An analytical solution for the two-sided Parisian stopping time, its asymptotics and the pricing of Parisian options

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Abstract

In this paper, we obtain a recursive formula for the density of the two-sided Parisian stopping time. This formula does not require any numerical inversion of Laplace transforms, and is similar to the formula obtained for the one-sided Parisian stopping time derived in Dassios and Lim [6]. However, when we study the tails of the two distributions, we find that the two-sided stopping time has an exponential tail, while the one-sided stopping time has a heavier tail. We derive an asymptotic result for the tail of the two-sided stopping time distribution and propose an alternative method of approximating the price of the two-sided Parisian option.

Keywords Brownian excursion, Double-sided Parisian options, Tail asymptotics

1 Introduction

Parisian options were first introduced by Chesney, Jeanblanc and Yor [4]. They are path dependent options whose payoff depends not only on the final value of the underlying asset, but also on the path trajectory of the underlying above or below a predetermined barrier L. For example, the owner of a Parisian down-and-out call loses the option when the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than D, while for a Parisian down-and-in call, the same event gives the owner the right to exercise the option. Parisian options are a kind of barrier option. However, they have the advantage of not being as easily manipulated by an influential agent as a simple barrier option, and thus is a guarantee against easy arbitrage.

Previous literature has largely focused on using Laplace transforms to price Parisian options. In Chesney et al. [4], Dassios and Wu [8], and Schröder [11], the problem is reduced to finding the Laplace transform of the Parisian stopping time, which is the first time the length of the excursion reaches level D. In Chesney et al. [4], the Laplace transform of the stopping time was obtained using the Brownian meander and Azema martingale while Dassios and Wu [7] introduced a perturbed Brownian motion and a semi-Markov model to obtain the Laplace transform. In both of these, an explicit form of the Laplace transform of the distribution of the Parisian stopping time was found. Other methods of pricing Parisian options include the PDE method, studied by Haber, Schönbucher and Wilmott [10], pricing by simulation, as in Anderluh [1] and Bernard and Boyle [3], and a combinatorial approach in Costabile [5]. In Dassios and Lim [6], a recursive solution for the density of the one-sided Parisian stopping time was found and a procedure for pricing Parisian options was proposed that does not require any numerical inversion of Laplace transforms.

There are also other types of Parisian options. Cumulative Parisian options, which are related to the total excursion time above (or below) a barrier, are studied in Chesney et al. [4], two-sided Parisian options are introduced in Dassios and Wu [8] and double barrier Parisian options in Dassios and Wu [7] and Anderluh and Weide [2]. In this paper, we look at the density of the two-sided Parisian stopping time. Using the same method as in Dassios and Lim [6], we obtain a recursive formula for its density. The formula we obtain has some similarity to that of the one-sided case. The advantage of this method compared to that in previous literature is that there is no Laplace transform to invert. This increases speed, and the formula is not an approximation, hence accuracy can be almost exact. Furthermore, we find that the two-sided stopping time has an exponential tail, and we derive an asymptotic result for it. However, a study of numerical results and graphs show that the one-sided stopping time has a heavier tail. Based on the asymptotic results, we propose a new method of approximating the two-sided stopping time.

Finally, we use the results to price two-sided Parisian options, which are options that get knocked in or out when the underlying either stays D amount of time above or below the barrier. These were first introduced in Dassios and Wu [8]. The Laplace transform of its pricing formula was given in their paper. Our approach does not require any numerical inversion of the Laplace transform.

Section 2 sets out some definitions and notations, Section 3 derives the density of the two-sided Parisian stopping time, Section 4 gives the result for the asymptotic tail probability of the above distribution, Section 5 gives some numerical results and graphs comparing the distributions of the one and two-sided Parisian stopping times, and Section 6 shows how we can use the results to price two-sided Parisian options.

2 Definitions

We will use the same definitions for the excursions as in Chesney et al. [4]. Let S be the underlying asset following a geometric Brownian motion, and $\mathcal{Q}$ denote the risk neutral probability measure. The dynamics of S under $\mathcal{Q}$ is

$$dS_t = S_t(rdt + \sigma dW_t), \qquad S_0 = x, \tag{2.1}$$

where W_t is a standard Brownian motion under $\mathcal{Q}$, and r and σ are positive constants. For simplicity, we have assumed zero dividends. Let K denote the strike price of the option and we introduce the notations

$$m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right), \qquad b = \frac{1}{\sigma}\ln\left(\frac{L}{x}\right), \qquad k = \frac{1}{\sigma}\ln\left(\frac{K}{x}\right),$$

so that the asset price $S_t = xe^{\sigma(mt+W_t)}$. We use the notation

$$g^S_{L,t} = \sup\{s \leq t | S_s = L\}, \qquad d^S_{L,t} = \inf\{s \geq t | S_s = L\},$$

with the usual convention that $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. The trajectory of S between $g^S_{L,t}$ and $d^S_{L,t}$ is the excursion which straddles time t. We are interested here in $t - g^S_{L,t}$, which is the age of the excursion at time t. We further denote by $g^W_{L,t}$ and $d^W_{L,t}$ the excursion lengths when the underlying process is the Brownian motion W. For $D > 0$, we now define

$$\tau^+_{L,D}(S) = \inf\{t \geq 0 | \mathbf{1}_{S_t > L}(t - g^S_{L,t}) \geq D\}, \tag{2.2}$$

$$\tau^-_{L,D}(S) = \inf\{t \geq 0 | \mathbf{1}_{S_t < L}(t - g^S_{L,t}) \geq D\}, \tag{2.3}$$

$$\tau_{L,D}(S) = \tau^+_{L,D}(S) \wedge \tau^-_{L,D}(S) \tag{2.4}$$

Thus, $\tau^+_{L,D}(S)$ denotes the first time that the length of the excursion of process S above the barrier L reaches level D, while $\tau^-_{L,D}(S)$ denotes the first time the length of the excursion of process S below level L reaches level D. We also introduce the following notation for the stopping times where we refer to the standard Brownian motion W instead of S. Furthermore, without loss of generality since any time t of interest can be expressed in units of the window length D, we let $D = 1$ from now on and drop its notation.

$$\tau^+_b = \inf\{t \geq 0 | \mathbf{1}_{W_t > b}(t - g^W_{b,t}) \geq 1\}, \tag{2.5}$$

$$\tau^-_b = \inf\{t \geq 0 | \mathbf{1}_{W_t < b}(t - g^W_{b,t}) \geq 1\}. \tag{2.6}$$

The two-sided stopping time is the minimum of the two one-sided stopping times with the same barrier. We denote it by τ_b and we have

$$\tau_b = \tau_b^+ \wedge \tau_b^- .$$

We note however that here we have taken the window length of both sides to be the same (ie. 1 in our case).

