

An axiomatization of difference-form contest success functions*

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Abstract

This paper presents an axiomatic characterization of difference-form contests, that is, contests where agents' probability of victory depends on the difference of their effective efforts. This axiomatization rests on a pairwise comparison axiom that relates the winning probabilities of any pair of participants to their winning probabilities in a contest between the two of them. The resulting difference-form contest success function overcomes some of the drawbacks of the widely-used ratio-form. Contrary to other difference-form functions, the family we characterize here can be scale invariant and have a positive elasticity of augmentation. By clarifying the properties of this family of contest success functions, this axiomatization can help researchers to find the functional form better suited to their application of interest.

Keywords: Contests, Groups, Contest success function, Axioms.

JEL codes: D31, D63, D72, D74.

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1 Introduction

Despite the relevance and ubiquity of contests in the real world, contest theory is often criticized for its great reliance on a particular construct: The Contest Success Function (Hirshleifer, 1989). This function maps the efforts made by contenders into their probability of attaining victory or, alternatively, their share of the contested prize. Critics argue that the contest success function (CSF henceforth) is too reduced form, too much of a black-box. For instance, the widely-used Tullock CSF (Tullock, 1967; 1980), under which success in the contest depends on relative efforts, might seem sensible. But there is no obvious reason why this functional form should govern most types of contests, from interstate wars to sport competitions.¹ Because of this, the predictions of contest theory might be seen as too reliant on very specific functional forms rather than on sound economic principles.

This view is somewhat unfair for two reasons: Firstly, because there are other areas of Economics where very specific functional forms are often assumed. Secondly, because there is an active and fruitful strand of research which in the last few years has provided foundations to the most frequently employed CSFs.² This literature has even addressed the econometric estimation of these functions.³ As a result of these efforts, economists have now at their disposal a growing menu of well-founded CSFs to choose from. The next natural question is which type of CSF is better suited to each specific application. A systematic study of the properties of the different families of CSFs can contribute to that aim.

One family of contests assumes that winning probabilities depend on the difference of contenders' efforts. These *difference-form* contests were introduced by Hirshleifer (1989; 1991) and explored later by Baik (1998) and Che and Gale (2000) for the case of two-player contests. Difference-form CSFs have been shown to emerge naturally in a number of settings. Gersbach and Haller (2009) show that a linear difference-form CSF is the result of intra-household bargaining when partners must decide how much time to devote to themselves or to their partner. Corchón and Dahm (2010) microfound a difference-form CSF as the result of a game where contenders are uncertain about the type of the contest designer; by interpreting the CSF as a share, they also show that the difference-form coincides with the claim-egalitarian bargaining solution. Corchón and Dahm (2011) obtain the

¹For excellent surveys of the contest literature see Corchon (2007) and Konrad (2009).

²These characterizations fall into four main categories: Axiomatic, stochastic, optimally-designed and microfounded (Jia, Skaperdas and Vaidya, 2013).

³For a detailed discussion of the econometric issues involved in the estimation of CSFs see Jia and Skaperdas (2011) and Jia et al. (2013).

difference-form as the result of a problem where the contest designer is unable to commit to a specific CSF once contenders have already exerted their efforts. Skaperdas and Vaydia (2012) derive a separable difference-form CSF in a Bayesian framework where contenders produce evidence stochastically in order to persuade an audience of the correctness of their respective views. Finally, Polishchuk and Tonis (2013) obtain a logarithmic difference-form CSF using a mechanism design approach when contestants have private information over their valuation of victory. In summary, it is fair to conclude that difference-form CSFs are well micro-founded. However, little is known about their properties and about how these differ from the properties of the more often used ratio-form CSFs, where winning probabilities are a function of the ratio of contenders' effective efforts.

The present paper offers an axiomatic characterization of difference-form CSFs. This axiomatization rests on a Pairwise Comparison axiom that describes the winning probabilities of any two participants in the contest as a function of their winning probabilities in the contest between the two of them. Under this axiom, if a contender has a zero winning probability in the grand contest, he/she can still have a positive probability of defeating another participant in a direct confrontation. This contrasts with the Consistency axiom employed in the characterizations of the ratio-form CSF. Under this axiom, a contender with no chance of winning the grand contest has no chance either of defeating another participant.

Our Theorem 1 shows that the Pairwise Comparison axiom, together with two other axioms already employed in the literature, characterize a separable difference-form CSF which generalizes the difference-form CSF introduced by Che and Gale (2000). This family of separable CSFs also encompasses as particular cases the ones micro-founded in the aforementioned literature as well as the ones employed by Levine and Smith (1995), Rohner (2006), Besley and Persson (2008, 2009) and Gartzke and Rohner (2011).

