to a known fourth order finite difference scheme.

1. Introduction

Consider the boundary value problem

$$y^{(4)}(x) + f(x)y(x) = g(x) \quad , \quad a<x<b \ , \tag{1.1}$$

$$y(a) = \alpha_0 \ , \quad y(b) = \alpha_1 \ , \tag{1.2}$$

$$y^{(1)}(a) = \beta_0 \ , \quad y^{(1)}(b) = \beta_1 \ , \tag{1.3}$$

where $\alpha_1., \beta_i.$; $i = 0,1$ are finite real constants and the functions f and g are continuous in [a,b].

In a recent paper Usmani [5] derives three finite difference schemes, of orders 2, 4 and 6 respectively, for the numerical solution of (1.1) - (1-3). In the present paper we describe a cubic spline method, similar to that proposed by Daniel and Swartz [2] for second order problems, and show that it is closely related to the fourth order finite difference method of [5], In particular, we show that the finite difference solution leads, with very little additional computational effort, to the construction of a cubic spline which gives order four approximations to the solution of (1.1) - (1.3) at any point of [a.b].

2. Preliminary Results

Let

$$x_i. = a + ih; \quad i = 0,1,..,k \quad , \quad h = (b-a)/k \ , \tag{2.1}$$

and assume that the function $y \varepsilon C^8[a,b]$ is given. Consider the problem of constructing a cubic spline S satisfying the interpolation conditions

$$S(x_i) = y_i \quad ; \quad i = 0,1,\ldots,k, \tag{2.2}$$

and the end conditions

$$\left.\begin{aligned} M_0 &= \frac{1}{h^2}\{\frac{-7}{2}y_0 + 4y_1 - \frac{1}{2}y_2 - 3hy_0^{(1)} \\ &\qquad + \frac{h^4}{840}(-24y_0^{(4)} + 107y_1^{(4)} - 16y_2^{(4)} + 3y_3^{(4)})\}, \\ M_k &= \frac{1}{h^2}\{\frac{-7}{2}y_k + 4y_{k-1} - \frac{1}{2}y_{k-2} + 3hy_k^{(1)} \\ &\qquad + \frac{h^4}{840}(-24y_k^{(4)} + 107y_{k-1}^{(4)} - 16y_{k-2}^{(4)} + 3y_{k-3}^{(4)})\}, \end{aligned}\right\} \tag{2.3}$$

where $y_i^{(j)} = y^{(j)}(x_i)$ and $M_i = S^{(2)}(x_i)$.

It is well known that the above interpolation problem has a unique solution S ; see e.g. Ahlberg et al. [1]. Furthermore, it follows easily from the results of Papamichael and Worsey [4] that the end conditions (2.3) satisfy the appropriate requirements of Lucas [3] so that

$$(M_{i-1} - 2M_i + M_{i+1,})/h^2 = y_i^{(4)} + 0(h^4); i = 1,2,\ldots k-1 \tag{2.4}$$

More specifically, the results of [4] show that (2.3) are the only end conditions of the form

$$\left.\begin{aligned} M_0 &= \frac{1}{h^2}\left\{\sum_{i=0}^{2} a_i y_i + hb_0 y_0^{(1)} + h^4 \sum_{i=0}^{3} e_i y_i^{(4)}\right\}, \\ M_k &= \frac{1}{h^2}\left\{\sum_{i=0}^{2} a_i y_{k-i} + hb_0 y_k^{(1)} + h^4 \sum_{i=0}^{3} e_i y_{k-i}^{(4)}\right\}, \end{aligned}\right\}$$

for which (2.4) holds.

3. Cubic Spline Method

Let $y \varepsilon C^8[a,b]$ be the solution of the boundary value problem (1.1) - (1.3). Then it follows at once, from (2.2) - (2.4), that the parameters $y_i = S(x_i)$ and $M_i = S^{(2)}(x_i)$ of the cubic spline of Section 2 satisfy the equations

$$M_0 = E_0\{y_{0,}y_{1,}y_2,y_3\}$$

$$= \frac{1}{h^2}\Big\{-\frac{7}{2}y_0 + 4y_1 - \frac{1}{2}y_2 - 3h\beta_0$$

$$+\frac{h^4}{840}(-24w_0 + 107w_1 - 16w_2 + 3w_3)\Big\}, \tag{3.1}$$

$$M_{i-1} - 2M_i + M_{i+1} + h^2 f_i y_i = h^2 g_i. + 0(h^6) \; ; \; i = 1,2,...,k\text{-}l, \tag{3.2}$$

$$M_k = E_k\{y_{k-3}, y_{k-2}, y_{k-1}, y_k\}$$

$$= \frac{1}{h^2}\Big\{-\frac{7}{2}y_K + 4y_{K+1} - \frac{1}{2}y_{K+2} - 3h\beta_1$$

$$+\frac{h^4}{840}(-24w_k + 107w_{k-1} - 16w_{k-2} + 3w_{k-3})\Big\}, \tag{3.3}$$

where, from (1.2),

$$y_0 = \alpha_0 \, , \, y_k = \alpha_1 \tag{3.4}$$

and, in (3.1) and (3.3),

$$w_i = g_i - f_i.y_i.$$

Naturally, the parameters of S also satisfy the consistency relations

$$M_{i-1} + 4M_i + M_{i+1} = 6(y_{i-1} - 2y_i + y_{i+1})/h^2 \quad ; \quad i = 1,2,...,k\text{-}1; \tag{3.5}$$

see e.g. Ahl berg et al. [1].