The owner of a Parisian min-in option receives the payoff only if there is an excursion below the level L or above level L which is of length greater than D. This will be the case if $\tau_L(S) \leq T$, where T is the maturity time of the option, and $\tau_L(S) = \tau_{L,D}(S)$. Denoting $C_i^{min}(x,T)$ as the price of a Parisian min-in call with initial underlying price x, maturity T, and parameters K, L, D, r fixed, we have the price formula

$$C_i^{min}(x,T) = E_{\mathcal{Q}}\left[e^{-rT}\mathbf{1}_{\{\tau_L(S)\leq T\}}(xe^{\sigma(mT+W_T)} - K)^+\right]. \tag{2.7}$$

We introduce a new probability measure $\mathcal{P}$, which makes $Z_t = W_t + mt$ a standard Brownian motion under $\mathcal{P}$. Applying Girsanov's Theorem, we have

$$C_i^{min}(x,T) = E_{\mathcal{P}}\left[e^{-(r+\frac{1}{2}m^2)T}\mathbf{1}_{\{\tau_b\leq T\}}e^{mZ_T}\left(xe^{\sigma Z_T} - K\right)^+\right]. \tag{2.8}$$

To simplify things, we also let

$$^*C_i^{min}(x,T) = e^{(r+\frac{1}{2}m^2)T}C_i^{min}(x,T). \tag{2.9}$$

We denote by $\mathcal{F}_t = \sigma(Z_s, s \leq t)$ the natural filtration of the Brownian motion $(Z_t, t \geq 0)$. Then τ_b is an $\mathcal{F}_t$-stopping time, and by the strong Markov property of Brownian motion

$$^*C_i^d(x,T) = E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b\leq T\}}E\left[e^{mZ_T}\left(xe^{\sigma Z_T} - K\right)^+ |\mathcal{F}_{\tau_b}\right]\right] \tag{2.10}$$

$$= \quad E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b\le T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^{+}\frac{1}{\sqrt{2\pi(T-\tau_b)}}e^{-\frac{(y-Z_{\tau_b})^2}{2(T-\tau_b)}}dy\right]. \quad (2.11)$$

We will first look at the density function of τ_b, which we will denote by $f_b(t)$, and then show how it can be used to obtain the prices of a Parisian min-in call option.

3 Density of the two-sided Parisian stopping time

In this section, we give an analytical formula for the density of the two-sided Parisian stopping time. The formula is very similar to that for the one-sided stopping time.

Theorem 3.1 *Denoting by $f_0(t)$ the probability density function of τ_0, we have*

$$f_0(t)=\sum_{k=0}^{n-1}(-1)^k L_k(t-1), \qquad \text{for } n<t\le n+1,\ n=1,2,..., \quad (3.1)$$

for $t>1$, where $L_k(t)$ is defined recursively as follows:

$$L_0(t) = \frac{1}{\pi\sqrt{t}}, \qquad \text{for } t>0, \quad (3.2)$$

$$L_{k+1}(t) = \int_1^{t-k} L_k(t-s)\frac{\sqrt{s-1}}{\pi s}ds, \qquad \text{for } t>k+1. \quad (3.3)$$

Proof. The Laplace transform of the density of τ_0 is (see Dassios and Wu [8])

$$\hat{f}_0(\beta)=\frac{1}{\Psi(\sqrt{2\beta})-e^{\beta}\sqrt{\pi\beta}},$$

where $\Psi(x)$ is

$$\Psi(x)=1+x\sqrt{2\pi}e^{\frac{x^2}{2}}\mathcal{N}(x),$$

and $\mathcal{N}(x)$ denotes the standard normal distribution function. Note however that the formula given in Dassios and Wu [8] differs from this because the function $\Psi(x)$ is defined differently.

Now, we have

$$
\begin{aligned}
\frac{1}{\beta}e^{-\beta}\left(\Psi\left(\sqrt{2\beta}\right)-e^{\beta}\sqrt{\pi\beta}\right) &= \frac{e^{-\beta}}{\beta}+2\sqrt{\frac{\pi}{\beta}}\int_{-\infty}^{\sqrt{2\beta}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx-\sqrt{\frac{\pi}{\beta}} \\
&= \frac{e^{-\beta}}{\beta}+\sqrt{\frac{\pi}{\beta}}\left(1+2\int_{0}^{\sqrt{2\beta}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\right)-\sqrt{\frac{\pi}{\beta}} \\
&= \frac{e^{-\beta}}{\beta}+\int_0^1\frac{e^{-\beta s}}{\sqrt{s}}ds \qquad (3.4)\\
&= \int_1^\infty e^{-\beta s}ds+\left(\int_0^\infty\frac{e^{-\beta s}}{\sqrt{s}}ds-\int_1^\infty\frac{e^{-\beta s}}{\sqrt{s}}ds\right) \\
&= \sqrt{\frac{\pi}{\beta}}+\frac{1}{\beta}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds \\
&= \sqrt{\frac{\pi}{\beta}}\left(1+\frac{1}{\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right),
\end{aligned}
$$

so

$$
\begin{aligned}
\hat{f}_0(\beta) &= \frac{e^{-\beta}}{\sqrt{\pi\beta}\left(1+\frac{1}{\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)} \qquad (3.5)\\
&= e^{-\beta}\sum_{k=0}^{\infty}(-1)^k\frac{1}{\sqrt{\pi\beta}}\left(\frac{1}{\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k. \qquad (3.6)
\end{aligned}
$$

We denote

$$
\hat{L}_k(\beta)=\frac{1}{\sqrt{\pi\beta}}\left(\frac{1}{\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k. \qquad (3.7)
$$

Since $\hat{L}_1(\beta)\to 0$ as $\beta\to\infty$, and $\hat{L}_k(\beta)$ is continuous and decreasing in β, there exists $\beta^*>0$ such that the above expansion from line (3.5) to (3.6) is valid for all $\beta>\beta^*$. Furthermore, let $\mathcal{L}$ denote the Laplace transform operator and we have the following Laplace inversions

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{\sqrt{\pi\beta}}\right) &= \frac{1}{\pi\sqrt{t}} \qquad (3.8)\\
\mathcal{L}^{-1}\left(\frac{1}{\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right) &= \frac{\sqrt{t-1}}{\pi t}\mathbf{1}_{\{t>1\}}, \qquad (3.9)
\end{aligned}
$$

since the LHS of (3.9) is the product of two functions whose inversion is known, so by taking their convolution we get

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right) &= \int_0^t \frac{1}{2\pi\sqrt{t-s}}\frac{1}{2s^{3/2}}\mathbf{1}_{\{s>1\}}ds \\ &= \left[-\frac{\sqrt{t-s}}{2\pi t\sqrt{s}}\right]_1^t = \frac{\sqrt{t-1}}{2\pi t}\mathbf{1}_{\{t>1\}}.\end{aligned}$$

Hence, taking the Laplace inversion of equation (3.7), we obtain that L_k is the k^{th} convolution of (3.9), and L_0 is the expression obtained in (3.8). Finally, we note that for $n < t < n+1$, $L_k(t)$ is zero for $k > n$, so we only need a finite sum up to n, where the series expansion is valid for $\beta > \beta^*$. Hence we have the recursive solution. ■

For $b > 0$, we are only interested in the case when $\{T_b < 1\}$, where T_b is the first hitting time of level b, since if $T_b \geq 1$, $\tau_b = 1$. We have the following recursive solution for the density of τ_b on the set $\{T_b < 1\}$.