With our axiomatization, we help to clarify the properties of difference-form CSFs. The family we characterize is different from the logistic difference-form function introduced by Hirshleifer (1989; 1991) and later generalized by Baik (1998). Under the logistic CSF winning probabilities are proportional to contenders' exponential efforts. This functional form belongs to the ratio family, as it satisfies the Consistency axiom and not our Pairwise Comparison axiom. We also show that contrary to the logistic CSF and to the Baik (1998) CSF, our difference-form CSF can be scale invariant, i.e. homogeneous of degree zero, and that it can have a positive elasticity of augmentation.⁴

⁴A positive elasticity of augmentation (Hwang, 2012) implies that the difference between the winning probabilities of two contenders diminishes when their efforts increase

This paper contributes to the axiomatic work pioneered by Skaperdas (1996) and Clark and Riis (1998). Later, Münster (2009) extended this characterization from individual to group contests. Arbatskaya and Milalon (2009) and Rai and Sarin (2009) axiomatized multi-investment contests, whilst Blavatsky (2010) did the same for contests with ties. More recently, Hwang (2012) axiomatized the family of CSF with constant elasticity of augmentation, which encompasses the logistic and the ratio forms as particular cases. Lu and Wang (2015) characterized success functions for contests producing strict rankings of players, whereas Vesperoni (2013) axiomatized an alternative success function producing rankings of any type. Finally, Bozbay and Vesperoni (2014) characterized a CSF for conflicts embedded in network architectures. Let us add that in our axiomatization we make connections with the income inequality literature. The literature on inequality measurement offers valuable insights on the properties of functional forms which we employ at several points of the text.⁵

2 Axiomatization

Let us start by considering a group of $K \geq 2$ individuals indexed by $k = 1, \dots, K$. Denote the set of individuals by \mathbb{K} . These K agents are in competition. They are engaged in a contest which can have only one winner. In Section 4 we generalize our analysis to the case of group contests.

Contenders can expend non-negative effort in order to alter in their favor the outcome of the contest. Depending on the specific type of contest, these efforts can be money, time, physical effort or weapons. Denote by $\mathbf{x} \equiv (x_1, \dots, x_K) \in \mathbb{R}_+^K$ the vector of efforts made by these contenders and by \mathbf{x}_{-k} the vector of efforts made by contenders other than k .

Efforts determine the winning probability of each contender according to a Contest Success Function (CSF) $p_k : \mathbb{R}_+^K \rightarrow \mathbb{R}_+$. The function $p_k(\mathbf{x})$ can also be thought of as the share of the prize or object being contested that participant k obtains in case of victory. We favor the former interpretation throughout the paper.

2.1 Two basic axioms: Let us present the first two axioms we would like to impose on our CSF. They were introduced by Skaperdas (1996) in his axiomatization of CSFs for individual contests and later generalized by

whilst keeping their difference constant.

⁵In this same spirit, Chakravarty and Maharaaj (2014) characterize a new family of individual contests success functions which satisfy properties akin to the intermediate inequality and ordinal consistency axioms employed in the income distribution literature.

Münster (2009) to group contests. These axioms are rather natural and should apply to the class of difference-form contests we study in this paper.

Axiom 1 (Probability) $\sum_{k=1}^K p_k(\mathbf{x}) = 1$ and $p_k(\mathbf{x}) \geq 0$ for any \mathbf{x} and all $k \in \mathbb{K}$.

Axiom 2 (Monotonicity) Consider two effort levels x_k and x'_k such that $x'_k > x_k$. Then,

- (i) $p_k(x'_k, \mathbf{x}_{-k}) \geq p_k(x_k, \mathbf{x}_{-k})$, with strict inequality if $p_k(x_k, \mathbf{x}_{-k}) \in (0, 1)$.
- (ii) $p_l(x'_k, \mathbf{x}_{-k}) \leq p_l(x_k, \mathbf{x}_{-k})$ for all $l \neq k$ and $l \in \mathbb{K}$.

The axiom of Probability states that the CSF generates a probability distribution over the set of contenders (or a proper sharing rule). The Monotonicity axiom implies that the winning probability of a contender is weakly increasing in her effort and weakly decreasing in the effort of others. Note that this axiom is slightly weaker than the Monotonicity axiom in Skaperdas (1996) and the analogous one in Clark and Riis (1998).

2.2 Pairwise comparisons: The next axiom is crucial in our axiomatization. Denote by $p_k^{\{k,j\}}$ the winning probability of contender k in a bilateral contest against contender j .

Axiom 3 (Pairwise Comparison) For any effort vector \mathbf{x} and any contender $k \in \mathbb{K}$ with $K \geq 3$ and such that $p_k(\mathbf{x}) > 0$

$$\ln \frac{p_k^{\{k,j\}}(x_j, x_k)}{p_j^{\{k,j\}}(x_j, x_k)} = p_k(\mathbf{x}) - p_j(\mathbf{x}). \quad (1)$$

This axiom relates the winning probability of any pair of participants in a multilateral contest to their winning probabilities in a bilateral confrontation between them. The functional form corresponds to the Bradley-Terry model of pairwise comparisons (Bradley and Terry, 1952). This is a very popular approach to describe and estimate the winning probability or overall "strength" of a contender within a set (e.g. a tournament) whose elements are repeatedly compared with one another (e.g. fights, games). It is used by the World Chess Federation to rank players, it has been applied to sports (Agresti, 2002), rankings of academic journals (Stigler, 1994), animal behavior (Whiting et al, 2006) and models of gene transmission (Sham and Curtis, 1995).

2.3 Comparing axioms: The property above relates to two axioms frequently employed in the axiomatic characterization of CSFs.