It follows easily, from (3.2) and (3.5), that

$$M_i = \{ 6y_{i-1} + (h^4 f_i - 12)y_i + 6y_{i+1} - h^4 g_i \} / 6h^2 + 0(h^6);$$

$$i = 1,2,\ldots,k-1, \qquad (3.6)$$

and hence, from (3.1)-(3.4), that the parameters y_i of S satisfy the matrix equation

$$(A + h_4 BF)y = \underline{c} + \underline{e} \qquad (3.7)$$

In (3.7) $\underline{y}$, $\underline{c}$, $\underline{e}$ are (k-1) dimensional column vectors and A, B, F are (k-1) x (k-1) matrices. These are respectively.

$$\underline{y} = (y_i), \ \underline{c} = (c_i), \ \underline{e} = (e_i),$$

where

$$\left.\begin{array}{l}
c_1 = \frac{11}{2}\alpha_0 + 3h\beta_0 + \frac{h^4}{280}\left\{ 8g_0 + 151g_1 + 52g_2 - g_3 - 8f_0\alpha_0 \right\} \\
c_{k-1} = \frac{11}{2}\alpha_1 + 3h\beta_1 + \frac{h^4}{280}\left\{ 8g_k + 151g_{k-1} + 52g_{k-2} - g_{k-3} - 8f_k\alpha_1 \right\} \\
c_2 = \frac{h^4}{6}\left\{ g_1 + 4g_2 + g_3 \right\} - \alpha_0, \ c_{k-2} = \frac{h^4}{6}\left\{ g_{k-3} + 4g_{k-2} + g_{k-1} \right\} - \alpha_1, \\
c_i = \frac{h^4}{6}\left\{ g_{i-1} + 4g_i + g_{i+1} \right\} \quad ; \quad i = 3,4,\ldots, \ k-3, \\
e_i = 0(h^8); \quad i = 1,2,\ldots, \ k-1,
\end{array}\right\}$$

$$A = \begin{bmatrix}
9 & -\frac{9}{2} & 1 & & & & & \\
-4 & 6 & -4 & 1 & & & & \\
1 & -4 & 6 & -4 & 1 & & & \\
 & . & . & . & . & . & & \\
 & & & 1 & -4 & 6 & -4 & 1 \\
 & & & & 1 & -4 & 6 & -4 \\
 & & & & & 1 & -\frac{9}{2} & 9
\end{bmatrix}
\quad
B = \begin{bmatrix}
\frac{151}{280} & \frac{13}{70} & -\frac{1}{280} & & & & \\
\frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & & \\
 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & \\
 & & . & . & . & & \\
 & & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \\
 & & & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\
 & & & & -\frac{1}{280} & \frac{13}{70} & \frac{151}{280}
\end{bmatrix}$$

$$and F = diag(f_i)$$

The above analysis shows that a cubic spline approximating the solution of (1.1) - (1.3) can be obtained, as in [2], by simply dropping the $0(h^6)$ terms from (3.2) and (3.6). Clearly the parameters $\tilde{Y}_i = S(x_i)$ and $\tilde{M}_i = \tilde{S}^{(2)}(x_i)$ of this spline may be determined from the equations

$$(A + h^4 BF)\,\underline{\tilde{y}} = \underline{c}, \tag{3,8}$$

and the equations, corresponding to (3.1), (3.6) and (3.3),

$$\left.\begin{aligned}
\tilde{M}_0 &= E_0\left\{\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3\right\},\\
\tilde{M}_i &= \left\{6\tilde{y}_{i-1} + (h^4 f_i - 12)\tilde{y}_i + 6\tilde{y}_{i+1} - h^4 g_i\right\}/6h^2;\ i = 1,2,\ldots,\ k-1,\\
\tilde{M}_k &= E_k\left\{\tilde{y}_{k-3}, \tilde{y}_{k-2}, \tilde{y}_{k-1}, \tilde{y}_k\right\}.
\end{aligned}\right\} \tag{3.9}$$

Comparison of the above with the results of [5] shows that (3.8) is precisely the linear system which determines the fourth order finite difference approximations $\tilde{Y}_i$ of Usmani; see [5, pp 1090-91]. This means that once this finite difference approximation is computed, equations (3.9) can be used to produce, with very little additional computational effort, the remaining parameters $\tilde{M}_i$; $i = 0,1,\ldots,k$, which determine the cubic spline $\tilde{S}$.