Theorem 3.2 *For $b > 0$, we denote by $f_b(t, T_b < 1)$ the probability density function of the two-sided stopping time τ_b on the set $\{T_b < 1\}$. We have*

$$f_b(t, T_b < 1) = \sum_{k=0}^{n-1} (-1)^k L_k(t-1), \qquad \text{for } n < t \leq n+1,\ n = 1, 2, ..., \tag{3.10}$$

for $t > 0$, where $L_k(t)$ is defined recursively as follows:

$$L_0(t) = \mathbf{1}_{\{0<t\leq 1\}}\frac{1}{\pi\sqrt{t}}e^{-\frac{b^2}{2t}} + \mathbf{1}_{\{t>1\}}\frac{2}{\pi\sqrt{t}}e^{-\frac{b^2}{2t}}\mathcal{N}\left(-b\sqrt{\frac{t-1}{t}}\right), \tag{3.11}$$

$$L_{k+1}(t) = \int_1^{t-k} L_k(t-s)\frac{\sqrt{s-1}}{\pi s}ds, \qquad \text{for } t > k+1. \tag{3.12}$$

Proof. We have

$$E\left[e^{-\beta\tau_b(t)}\mathbf{1}_{\{T_b<1\}}\right] = E\left[e^{-\beta(T_b+\tau_0)}\mathbf{1}_{\{T_b<1\}}\right]$$

$$
\begin{aligned}
&= E\left[e^{-\beta T_b}\mathbf{1}_{\{T_b<1\}}\right]\frac{1}{\Psi(\sqrt{2\beta})-e^{\beta}\sqrt{\pi\beta}} \\
&= e^{-\beta}\sum_{k=0}^{\infty}(-1)^k\frac{E\left[e^{-\beta T_b}\mathbf{1}_{\{T_b<1\}}\right]}{\sqrt{\pi\beta}}\left(\frac{1}{\sqrt{\pi\beta}}\int_1^{\infty}\frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
L_0(t) &= \mathcal{L}^{-1}\left(\frac{E\left[e^{-\beta T_b}\mathbf{1}_{\{T_b<1\}}\right]}{\sqrt{\pi\beta}}\right) \\
&= \mathbf{1}_{\{0<t\leq 1\}}\int_0^t\frac{b}{\sqrt{2\pi s^3}}e^{-\frac{b^2}{2s}}\frac{1}{\pi\sqrt{t-s}}ds+\mathbf{1}_{\{t>1\}}\int_0^1\frac{b}{\sqrt{2\pi s^3}}e^{-\frac{b^2}{2s}}\frac{1}{\pi\sqrt{t-s}}ds \\
&= \mathbf{1}_{\{0<t\leq 1\}}\int_{\frac{b}{\sqrt{t}}}^{\infty}\frac{2}{\pi\sqrt{2\pi}}e^{-\frac{x^2}{2}}\sqrt{\frac{x^2}{tx^2-b^2}}dx+\mathbf{1}_{\{t>1\}}\int_b^{\infty}\frac{2}{\pi\sqrt{2\pi}}e^{-\frac{x^2}{2}}\sqrt{\frac{x^2}{tx^2-b^2}}dx \\
&= \mathbf{1}_{\{0<t\leq 1\}}\int_{\frac{b^2}{t}}^{\infty}\frac{1}{\pi\sqrt{2\pi}}e^{-\frac{y}{2}}\sqrt{\frac{1}{ty-b^2}}dy+\mathbf{1}_{\{t>1\}}\int_{b^2}^{\infty}\frac{1}{\pi\sqrt{2\pi}}e^{-\frac{y}{2}}\frac{1}{\sqrt{ty-b^2}}dy \\
&= \mathbf{1}_{\{0<t\leq 1\}}\frac{2}{\pi\sqrt{t}}e^{-\frac{b^2}{2t}}\int_0^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}dx+\mathbf{1}_{\{t>1\}}\frac{2}{\pi\sqrt{t}}e^{-\frac{b^2}{2t}}\int_{b\sqrt{t-1}}^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}dx \\
&= \mathbf{1}_{\{0<t\leq 1\}}\frac{1}{\pi\sqrt{t}}e^{-\frac{b^2}{2t}}+\mathbf{1}_{\{t>1\}}\frac{2}{\pi\sqrt{t}}e^{-\frac{b^2}{2t}}\mathcal{N}\left(-b\sqrt{\frac{t-1}{t}}\right).
\end{aligned}
$$

The L_k for $k=1,2,...$ is derived the same way as the previous case.

And for $b<0$, we have

$$
f_b(t, T_b<1)=f_{-b}(t, T_{-b}<1),
$$

due to the symmetry of the standard Brownian motion. ■

4 Tail distribution of the two-sided Parisian stopping time

In this section, we prove that the two-sided stopping time τ_0 has an exponential tail, unlike the distribution of the one-sided stopping time τ_0^-. This is as expected because the one-sided case involves the hitting time of a Brownian motion, which is a heavy tailed distribution with infinite expectation, while the two-sided one involves the hitting time of a Brownian motion reflected in zero, which has an exponential tail. We present some numerical results and graphs to see what happens when $t \to \infty$. Furthermore, we compare this to the one-sided stopping time τ_0^-, which has a heavier tail as we will see.