Skaperdas (1996) and Munster (2009) employ an axiom of Independence stating that the outcome of any contest among a subset of participants should not depend on the efforts exerted by contenders outside the subset. Skaperdas (1996) points out that this property relates to the axiom of Independence of Irrelevant Alternatives in probabilistic individual choice. In fact, Rai and Sarin (2009) label this axiom as IIA. Our Pairwise Comparison axiom also implies a form of IIA but only in bilateral confrontations (e.g. sport matches, combat) where it would seem natural to hold.

Another axiom employed in the literature is Consistency (Skaperdas, 1996; Münster, 2009; Rai and Sarin, 2009). It states

Axiom 3' (Consistency) *For any vector \mathbf{x} and any subcontest $\mathbb{S} \subset \mathbb{K}$ such that $p_k(x) > 0$ for at least one $l \in \mathbb{S}$*

$$p_k^{\mathbb{S}}(\mathbf{x}) = \frac{p_k(\mathbf{x})}{\sum_{l \in \mathbb{S}} p_l(\mathbf{x})}. \quad (2)$$

This axiom posits that the winning probabilities of members of the smaller contest \mathbb{S} are proportional to their winning probability in \mathbb{K} . As pointed out by Clark and Riis (1998), Anonymity and Consistency together imply that the resulting CSF satisfies Luce's Choice Axiom. When applied to bilateral contests it implies

$$\frac{p_k^{\{k,j\}}(\mathbf{x})}{p_j^{\{k,j\}}(\mathbf{x})} = \frac{p_k(\mathbf{x})}{p_j(\mathbf{x})}.$$

However, this property presents some drawbacks. First, it is not well defined when $p_k(\mathbf{x}) = 0$ for all contenders in \mathbb{S} . Second, it forces contenders with zero probability in the grand contest to have a zero winning probability in any pair contest against another player. For instance, suppose that a contender k is very weak and has a zero winning probability in the grand contest, whereas contender l is marginally stronger and has a winning probability ε arbitrarily close to zero. Then contender k must have a zero winning probability in the bilateral contest against the similarly weak contender l . This may be undesirable in a number of applications, such as sport competitions where a team or an individual player may have no chance of winning a tournament but can still defeat other players.

Our Pairwise Comparison axiom does not make contenders with a zero winning probability in \mathbb{K} to have a zero winning probability in *all* pair contests. Still, we can compare both axioms when a contender has a winning probability arbitrarily close to zero. Take two contenders, one with winning probability $p \in (0, 1)$ and a weaker one with winning probability $\varepsilon \in (0, p)$.

Under Consistency, the latter would win their bilateral contest with probability $\frac{\varepsilon}{\varepsilon+p}$, which tends to zero as ε goes to zero. Under the Pairwise Comparison axiom though, the weaker contender would win the pair contest with probability $\frac{\exp\{\varepsilon\}}{\exp\{\varepsilon\}+\exp\{p\}}$ which tends to $\frac{1}{1+\exp\{p\}}$ as ε goes to zero.

The reader may argue that one drawback of the Pairwise Comparison axiom could be the following: Consider a scenario where one contender is much weaker than the rest of contenders, who are all equally strong. But because there is a large number of these strong agents, each of them enjoys a winning probability of just $q\varepsilon$ where $q > 1$ and ε is the (very small) winning probability of the weak contender. The Pairwise Comparison axiom would imply that the weak agent would have a winning probability of $\frac{1}{1+\exp\{(q-1)\varepsilon\}}$ in a bilateral contest against any of the strong agents. This seems an unrealistically large probability since this contender is substantially weaker than the rest. But as we will see below, the CSF that we axiomatize would bound to zero the winning probability of the weak contender; if this agent had a positive winning probability in the grand contest then it could not be much weaker than the rest of contenders.

2.4 The main theorem: We are now in the position to state our main theorem characterizing the family of the difference-form CSFs. This family emerges from using the basic axioms of Probability, Monotonicity and the Pairwise Comparison axiom.

Theorem 1 *If the CSF $p_k(\mathbf{x})$ is continuous and satisfies axioms A1-A3 then for each vector \mathbf{x} there exists a set \mathbb{K}^* of $K^* \leq K$ contenders such that*

$$p_k(\mathbf{x}) = \begin{cases} \frac{1}{K^*} + h_k(x_k) - \frac{1}{K^*} \sum_{l \in \mathbb{K}^*} h_l(x_l) & \text{for } k \in \mathbb{K}^* \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

where each $h_k : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuous and increasing function.