It is shown in [5, p 1092] that if

$$\|f\| = \max_{a \le x \le b} |f(x)| \langle 384/\left\{(b-a)^4 + 8h^3(b-a)\right\} \tag{3.10}$$

then

$$\left|y_i - \tilde{y}_i\right| = o(h^4);\ i = 1,2,\ldots,\ k-1. \tag{3.11}$$

It follows easily from (3.11), by using (3.1), (3.3), (3.6), (3.9) and certain standard cubic spline results, that if (3.10) holds then

$$\| \tilde{S}^{(j)} - Y^{(j)} \| = 0(h^{4-j}); \quad j = 0,1,2. \tag{3.12}$$

It is of interest to note that the cubic spline method can be extended easily to the case where the boundary conditions (1.3) are replaced by the second derivative conditions

$$y^{(2)}(a) = \gamma_0, \; y(2)(b) = \gamma_1. \qquad (1.3(a))$$

The analysis remains the same except that the end conditions (2.3) are replaced by the conditions

$$\left.\begin{aligned} M_0 &= \left\{ y_0^{(2)} + \frac{h^2}{720}(-56y_0^{(4)} - 10y_1^{(4)} + 8y_2^{(4)} - 2y_3^{(4)} \right\}, \\ M_k &= \left\{ y_k^{(2)} + \frac{h^2}{720}(-56y_k^{(4)} - 10y_{k-1}^{(4)} + 8y_{k-2}^{(4)} - 2y_{k-3}^{(4)} \right\}, \end{aligned}\right\}$$

which, by using the results of [4], are the only conditions of the form

$$\left.\begin{aligned} M_0 &= \frac{1}{h^2}\left\{ \sum_{i=0}^{2} a_i y_i + h^2 c_0 y_0^{(2)} + h^4 \sum_{i=0}^{3} e_i y_i^{(4)} \right\}, \\ M_k &= \frac{1}{h^2}\left\{ \sum_{i=0}^{2} a_i y_{k-i} + h^2 c_0 y_k^{(2)} + h^4 \sum_{i=0}^{3} e_i y_{k-i}^{(4)} \right\}, \end{aligned}\right\}$$

for which (2.4) holds. This change of the end conditions leaves all the results unaltered except for the first and last equations in (3.7), in which the following changes occur, in the elements of A,B and <u>c.</u>

$a_{11} = a_{k-1,k-1} = 5$, $a_{12} = a_{k-1,k-2} = -4$,

$$b_{11} = b_{k-1,k-1} = \frac{490}{720}, b_{12} = b_{k-1,k-2} = \frac{112}{720}, b_{13} = b_{k-1,k-3} = \frac{2}{720},$$

$$\left.\begin{aligned} c_1 &= 2\alpha_0 - h^2\gamma_0 + \frac{h^4}{720}[56\, g_0 + 490\, g_1 + 112\, g_2 + 2g_3 - 56\, f_0\alpha_0\,] \\ &\textit{and} \\ c_{k-1} &= 2\alpha_1 - h^2\gamma_1 + \frac{h^4}{720}[56\, g_k + 490\, g_{k-1} + 112\, g_{k-2} + 2g_{k-3} - 56\, f_k\alpha_1\,]. \end{aligned}\right\}$$

The convergence results (3.11), (3.12) follow as before, provided that (3.10) is replaced by the condition

$$|| f || = \max_{a=x=b} |f(x)| < 384/\{5(b-a)^4 + 4h^2(b-a)^2\}.$$

Finally, we observe that the cubic spline method described in this paper is more efficient than the quintic and sextic spline methods proposed by Usmani in [6]. These two spline methods are respectively of order 2 and 4. However, as is pointed out in [6, p97], the approximations produced by the fourth order sextic spline method at the knots (2.1) are, in general, less accurate than those obtained from the solution of the linear system (3.8).

REFERENCES

1. J.H. AHLBERG, E.N. NILSON and J.L. WALSH, The Theory of Splines and their Applications, London: Academic Press, 1967.

2. J.W. DANIEL and B.K. SWARTZ, "Extrapolated collocation for two-point boundary - value problems using cubic splines," J, Inst. Maths Applics., v.16, 1975, pp 161-174.

3. T.R. LUCAS, "Error bounds for interpolating cubic splines under various end conditions," SIAM J. Numer, Anal., v.11, 1974, pp 569-584.

4. N. PAPAMICHAELandA.J. WORSEY, "End conditions for improved cubic spline derivative approximations, " TR/100, Dept. of Mathematics, Brunel University, 1980.

5. R.A. USMANI, "Discrete variable methods for a boundary value problem with engineering applications, " Math. Comp. , v.32, 1978, pp 1087-1096.

6. R.A. USMANI, "Smooth spline approximations for the solution of a boundary value problem with engineering applications," J. Comp. and Appl. Maths., v. 6, 1980, pp 93-98.