Theorem 4.1 *We denote $\bar{F}_0(t)$ as the tail of the distribution of the two-sided Parisian stopping time τ_0 with barrier 0. It has an exponential tail. As $t \to \infty$, we have*

$$\bar{F}_0(t) \sim 2e^{-\beta^*} e^{-\beta^*(t-1)}, \tag{4.1}$$

for some constant $\beta^* > 0$ such that $-\beta^*$ is the unique solution of the equation

$$\int_0^1 \frac{e^{-\beta s}}{\sqrt{s}} ds + \frac{e^{-\beta}}{\beta} = 0. \tag{4.2}$$

Proof. First, we have

$$\begin{aligned} \hat{f}_0(\beta) &= \frac{1}{\Psi(\sqrt{2\beta}) - e^{\beta}\sqrt{\pi\beta}} \\ &= \frac{e^{-\beta}}{\beta\left(\int_0^1 \frac{e^{-\beta s}}{\sqrt{s}} ds + \frac{e^{-\beta}}{\beta}\right)}. \end{aligned}$$

For simplicity, since for any function $h(t)$ with Laplace transform $\hat{h}(\beta)$, when shifted by 1, has Laplace transform

$$\mathcal{L}(h(t-1)) = e^{-\beta}\hat{h}(\beta), \qquad \text{for } t > 1,$$

we exclude $e^{-\beta}$ in the numerator from our calculations, and shift the resulting function at the end of the calculations by the window length 1 to obtain the actual tail. The intuition behind this is that since the window length is 1, the stopping time will not occur before time 1, and hence it is only useful to study the density for $t > 1$. We then have

$$\frac{1}{\beta\left(\int_0^1 \frac{e^{-\beta s}}{\sqrt{s}} ds + \frac{e^{-\beta}}{\beta}\right)} = \frac{1}{1 + \int_0^1 (1 - e^{-\beta v}) \frac{1}{2v^{3/2}} dv}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-u} e^{-u\int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv} du \\
&= E\left(e^{-\beta X_T}\right),
\end{aligned}
$$

where X_T is a subordinator (non-decreasing Lévy process) with Lévy measure $\frac{1}{2v^{3/2}}$ for $v < 1$ at an independent exponential time $T \sim Exp(1)$. Hence, we observe an interesting connection between the distributions of the Parisian stopping time and that of the Lévy process X_T. This suggests possibilities for further study. The first step above follows from (3.4) and the second step can be derived as below:

$$
\begin{aligned}
\int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv &= \int_0^1 \int_0^v \beta e^{-\beta u} du \frac{1}{2v^{3/2}} dv \\
&= \int_0^1 \beta e^{-\beta u} \int_u^1 \frac{1}{2v^{3/2}} dv du \\
&= \int_0^1 \beta e^{-\beta u} \left(\frac{1}{\sqrt{u}} - 1\right) du \\
&= \beta \left(\int_0^1 \frac{e^{-\beta s}}{\sqrt{s}} ds + \frac{e^{-\beta}}{\beta}\right) - 1.
\end{aligned}
$$

Next, we define two new discrete random variables $\overline{T}$ and $\underline{T}$:

$$
\begin{aligned}
\underline{T} &= \sum_{k=1}^{\infty} (k-1)h \mathbf{1}_{\{(k-1)h < T \le kh\}} \\
\overline{T} &= \sum_{k=0}^{\infty} kh \mathbf{1}_{\{(k-1)h < T \le kh\}}
\end{aligned}
$$

so that $\underline{T} \le T \le \overline{T}$. They have probability functions

$$
\begin{aligned}
P(\overline{T} = kh) &= e^{-kh}(1-e^{-h}) \qquad k = 0, 1, ..., \\
P(\underline{T} = kh) &= e^{-(k-1)h}(1-e^{-h}) \qquad k = 1, 2, ...
\end{aligned}
$$

We note that $\overline{T}$ is the upper bound for T and $\underline{T}$ is its lower bound. Hence, we have that $P(\underline{T} \leq t) \leq P(T \leq t) \leq P(\overline{T} \leq t)$, and thus

$$P(X_{\overline{T}} > x) \leq P(X_T > x) \leq P(X_{\underline{T}} > x),$$

since X_t is a subordinator and hence increasing. We then proceed to show that as $h \to 0$, both $P(X_{\overline{T}} > x)$ and $P(X_{\underline{T}} > x)$ converges to the same limit.

We have

$$E\left(e^{-\beta X_{\overline{T}}}\right) = \sum_{k=0}^{\infty} e^{-kh}(1 - e^{-h})e^{-kh\int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv},$$

and

$$E\left(e^{-\beta X_{\underline{T}}}\right) = \sum_{k=1}^{\infty} e^{-(k-1)h}(1 - e^{-h})e^{-kh\int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv}.$$

We look first at $E\left(e^{-\beta X_{\overline{T}}}\right)$. We define the function $\hat{g}_h(\beta)$ as

$$\hat{g}_h(\beta) = e^{-h\int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv},$$

and note that $\hat{g}_h(\beta)$ is the Laplace transform of X_h,

$$\hat{g}_h(\beta) = E\left(e^{-\beta X_h}\right).$$

We also denote $G_h(x)$ as the distribution of X_h, and $\overline{G}_h(x)$ as its survival function. Then

$$\begin{aligned} \sum_{k=0}^{\infty} e^{-kh}(1 - e^{-h})e^{-kh\int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv} &= \sum_{k=0}^{\infty} e^{-kh}(1 - e^{-h})\left(\hat{g}_h(\beta)\right)^k \\ &= \frac{1 - e^{-h}}{1 - \hat{g}_h(\beta)e^{-h}}. \end{aligned}$$

Now, we denote by $\overline{L}(x)$ the tail distribution $P(X_{\overline{T}} > x)$, and $\hat{\overline{L}}(\beta)$ its Laplace transform. So we have

$$\begin{aligned} \hat{\overline{L}}(\beta) &= \frac{1 - \frac{1-e^{-h}}{1-\hat{g}_h(\beta)e^{-h}}}{\beta} \\ &= \frac{e^{-h}\frac{1-\hat{g}_h(\beta)}{\beta}}{1-\hat{g}_h(\beta)e^{-h}}, \end{aligned}$$

and thus,

$$\begin{aligned} \hat{\overline{L}}(\beta)\left(1-\hat{g}_h(\beta)e^{-h}\right) &= e^{-h}\frac{1-\hat{g}_h(\beta)}{\beta} \\ \hat{\overline{L}}(\beta) - \hat{\overline{L}}(\beta)\hat{g}_h(\beta)e^{-h} &= e^{-h}\frac{1-\hat{g}_h(\beta)}{\beta} \end{aligned}$$

Inverting the Laplace transform on both sides and writing the second term as a convolution, we have

$$\overline{L}(x) - \int_0^x \overline{L}(x-y)dG_h(y)e^{-h} = e^{-h}\overline{G}_h(x).$$