Proof. Denote by \mathbb{K}^* the set of contenders who enjoy a positive winning probability in the grand contest, that is, those A3 applies to. Let K^* be the cardinality of this set. Consider the case where at least three contenders j , k and l have strictly positive winning probabilities in the grand contest \mathbb{K} , i.e. $K^* \geq 3$. By A3

$$\begin{aligned} p_k(\mathbf{x}) - p_l(\mathbf{x}) &= p_k(\mathbf{x}) - p_j(\mathbf{x}) - (p_l(\mathbf{x}) - p_j(\mathbf{x})) \\ &= \ln \frac{p_k^{\{k,j\}}(x_k, x_j)}{1 - p_k^{\{k,j\}}(x_k, x_j)} - \ln \frac{p_l^{\{l,j\}}(x_l, x_j)}{1 - p_l^{\{l,j\}}(x_l, x_j)}, \end{aligned}$$

Now define for each $k \in \mathbb{K}^*$ a function $h_k : \mathbb{R}_+^2 \rightarrow \mathbb{R}$

$$h_k(x_k, x_j) = \ln \frac{p_k^{\{k,j\}}(x_k, x_j)}{1 - p_k^{\{k,j\}}(x_k, x_j)}. \quad (4)$$

This implies that

$$p_k(\mathbf{x}) - p_l(\mathbf{x}) = h_k(x_k, x_j) - h_l(x_l, x_j) \quad \text{for any } k, l \in \mathbb{K}^*. \quad (5)$$

Adding up across all contenders $l \in \mathbb{K}^*$ yields

$$K^* p_k(\mathbf{x}) - 1 = K^* h_k(x_k, x_j) - \sum_{l \in \mathbb{K}^*} h_l(x_l, x_j),$$

which yields

$$p_k(\mathbf{x}) = \frac{1}{K^*} + h_k(x_k, x_j) - \frac{1}{K^*} \sum_{l \in \mathbb{K}^*} h_l(x_l, x_j) \quad \text{for any } k, l \in \mathbb{K}^*. \quad (6)$$

We next need to show that the function $h_k(x_k, x_j)$ does not depend on x_j . Take contender k and l . By A3 and (5) we know that

$$h_k(x_k, x_j) - h_l(x_l, x_j) = \ln \frac{p_k^{\{k,l\}}(x_k, x_l)}{1 - p_k^{\{k,l\}}(x_k, x_l)}.$$

Since the right hand side is independent of x_j then the left hand side must also be independent of x_j . Define $h_k(x_k) = h_k(x_k, 0)$. Therefore we can write

$$h_k(x_k) - h_l(x_l) = \ln \frac{p_k^{\{k,l\}}(x_k, x_l)}{1 - p_k^{\{k,l\}}(x_k, x_l)} = p_k(\mathbf{x}) - p_l(\mathbf{x}). \quad (7)$$

So adding up again across $l \in \mathbb{K}^*$ yields the expression in (3).

Let us now show that the function $h_k(x_k)$ must be increasing. Consider a pair of vectors \mathbf{x}' and \mathbf{x} such that $\mathbf{x}' = (x_1, \dots, x'_k, \dots, x_K)$ where $x'_k > x_k$. That is, vector \mathbf{x}' is identical to vector \mathbf{x} except for contender k . By A2 it must be that $p_k(\mathbf{x}) \leq p_k(\mathbf{x}')$ and $p_l(\mathbf{x}) \geq p_l(\mathbf{x}')$ for any $l \neq k$. The property holds trivially if $p_k(\mathbf{x}) = 0$ or $p_l(\mathbf{x}) = 0$. If both probabilities $p_k(\mathbf{x})$ and $p_l(\mathbf{x})$ are strictly positive, expression (7) implies

$$\begin{aligned} h_k(x_k) - h_l(x_l) &= p_k(\mathbf{x}) - p_l(\mathbf{x}) \\ &\leq p_k(\mathbf{x}') - p_l(\mathbf{x}') = h_k(x'_k) - h_l(x_l), \end{aligned}$$

thus proving that $h_k(x_k)$ is increasing.

The last remaining step is to characterize the set \mathbb{K}^* . Define the set \mathbb{K}^* as the set of contenders in \mathbb{K} such that

1. $\frac{1}{K^*} + \min_{k \in \mathbb{K}^*} h_k(x_k) - \frac{1}{K^*} \sum_{l \in \mathbb{K}^*} h_l(x_l) > 0$;
2. $\frac{1}{K^*+1} + \max_{j \in \mathbb{K} \setminus \mathbb{K}^*} h_j(x_j) - \frac{1}{K^*+1} [\max_{j \in \mathbb{K} \setminus \mathbb{K}^*} h_j(x_j) + \sum_{l \in \mathbb{K}^*} h_l(x_l)] < 0$.

In other words, the set \mathbb{K}^* is formed by the K^* contestants with the largest impacts such that

$$\frac{1}{K^*} + h_k(x_k) - \frac{1}{K^*} \sum_{l=1}^{K^*} h_l(x_l) > 0, \quad (8)$$

holds for all of them and does not hold for any other contestant if it were to be included in \mathbb{K}^* . Next we show that a contender is a member of \mathbb{K}^* if and only if it has a positive winning probability.

Let us work out the first implication. By contradiction, suppose that $j \in \mathbb{K}^*$, where \mathbb{K}^* satisfies the two conditions above, but $p_j(\mathbf{x}) = 0$. Adding up (3) for the members of \mathbb{K}^* except j yields

$$\begin{aligned} 1 &= \frac{K^* - 1}{K^*} + \sum_{l \in \mathbb{K}^*, l \neq j} h_l(x_l) - \frac{K^* - 1}{K^*} \sum_{l \in \mathbb{K}^*} h_l(x_l) \\ &= \frac{K^* - 1}{K^*} - h_j(x_j) + \frac{1}{K^*} \sum_{l \in \mathbb{K}^*} h_l(x_l), \end{aligned}$$

implying that

$$\frac{1}{K^*} + h_j(x_j) - \frac{1}{K^*} \sum_{l \in \mathbb{K}^*} h_l(x_l) = 0,$$

thus contradicting that j belongs to \mathbb{K}^* . Therefore, all players in \mathbb{K}^* must have a positive winning probability.