Let $\beta^* > 0$ be such that $-\beta^*$ is the solution to the equation

$$1 + \int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv = 0.$$

We note that this equation has a unique negative solution, because the expression on the left hand side of this equation is decreasing for negative β. Furthermore, as $\beta \to 0$, the expression approaches 1, and as $\beta \to -\infty$, the expression approaches $-\infty$. Next, we define $\overline{L}^*(x)$ as

$$\overline{L}(x)e^{\beta^* x} = \overline{L}^*(x).$$

Then we have

$$\overline{L}^*(x)e^{-\beta^* x} - \int_0^x \overline{L}^*(x-y)e^{-\beta^*(x-y)}dG_h(y)e^{-h} \quad = \quad e^{-h}\overline{G}_h(x)$$

$$\overline{L}^*(x) - \int_0^x \overline{L}^*(x-y) e^{\beta^* y} dG_h(y) e^{-h} \quad = \quad e^{-h} e^{\beta^* x} \overline{G}_h(x).$$

By the key renewal theorem (see Feller [9] Chapter 11), we have that as $x \to \infty$,

$$\begin{aligned}
\overline{L}^*(x) \quad &\to \quad \frac{\int_0^\infty e^{-h} e^{\beta^* y} \overline{G}_h(y) dy}{\int_0^\infty y e^{\beta^* y} dG_h(y)} \\
&= \quad \frac{e^{-h}\left(1 - e^{-h\int_0^1 (1-e^{\beta^* v})\frac{1}{2v^{3/2}}dv}\right)}{-\beta^* \frac{d}{d\beta^*}\hat{g}_h(\beta^*)} \\
&= \quad \frac{e^{-h}\left(1 - e^{-h\int_0^1 (1-e^{\beta^* v})\frac{1}{2v^{3/2}}dv}\right)}{-\beta^*\left(h\int_0^1 e^{\beta^* v}\frac{1}{2\sqrt{v}}dv\right) e^{-h\int_0^1 (1-e^{\beta^* v})\frac{1}{2v^{3/2}}dv}}.
\end{aligned}$$

We denote this by $\overline{C}_h$. When $h \to 0$, we get

$$\begin{aligned}
\overline{C}_h \quad &= \quad \frac{\int_0^1 (1-e^{\beta^* v})\frac{1}{2v^{3/2}}dv}{-\beta^* \int_0^1 e^{\beta^* v}\frac{1}{2\sqrt{v}}dv} \\
&= \quad \frac{-1}{-\frac{\beta^*}{2}\left(\int_0^1 e^{\beta^* v}\frac{1}{\sqrt{v}}dv - \frac{e^{\beta^*}}{\beta^*}\right) - \frac{e^{\beta^*}}{2}} \\
&= \quad 2e^{-\beta^*}.
\end{aligned}$$

Likewise, we denote by $\bar{l}(x)$ the tail distribution $P(X_{\underline{T}} > x)$, and $\hat{\bar{l}}(\beta)$ its Laplace transform. Similarly, we can compute

$$\begin{aligned}
\hat{\bar{l}}(\beta) \quad &= \quad \frac{1 - \frac{1-e^{-h}}{1-\hat{g}_h(\beta)e^{-h}}\hat{g}_h(\beta)}{\beta} \\
&= \quad \frac{\frac{1-\hat{g}_h(\beta)}{\beta}}{1-\hat{g}_h(\beta)e^{-h}},
\end{aligned}$$

and thus,

$$\hat{\bar{l}}(\beta)\left(1 - \hat{g}_h(\beta)e^{-h}\right) = \frac{1-\hat{g}_h(\beta)}{\beta}.$$

Inverting the Laplace transform, we have

$$\bar{l}(x) - \int_0^x \bar{l}(x-y) dG_h(y) e^{-h} = \bar{G}_h(x),$$

and we define $\bar{l}^*(x)$ as

$$\bar{l}(x) e^{\beta^* x} = \bar{l}^*(x).$$

Then we have

$$\bar{l}^*(x) - \int_0^x \bar{l}^*(x-y) e^{\beta^* y} dG_h(y) = e^{\beta^* x} \overline{G}_h(x).$$

By the key renewal theorem,

$$\begin{aligned} \bar{l}^*(x) &\to \frac{\int_0^\infty e^{\beta^* y} \overline{G}_h(y) dy}{\int_0^\infty y e^{\beta^* y} dG_h(y)} \\ &= \frac{1 - e^{-h \int_0^1 (1-e^{\beta^* v}) \frac{1}{2v^{3/2}} dv}}{-\beta^* \left(h \int_0^1 e^{\beta^* v} \frac{1}{2\sqrt{v}} dv \right) e^{-h \int_0^1 (1-e^{\beta^* v}) \frac{1}{2v^{3/2}} dv}}. \end{aligned}$$

We denote this by $\underline{C}_h$. When $h \to 0$, we get

$$\underline{C}_h = 2e^{-\beta^*}.$$

Finally, we note that since we have

$$\begin{aligned} e^{\beta^* x} P(X_{\overline{T}} > x) \leq \quad & e^{\beta^* x} P(X_T > x) & \leq e^{\beta^* x} P(X_{\underline{T}} > x) \\ \overline{L}^*(x) \leq \quad & e^{\beta^* x} \bar{F}_0(x) & \leq \bar{l}^*(x), \end{aligned}$$

and as $h \to 0$, $\overline{L}^*(x)$ and $\bar{l}^*(x)$ converges to the same limit as $x \to \infty$, we have that $e^{\beta^* x} \bar{F}_0(x)$ also converges to this limit as $x \to \infty$. Shifting the resulting function to the right by 1, we thus have the result

$$\bar{F}_0(t) \to 2e^{-\beta^*} e^{-\beta^*(t-1)},$$

as $t \to \infty$. ■

Remark 4.2 *We can compute β^* numerically to be 0.854.*

5 Numerical Results

The table below presents the survival function for both τ_0 and τ_0^-, computed using a time step of $h = 0.001$ with R.

Table 1: One and two-sided survival functions for $0 < t \leq 10$

t	$\bar{F}_0(t)$	$\bar{F}_0^-(t)$	t	$\bar{F}_0(t)$	$\bar{F}_0^-(t)$
1.5	0.555931	0.775033	6.0	0.015114	0.403422
2.0	0.369469	0.681770	6.5	0.010779	0.387956
2.5	0.242144	0.614236	7.0	0.007910	0.374142
3.0	0.159600	0.563552	7.5	0.006003	0.361704
3.5	0.105503	0.523602	8.0	0.004726	0.350429
4.0	0.070093	0.491082	8.5	0.003866	0.340146
4.5	0.046893	0.463944	9.0	0.003278	0.330718
5.0	0.031679	0.440854	9.5	0.002872	0.322033
5.5	0.021687	0.420896	10.0	0.002586	0.313997

We can see that the two-sided survival function goes to 0 much faster than the one-sided case.

The following graph compares the density functions of the one and two-sided case. The red line represents $f_0(t)$ while the black line $f_0^-(t)$, plotted against time. This graph suggests that $f_0^-(t)$ has a heavier tail.

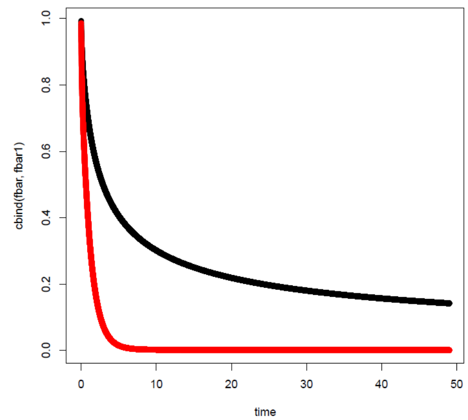

Figure 1: Graph of $f_0(t)$ and $f_0^-(t)$ vs t for $0 < t \leq 50$

The following graph depicts the tails $\bar{F}_0(t)$ (black) and the approximation $C_{\beta^*}e^{-\beta^* t}$ (red).

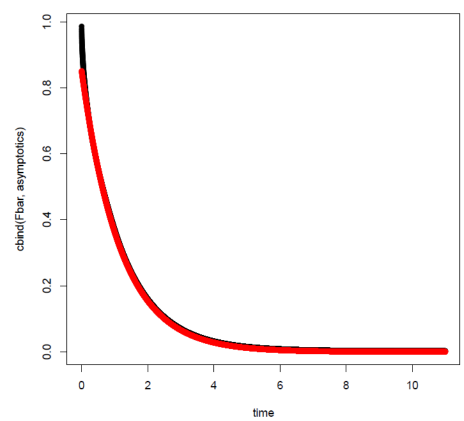

Figure 2: Graph of $\bar{F}_0(t)$ and $C_{\beta^*}e^{-\beta^* t}$ vs t for $0 < t \leq 20$

It suggests that the asymptotic provides a good approximation for the survival function. The following graph plots $\bar{F}_0^-(t)$ against $\ln(t)$.

Remark 5.1 *Based on the asymptotics in Theorem 3.4, we propose a new method of obtaining the density of the two-sided stopping time when* $b = 0$*. From the recursions in Theorem 3.1, we can compute the closed form formulas for the density* $f_0(t)$ *for* $1 < t \leq 4$*. For* $t > 4$*, we* *approximate the density with the asymptotics. We have from Theorem 3.1*

$$L_0(t) = \frac{1}{\pi\sqrt{t}}, \qquad \textit{for } t > 0.$$

$$\begin{aligned}
L_1(t) &= \int_1^t L_1(t-s)\frac{s-1}{\pi s}ds \\
&= \int_0^{t-1} \frac{\sqrt{t-s-1}}{\pi^2(t-s)\sqrt{s}}ds \\
&= \left[2\arctan\left(\sqrt{\frac{s}{t-s-1}}\right) - \frac{2}{\sqrt{t}}\arctan\left(\sqrt{\frac{s}{t(t-s-1)}}\right)\right]_0^{t-1} \\
&= \frac{1}{\pi} - \frac{1}{\pi\sqrt{t}}.
\end{aligned}$$

$$\begin{aligned}
L_2(t) &= \int_1^{t-1} L_2(t-s)\frac{\sqrt{s-1}}{\pi s}ds \\
&= \frac{1}{\pi^2}\int_1^{t-1}\left(\frac{\sqrt{t-s-1}}{t-s}ds - \frac{\sqrt{t-s-1}}{\sqrt{s}(t-s)}\right)ds \\
&= \frac{1}{\pi^2}\left[2\arctan(\sqrt{t-s-1}) - 2\sqrt{t-s-1}\right]_1^{t-1} \\
&\quad -\frac{1}{\pi^2}\left[2\arctan\sqrt{\frac{s}{t-s-1}} - \frac{2\arctan\left(\sqrt{\frac{s}{t(t-s-1)}}\right)}{\sqrt{t}}\right]_1^{t-1} \\
&= \frac{2\sqrt{t-2}}{\pi^2} - \frac{4\arctan\sqrt{\frac{t-2}{t}}}{\pi^2\sqrt{t}}.
\end{aligned}$$

Hence, we have an approximation for the density:

$$f_0(t) = \begin{cases}
\frac{1}{\pi\sqrt{t}} & \textit{for } 1 < t \leq 2 \\
\frac{2}{\pi\sqrt{t}} - \frac{1}{\pi} & \textit{for } 2 < t \leq 3 \\
\frac{2}{\pi\sqrt{t}} - \frac{1}{\pi} - \frac{4}{\pi^2}\tan^{-1}\sqrt{t-2} + \frac{4}{\pi^2\sqrt{t}}\tan^{-1}\sqrt{\frac{t-2}{t}} + \frac{2}{\pi^2}\sqrt{t-2} & \textit{for } 3 < t \leq 4 \\
2\beta^* e^{-\beta^* t} & \textit{for } t > 4
\end{cases},$$

where $\beta^* = 0.854$. For $b > 0$, closed form formulas cannot be found, but we can similarly approximate the density $f_b(t)$ with the asymptotic for large values of t. For $f_b(t)$, the asymptotics as $t \to \infty$ is as follows:

$$f_b(t, T_b < 1) \sim 2\beta^* e^{-\beta^*(t+1)} \int_0^1 e^{\beta^* s - \frac{b^2}{2s}} \frac{b}{\sqrt{2\pi s^3}} dt.$$

6 Pricing two-sided Parisian Options

6.1 Min-call-in Parisian call

A min-call-in Parisian call is a call option that gets knocked in, as the name suggests, when either the underlying makes an excursion above the barrier or an excursion below the barrier of a certain length. Here, we price a min-call-in Parisian call with the same window length $D = 1$ above and below the barrier.