Next, we need to prove the opposite implication, that is, that $p_j(\mathbf{x}) > 0$ implies that j must belong to \mathbb{K}^* . Suppose on the contrary that $j \notin \mathbb{K}^*$. If $p_j(\mathbf{x}) > 0$ then

$$\frac{1}{K^* + 1} + h_j(x_j) - \frac{1}{K^* + 1} \sum_{l \in \mathbb{K}^* \cup \{j\}} h_l(x_l) > 0,$$

thus contradicting that j does not belong to \mathbb{K}^* . This establishes that all contenders with positive winning probability must belong to \mathbb{K}^* . This together with the previous implication demonstrates that the set \mathbb{K}^* is the set of contenders with positive probability. ■

The function $h_k(x_k)$ is commonly known as the *impact function*. In individual contests, it can easily be interpreted as a function determining the effectiveness of contenders' raw efforts.

One obvious feature of this CSF is that it is additively separable in the impact of the contestants. The marginal productivity of individual efforts does not depend on the efforts of outsiders. This implies that any equilibrium in an individual contest under this CSF must be in dominant strategies under two sets of circumstances: 1) when contenders are risk neutral, so $p_k(\mathbf{x})$ can also be interpreted as a share; or 2) when individual utilities are non-linear and $p_k(\mathbf{x})$ is a winning probability. It is thus natural that Beviá and Corchón (2015) microfound this type of CSFs by means of dominant strategy implementation. Dominance solvability does not apply however when utilities are non-linear and $p_k(\mathbf{x})$ is instead a share of the prize contested, as in Levine and Smith (1995).⁶

The difference-form CSF in (3) relates the success of a contender to the difference between its impact and the average impact of all contenders. If the impact of a contender is above (below) average impact, its winning probability must be above (below) the probability of winning the contest under a fair lottery. This implies that if the impact of a contestant is positive but sufficiently low, its winning probability is zero. By the same token, a contender can attain a sure victory if her impact is sufficiently large. Take for instance, the case of two-player contests, i.e. $K = 2$, where (3) boils down to

$$p_k(\mathbf{x}) = \min \left\{ \max \left\{ \frac{1}{2} + \frac{1}{2}[h_k(x_k) - h_j(x_j)], 0 \right\}, 1 \right\}.$$

This CSF generalizes the linear difference-form CSF introduced by Che and Gale (2000) and later employed by Rohner (2006), Besley and Persson (2008, 2009) and Gartzke and Rohner (2011). Contender k can obtain victory with certainty if $h_k(x_k) \geq 1 + h_j(x_j)$. This shows that, as highlighted by Che and Gale (2000), the difference-form CSF has strong connections with auctions: contenders can obtain a sure win by outbidding others by a wide enough margin. On the other hand, a contender with zero impact can still enjoy a positive winning probability if the other contestants have moderate impacts. That would be the case for k in the example above if $h_j(x_j) < 1$.

⁶We thank Alberto Vesperoni for pointing this out.

3 Invariance

3.1 Scale invariance

In this section, we study two other properties employed in previous axiomatic characterizations of CSFs. These properties impose the invariance of winning probabilities to certain changes in the profile of contestants' efforts. The first one, and most-commonly used, is homogeneity of degree zero, which we refer to here as scale invariance.

Axiom 4 (Scale Invariance) *For all $\lambda > 0$ and all $k \in \mathbb{K}$*

$$p_k(\lambda \mathbf{x}) = p_k(\mathbf{x}).$$

This axiom states that winning probabilities must remain constant to equiproportional changes in all contenders' efforts. Scale invariance implies that units of measurement of effort do not matter. This is a desirable property when efforts are measured in money or military units (battalions, regiments, etc.). It is a property which is also satisfied by the indices of relative inequality introduced by Atkinson (1970). The interest of this analogy will become clear below.

Münster (2009) proved that if a CSF satisfies axioms A1, A2, A3', A4 together with an Independence axiom, impact functions must be all homogeneous of the same degree. Let us perform the analogous exercise in our setting and characterize the family of scale invariant difference-form CSFs.

Theorem 2 *If a CSF satisfies axioms A1-A4, then it satisfies (3) and the impact functions $h_k(x_k)$ satisfy*

$$h_k(x_k) = \alpha_k + \beta \ln x_k, \tag{9}$$

where α_k and $\beta > 0$ are parameters.

Proof. A4 implies that $p_k(\lambda \mathbf{x}) = p_k(\mathbf{x})$. Hence, if $p_k(\mathbf{x}) = 0$ then $p_k(\lambda \mathbf{x}) = 0$ and viceversa, so the set \mathbb{K}^* does not change.

In the next step of the proof we follow a similar procedure to the proof of Theorem 2 in Rai and Sarin (2009, p. 147). Take any two contenders $k, j \in \mathbb{K}^*$. By Theorem 1 and A4 their impact functions satisfy

$$h_k(\lambda x_k) - h_k(x_k) = h_j(\lambda x_j) - h_j(x_j) = \frac{1}{K^*} \left[\sum_{l \in \mathbb{K}} h_l(\lambda x_l) - \sum_{l \in \mathbb{K}} h_l(x_l) \right],$$

for any $x_k \in \mathbb{R}_{++}$. Since the last term in the above equality is just a constant, the difference $h_k(\lambda x_k) - h_k(x_k)$ is the same for all $k \in \mathbb{K}^*$ and we can conclude that this difference depends on λ but not on x_k . Hence it must hold true that

$$h_k(\lambda x_k) - h_k(x_k) = h_k(\lambda) - h_k(1).$$

Now add and subtract $h_k(1)$ to the left hand side of this expression and denote $H(x_k) = h_k(x_k) - h_k(1)$. It can then be rewritten as

$$H(\lambda x_k) = H(\lambda) + H(x_k).$$

If $x_k = t$ for $t > 0$ then

$$H(\lambda t) = H(\lambda) + H(t). \tag{10}$$

$H(\lambda)$ is a function of one variable and it is increasing since by Theorem 1 we know that $h_k(x_k)$ must be increasing. Expression (10) is one of the Cauchy functional equations whose only solution is given by $G(z) = \beta \ln z$ where β is an arbitrary constant (Aczél, 1966, p. 41). This implies

$$H(\lambda) = \ln \lambda^\beta,$$

and by the same token that

$$H(\lambda x_k) = \ln \lambda^\beta + H(x_k).$$

Given our definition of the function $H(\cdot)$, this implies that

$$h_k(\lambda x_k) = \ln \lambda^\beta + h_k(x_k).$$

By A4, β must be identical across participants.

Now, consider the case $x_k = 0$. Fix $x_j \neq 0$ for each $j \neq k$ in \mathbb{K}^* and $\lambda \neq 1$. Because $p_k(\lambda \mathbf{x}) = p_k(\mathbf{x})$ it must be that

$$\begin{aligned} & \frac{1}{K^*} + \frac{K^* - 1}{K^*} h_k(0) - \frac{1}{K^*} \sum_{l \in \mathbb{K}^*, l \neq k} (\ln \lambda^\beta + h_l(x_l)) \\ &= \frac{1}{K^*} + \frac{K^* - 1}{K^*} h_k(0) - \frac{1}{K^*} \sum_{l \in \mathbb{K}^*, l \neq k} h_l(x_l). \end{aligned}$$

This identity can only hold under two circumstances: First, if $\beta = 0$, which in turn implies that the impact functions must be homogeneous of degree zero, i.e. $h_k(\lambda x_k) = h_k(x_k)$, and thus constant. This violates A2. So we are left with the only other possible case, that is, $\lim_{x_k \rightarrow 0^+} h_k(x_k) = -\infty$.

Next define $F_k(x_k) = \exp\{h_k(x_k)\}$. It is clear that the function $F_k(x_k)$ is homogeneous of degree β since $F_k(\lambda x_k) = \lambda^\beta F_k(x_k)$. This is a function of one variable, which in turn must be a multiple of a power function, i.e. $F(s) = as^\beta$ with $\beta > 0$ and $a = F(1)$ (Münster, 2009; p 352). Hence it is possible to rewrite

$$F_k(x_k) = a_k \cdot (g_k(x_k))^\beta,$$

where $g_k : \mathbb{R}_+ \rightarrow \mathbb{R}$. The function $g_k(x_k)$ must be homogeneous of degree one since

$$F_k(\lambda x_k) = \lambda^\beta F_k(x_k) \Rightarrow g_k(\lambda x_k) = \lambda g_k(x_k).$$

Again, because $g(x_k)$ is a function of one variable and homogeneous of degree one, it must be of the form $g(s) = cs$ where $c > 0$.

Finally, tracing back our steps

$$h_k(x_k) = \ln F_k(x_k) = \ln a_k + \beta \ln g(x_k) = \alpha_k + \beta \ln cx_k.$$

Finally, observe that it must also be that $g_k(1) = 1$ given that

$$a_k = F_k(1) = \exp\{h_k(1)\} = a_k c^\beta,$$

implying that either $\beta = 0$ or $c = 1$. Since the former violates A2, it must be that $h_k(x_k) = \alpha_k + \beta \ln x_k$. Note that, as stated above, $\lim_{x_k \rightarrow 0^+} h_k(x_k) = -\infty$.