Theorem 6.1 *The price of a two-sided Parisian-in option on the underlying S as defined in (2.1), with barrier L, strike price K, maturity time $T > 1$, and window length $D = 1$, is*

$$\begin{aligned} {}^*C_i^{min}(x, T) &= x\phi(\sigma + m) - K\phi(m) \\ &+ \sqrt{\frac{\pi}{2}} \int_0^T f_b(t; T_b < 1) \left(x\psi(\sigma + m, h_b, b, \rho, t) + \psi(-(\sigma + m), h_b, -b, -\rho, t) \right. \\ &\left. -K \left(\psi(-m, h_b', -b, -\rho, t) + \psi(m, h_b', b, \rho, t) \right) \right) dt, \end{aligned} \tag{6.1}$$

where $f_b(t; T_b < 1)$ is the density function of the two-sided Parisian stopping time with barrier b as in Theorem 3.3, and we define the functions

$$\begin{aligned} \psi(x, y, b, \rho, t) &= e^{\frac{x^2(1+T-t)+2bx}{2}} \left(Z(-x)\mathcal{N}\left(\frac{-x\rho - y}{\sqrt{1-\rho^2}}\right) - \rho Z(y)\mathcal{N}\left(\frac{-x - \rho y}{\sqrt{1-\rho^2}}\right) \right. \\ &\left. -x\left(\mathcal{N}(-x) - \mathcal{N}_\rho(-x, y)\right) \right), \end{aligned} \tag{6.2}$$

$$\phi(x) = e^{\frac{x^2 T}{2}} \left(\mathcal{N}(b - x) - \mathcal{N}_{\frac{1}{\sqrt{T}}}\left(b - x, \frac{k - xT}{\sqrt{T}}\right) \right)$$

$$-e^{\frac{x^2T+4bx}{2}}\left(\mathcal{N}(-b-x)-\mathcal{N}_{\frac{1}{\sqrt{T}}}\left(-b-x,\frac{k-2b-xT}{\sqrt{T}}\right)\right), \tag{6.3}$$

and

$$\begin{aligned} h_b &= \frac{1}{\sqrt{1+T-t}}\left(k-b-(\sigma+m)(1+T-t)\right) && (6.4)\\ h'_b &= \frac{1}{\sqrt{1+T-t}}\left(k-b-m(1+T-t)\right) && (6.5)\\ \rho &= \frac{1}{\sqrt{1+T-t}}. && (6.6)\end{aligned}$$

Proof. First, we note that τ_b is an $\mathcal{F}_t$-stopping time, and by the strong Markov property of Brownian motion

$$\begin{aligned} {}^*C_i^{min}(x,T) &= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b\le T\}}E\left[e^{mZ_T}\left(xe^{\sigma Z_T}-K\right)^+|\mathcal{F}_{\tau_b}\right]\right]\\ &= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b\le T\}}\int_{-\infty}^{\infty}e^{my}\left(xe^{\sigma y}-K\right)^+\frac{1}{\sqrt{2\pi(T-\tau_b)}}e^{-\frac{(y-Z_{\tau_b})^2}{2(T-\tau_b)}}dy\right]\\ &= E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b>1\}}\mathbf{1}_{\{\tau_b\le T\}}\int_{-\infty}^{\infty}e^{my}\left(xe^{\sigma y}-K\right)^+\frac{1}{\sqrt{2\pi(T-\tau_b)}}e^{-\frac{(y-Z_{\tau_b})^2}{2(T-\tau_b)}}dy\right]\\ &\quad +E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b\le 1\}}\mathbf{1}_{\{\tau_b\le T\}}\int_{-\infty}^{\infty}e^{my}\left(xe^{\sigma y}-K\right)^+\frac{1}{\sqrt{2\pi(T-\tau_b)}}e^{-\frac{(y-Z_{\tau_b})^2}{2(T-\tau_b)}}dy\right].\end{aligned}$$

If $T_b>1$, $\tau_b=1$, so we have

$$\begin{aligned} &E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b>1\}}\mathbf{1}_{\{\tau_b\le T\}}\int_{-\infty}^{\infty}e^{my}\left(xe^{\sigma y}-K\right)^+\frac{1}{\sqrt{2\pi(T-\tau_b)}}e^{-\frac{(y-Z_{\tau_b})^2}{2(T-\tau_b)}}dy\right]\\ =\ &E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b>1\}}\mathbf{1}_{\{1\le T\}}\int_{-\infty}^{\infty}e^{my}\left(xe^{\sigma y}-K\right)^+\frac{1}{\sqrt{2\pi(T-1)}}e^{-\frac{(y-Z_1)^2}{2(T-1)}}dy\right]\\ =\ &\frac{1}{\sqrt{2\pi(T-1)}}\int_{-\infty}^{b}\int_{k}^{\infty}e^{my}(xe^{\sigma y}-K)e^{-\frac{(y-z)^2}{2(T-1)}}\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{z^2}{2}}-e^{-\frac{(z-2b)^2}{2}}\right)dzdy\\ =\ &x\phi(\sigma+m)-K\phi(m).\end{aligned}$$

For $T_b\le 1$, Z_{τ_b} is independent of τ_b. This is because of the strong Markov property of Brownian motion, and we can think of Z_{τ_b} as a Brownian meander, since it is a Brownian motion starting at b and conditioned to stay above b up to time 1. The value at time 1 of the Brownian meander is independent of the last hitting time of level b. Furthermore, we denote the density of Z_{τ_b} by $v(dz)$. The density of Z_{τ_b} is that of the Brownian meander and we have (see Yor [**?**] for more detail), and we have

$$\begin{aligned} v(dz) &= P(Z_{\tau_b} \in dz) \\ &= \frac{b-z}{2} e^{-\frac{(z-b)^2}{2}} \mathbf{1}_{\{z<b\}} dz + \frac{z-b}{2} e^{-\frac{(z-b)^2}{2}} \mathbf{1}_{\{z>b\}}. \end{aligned}$$