■

When impact functions are of the form in (9), changes in the unit of measurement of efforts do not change contenders' winning probabilities, albeit their impact changes in absolute terms. Therefore, despite the received wisdom (Skaperdas, 1996; Hirshleifer, 2000; Alcalde and Dahm, 2007, p. 103; Corchón, 2007, p. 74),⁷ difference-form CSFs do not necessarily violate Scale Invariance. Such violation would be undesirable if the difference between the efforts of two contenders is kept fixed, the weaker side is expected to be more likely to win as contenders' absolute efforts increase. More formally, $p_k^{\{k,j\}}(x_k, x_k + c)$ should be increasing in x_k , where $c > 0$. This property is called positive elasticity of augmentation by Hwang (2012). It is indeed not satisfied by either the logistic ratio-form (Hirshleifer, 1991), the Baik (1998) CSF or the linear CSF introduced by Che and Gale (2000). This is because these CSFs assume a linear mapping from effort to impact implying that

$$p_k(x_k, x_k + c) = p_k(x_k + t, x_k + t + c),$$

⁷“It might be thought a fatal objection against the difference form of the CSF that a force balance of 1,000 soldiers versus 999 implies the same outcome (in terms of relative success) as 3 soldiers versus 2! [...] Any reasonable provision for randomness would imply a higher likelihood of the weaker side winning the 1,000:999 comparison than in the 3:2 comparison.” (Hirshleifer, 2000, p 779)

for $t > 0$. This in turn implies that $p_k(x_k, x_k + c)$ is constant in x_k , so the elasticity of augmentation is zero.

Our Theorem 2 proves that our separable difference-form CSF can be scale invariant and have a positive elasticity of augmentation. More formally, when the impact function is of the form (9), winning probabilities under the family of difference-form CSFs in (3) satisfy

$$\begin{aligned} p_k(x_k, c + x_k) &= p_k(\lambda x_k, \lambda c + \lambda x_k) \\ &\leq p_k(\lambda x_k, c + \lambda x_k), \end{aligned}$$

for any $\lambda > 1$, where the last inequality follows from the Monotonicity axiom. Therefore, the weaker contestant has a higher winning probability as the efforts of the two contenders increase whilst keeping their absolute difference between them constant; that is, the elasticity of augmentation is positive.

To the best of our knowledge, this family of scale invariant difference-form CSFs has only been studied in Polishchuk and Tonis (2013, p. 218). They microfound a CSF of the form

$$p_k(\mathbf{x}) = \frac{1}{K} + \ln g(x_k) - \frac{1}{K} \sum_{l=1}^K \ln g(x_l),$$

by using a mechanism design approach when contenders are individuals who have private information over their valuation of victory.

3.2 Translation invariance

If a CSF is defined as a function of the difference between contenders' efforts, another natural invariance property is the following: Winning probabilities should remain constant when the effort of all contenders increase by the same amount. This is equivalent to the following property.

Axiom 5 (Translation Invariance) *For all $\lambda > 0$ and all $k \in \mathbb{K}$,*

$$p_k(\mathbf{x} + \lambda \cdot \mathbf{1}) = p_k(\mathbf{x}),$$

where $\mathbf{1} = (1, \dots, 1)$ is the vector of appropriate length whose components are all equal to one.

Skaperdas (1996) and Münster (2009) used this property as an alternative to homogeneity of degree zero in their axiomatization of ratio-form CSFs. Actually, Translation Invariance can be traced back to the income distribution literature, and in particular to the concept of absolute inequality introduced

by Kolm (1976a,b). Absolute inequality states that the level of inequality in a distribution should not vary when the income of every individual increases by the same fixed amount. Hence, any measure of absolute income inequality must be translation invariant.

As a matter of fact we borrow the following concept from the study of absolute inequality by Blackorby and Donaldson (1980).

Definition The impact function $h_k(x_k)$ is said to be *translatable* if

$$h_k(x_k+\lambda) = h_k(x_k) + \beta_k \lambda \quad \text{where } \beta_k, \lambda > 0.$$

We will refer to the scalar β_k as the degree of (linear) translatability of the impact function. Translatability is analogous to linear homogeneity when a fixed amount is added to the arguments of a function.

We are finally ready to state our theorem characterizing the family of translation invariant difference-form CSFs.

Theorem 3 *If a CSF satisfies axioms A1-A3 and A5, then it is of the form (3) and the impact functions $h_k(x_k)$ satisfy*

$$h_k(x_k) = \alpha + \beta x_k.$$

where $\alpha, \beta > 0$ are parameters.

Proof. A5 implies that $p_k(\mathbf{x} + \lambda \cdot \mathbf{1}) = p_k(\mathbf{x})$ so we can use the same reasoning as in the proof of Theorem 2 to establish that \mathbb{K}^* does not change.

Now, combining Theorem 1 with A5 for any $k, l \in \mathbb{K}^*$ we obtain,

$$h_k(x_k + \lambda) - h_k(x_k) = h_l(x_l + \lambda) - h_l(x_l).$$

Since this holds for any $l, k \in \mathbb{K}^*$, the difference in impacts must depend only on λ so

$$h_k(x_k + \lambda) - h_k(x_k) = \phi(\lambda),$$

where $\phi(\cdot)$ is a continuous function because it is equal to the difference of two continuous functions. This expression holds for any $\lambda > 0$ so

$$h_k(x_k + (\lambda + \mu)) = h_k(x_k + \lambda) + \phi(\mu) = h_k(x_k) + \phi(\lambda) + \phi(\mu),$$

implying that

$$\phi(\lambda + \mu) = \phi(\lambda) + \phi(\mu).$$

This is just the Cauchy functional equation whose only solution is of the form $\phi(\lambda) = \beta\lambda$ where $\beta > 0$ is an arbitrary real number. Now define $H(t) = \exp\{\phi(t)\}$. Then

$$H(x_k + \lambda) = \exp\{\beta(x_k + \lambda)\} = \exp\{\beta\lambda\}H(x_k).$$