Hence,

$$\begin{aligned} & E_{\mathcal{P}} \left[\mathbf{1}_{\{T_b \le 1\}} \mathbf{1}_{\{\tau_b \le T\}} \int_{-\infty}^{\infty} e^{my} \left(x e^{\sigma y} - K \right)^+ \frac{1}{\sqrt{2\pi(T-\tau_b)}} e^{-\frac{(y-Z_{\tau_b})^2}{2(T-\tau_b)}} dy \right] \\ = & \int_0^T \int_{-\infty}^{\infty} f_b(t; T_b < 1) v(dz) \int_{-\infty}^{\infty} e^{my} \left(x e^{\sigma y} - K \right)^+ \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-z)^2}{2(T-t)}} dy dt \\ = & \int_0^T \int_{-\infty}^{b} \int_k^{\infty} f_b(t; T_b < 1) \frac{b-z}{2} e^{\frac{(b-z)^2}{2}} e^{my} (x e^{\sigma y} - K) \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-z)^2}{2(T-t)}} dy dz dt \\ & + \int_0^T \int_b^{\infty} \int_k^{\infty} f_b(t; T_b < 1) \frac{z-b}{2} e^{\frac{(z-b)^2}{2}} e^{my} (x e^{\sigma y} - K) \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-z)^2}{2(T-t)}} dy dz dt \\ = & \sqrt{\frac{\pi}{2}} \int_0^T \int_{-\infty}^{b} \int_k^{\infty} f_b(t; T_b < 1) \frac{b-z}{2} e^{\frac{(b-z)^2}{2}} e^{my} (x e^{\sigma y} - K) \frac{1}{2\pi\sqrt{T-t}} e^{-\frac{(y-z)^2}{2(T-t)}} dy dz dt \\ & + \sqrt{\frac{\pi}{2}} \int_0^T \int_{-\infty}^{b} \int_k^{\infty} f_b(t; T_b < 1) \frac{b-z}{2} e^{\frac{(b-z)^2}{2}} e^{my} (x e^{\sigma y} - K) \frac{1}{2\pi\sqrt{T-t}} e^{-\frac{(y-2b+z)^2}{2(T-t)}} dy dz dt \\ = & \sqrt{\frac{\pi}{2}} \int_0^T \int_{-\infty}^{b} \int_k^{\infty} f_b(t; T_b < 1) \frac{b-z}{2} e^{\frac{(b-z)^2}{2}} e^{my} (x e^{\sigma y} - K) \frac{1}{2\pi\sqrt{T-t}} e^{-\frac{(y-z)^2}{2(T-t)}} dy dz dt \\ & + \sqrt{\frac{\pi}{2}} \int_0^T \int_{-\infty}^{b} \int_k^{\infty} f_b(t; T_b < 1) \frac{b-z}{2} e^{\frac{(b-z)^2}{2}} e^{my} (x e^{\sigma y} - K) \frac{1}{2\pi\sqrt{T-t}} e^{-\frac{(y-2b+z)^2}{2(T-t)}} dy dz dt \\ = & \sqrt{\frac{\pi}{2}} \int_0^T f_b(t; T_b < 1) (x \psi(\sigma + m, h, b, \rho, t) - K \psi(m, h', b, \rho, t)) dt \\ & + \sqrt{\frac{\pi}{2}} \int_0^T f_b(t; T_b < 1) (x \psi(-(\sigma + m), h_b, b, -\rho, t) - K \psi(-m, h_b', b, -\rho, t)) dt, \end{aligned}$$

where the last step follows from:

$$
\begin{aligned}
& \sqrt{\frac{\pi}{2}}\int_0^T\int_{-\infty}^b\int_k^\infty f_b(t;T_b<1)\frac{b-z}{2}e^{\frac{(b-z)^2}{2}}e^{my}(xe^{\sigma y}-K)\frac{1}{2\pi\sqrt{T-t}}e^{-\frac{(y-2b+z)^2}{2(T-t)}}dydzdt \\
= & \sqrt{\frac{\pi}{2}}\int_0^T\int_{-\infty}^b\int_k^\infty f_b(t;T_b<1)e^{2b(\sigma+m)}\frac{b-z}{2}e^{\frac{(b-z)^2}{2}}xe^{(\sigma+m)(y-2b)}\frac{1}{2\pi\sqrt{T-t}}e^{-\frac{(y-2b+z)^2}{2(T-t)}}dydzdt \\
= & \sqrt{\frac{\pi}{2}}\int_0^T\int_{-\infty}^b\int_{k-2b}^\infty f_b(t;T_b<1)e^{2b(\sigma+m)}\frac{b-z}{2}e^{\frac{(b-z)^2}{2}}xe^{(\sigma+m)y}\frac{1}{2\pi\sqrt{T-t}}e^{-\frac{(y+z)^2}{2(T-t)}}dydzdt,
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{2\pi\sqrt{T-t}}\int_{-\infty}^b\int_k^\infty xe^{(\sigma+m)y}e^{-\frac{(y+z)^2}{2(T-t)}}(b-z)e^{-\frac{(z-b)^2}{2}}dzdy \qquad (6.7)\\
= & e^{\frac{(\sigma+m)^2(1+T-t)-2b(\sigma+m)}{2}}\frac{x}{2\pi\sqrt{T-t}}\int_{-\infty}^b\int_{k-2b}^\infty(b-z)\exp\{-\frac{(y+(b-(\sigma+m)(1+T-t)))^2}{2(T-t)}\} \qquad (6.8)\\
& \exp\{-\frac{(z-(b-(\sigma+m)))^2}{2(T-t)/(1+T-t)}\}\exp\{-\frac{2(y+(b-(\sigma+m)(1+T-t)))(z-(b-(\sigma+m)))}{2(T-t)}\}dydzdt \\
= & xe^{\frac{(\sigma+m)^2(1+T-t)-2b(\sigma+m)}{2}}\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\sigma+m}\int_{h_b}^\infty(-v+(\sigma+m))e^{-\frac{u^2+2\rho uv+v^2}{2}}dudv \qquad (6.9)\\
= & xe^{-2b(\sigma+m)}\psi(-(\sigma+m),h_b,b,-\rho,t).
\end{aligned}
$$

In the above, expression (6.3) is obtained from (6.2) by a manipulation of the exponents, and from expression (6.3) to (6.4), we have used the transformation $u=\frac{y+(b-(\sigma+m)(1+T-t))}{\sqrt{1+T-t}}$ and $v=z-(b-(\sigma+m))$. ■

6.2 Min-call-out Parisian Call

For the knock-out call with the same parameters, we have

$$
\begin{aligned}
C_i^{min}(x,T) &= E_{\mathcal{Q}}\left[\mathbf{1}_{\{\tau_b>t\}}(S_T-K)^+e^{-rT}\right] \\
&= E_{\mathcal{Q}}\left[\mathbf{1}_{\{T_b<1\}}(S_T-K)^+e^{-rT}\right]-E_{\mathcal{Q}}\left[\mathbf{1}_{\{T_b<1\}}\mathbf{1}_{\{\tau_b\le T\}}(S_T-K)^+e^{-rT}\right] \\
&= \int_0^1\frac{b}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}}C_{BS}(L,T-t)dt-\left(C_i^{min}(x,T)-(x\phi(\sigma+m)-K\phi(m))\right).
\end{aligned}
$$

6.3 Numerical results

The following table gives the prices of the two-sided Parisian option for different values of initial asset price S_0 and window length D, for parameters $K = 95$, $L = 90$, $T = 1$ year, $r = 0.05$ and $\sigma = 0.2$. These values are obtained using the recursive formula for $t \leq 4$, and for $t > 4$, the asymptotics is used.

Table 2: Price of Parisian min-in call

S_0	$D = 1$ week	$D = 2$ weeks	$D = 1$ month	$D = 2$ months
80	2.817708	2.809610	2.660829	2.123282
82	3.471103	3.430688	3.145066	2.482966
84	4.203278	4.101558	3.737759	3.096815
86	5.050461	4.978642	4.724678	4.261088
88	6.535228	6.639547	6.589191	6.342500
90	6.897115	6.895460	6.891562	6.872088.

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