Let $x_k = 0$. In that case, $H(\lambda) = \exp\{\beta\lambda\}H(0)$. Substituting λ by t yields

$$H(t) = a \exp\{\beta t\},$$

where $a = H(0) = \exp\{\phi(0)\}$. Now, tracing back our steps,

$$\phi(x_k) = \ln H(x_k) = \ln a + \beta x_k = \alpha + \beta x_k,$$

which completes the proof. ■

One remark is in order at this point: In his Theorem 3, Skaperdas (1996) characterizes the class of ratio-form CSF which are also translation invariant. He shows that for individual contests, this class boils down to the logistic difference-form CSF introduced by Hirshleifer (1989; 1991). Our Pairwise Comparison axiom does not lead to a logistic CSF but to a separable one, and with linear impact functions rather than exponential. Both the logistic and our separable CSF share the property that contenders with zero effort may still enjoy a positive winning probability. Contrary to our CSF, no participant under the logistic CSF can win the contest with certainty, no matter how much more effort he/she exerts compared to the rest of participants.

4 Group contests

Let us extend our previous analysis to group contests. Now consider that society is divided into $K \geq 2$ disjoint groups formed by a number $n_k \geq 1$ of individuals each, adding up to a total of N . Denote the set of groups by \mathbb{K} , by $\mathbf{x}_k \equiv (x_{1k}, \dots, x_{n_k k}) \in \mathbb{R}_+^{n_k}$ the vector of efforts made by members of group k and by \mathbf{x} the vector $(\mathbf{x}_1, \dots, \mathbf{x}_K)$. For convenience denote by \mathbf{x}_{-k} the vector of efforts in groups other than k . Impact functions must now aggregate members' efforts into a measure of group influence, i.e. $h_k : \mathbb{R}_+^{n_k} \rightarrow \mathbb{R}$.

All theorems above can be re-stated in this set-up after modifying Axioms A1 to A5 in order to account for groups.⁸ In this Section, we explore two sets of results that are specific to group contests. The first one refers to an important modification of the Translation Invariance axiom. The second set of results stems from imposing additional properties to the aggregation of members' efforts.

⁸These results are available from the authors upon request.

4.1 Group Translation Invariance

In his axiomatization of group CSFs, Münster (2009) employed the Translation Invariance property explored in Section 3.2 as an alternative to homogeneity of degree zero. However, this axiom builds in an implicit bias against big groups. Adding a constant λ to the effort of each member means that the total effort of each group increases by λn_k . Therefore, bigger groups increase their total effort more than smaller groups. Translation Invariance implies though that winning probabilities should remain invariant after that change. In order to correct this bias, we introduce the following axiom:

Axiom 6 (Group Translation Invariance) *For all $\lambda > 0$ and all $k \in \mathbb{K}$*

$$p_k(\mathbf{x}_1 + \frac{\lambda}{n_1} \cdot \mathbf{1}, \dots, \mathbf{x}_K + \frac{\lambda}{n_K} \cdot \mathbf{1}) = p_k(\mathbf{x}).$$

This property implies that if total effort increases across all groups by the same positive amount λ , because members increase their effort by a fix amount $\frac{\lambda}{n_k}$, winning probabilities should remain constant. Group Translation Invariance thus eliminates the bias against big groups implicitly built in the standard Translation Invariance property, a bias which has been so far overlooked by the literature.

The following Theorem characterizes the group impact functions satisfying Group Translation Invariance.

Theorem 4 *If a group CSF satisfies axioms A1-A3 and A6, then it is of the form (3) and each impact function $h_k(\mathbf{x}_k)$ is translatable of degree βn_k .*

Proof. The first part of the Theorem proceeds along the same lines as the proof of Theorem 1 in order to show that the CSF is of the form (3) but where each impact function is now a mapping $h_k : \mathbb{R}_+^{n_k} \rightarrow \mathbb{R}$.

Next, by A6 we know that

$$p_k\left(\left\{\mathbf{x}_j + \frac{\lambda}{n_j}\right\}_{j \in \mathbb{K}}\right) = p_k(\mathbf{x}) \quad \forall \lambda > 0 \text{ and } \forall k \in \mathbb{K}.$$

Take two vectors \mathbf{x}' and \mathbf{x} such that $\mathbf{x}' = (\mathbf{x}_1, \dots, \mathbf{x}'_k, \dots, \mathbf{x}_K)$. That is, vector \mathbf{x}' is identical to vector \mathbf{x} except for group k . Therefore

$$p_k\left(\left\{\mathbf{x}_j + \frac{\lambda}{n_j}\right\}_{j \in \mathbb{K}}\right) - p_k(\mathbf{x}) = p_k\left(\mathbf{x}'_k + \frac{\lambda}{n_k} \cdot \mathbf{1}, \left\{\mathbf{x}_j + \frac{\lambda}{n_j}\right\}_{j \neq k}\right) - p_k(\mathbf{x}'). \quad (11)$$

For the case where $k \in \mathbb{K}^*$ and the set \mathbb{K}^* is the same under \mathbf{x}_k and \mathbf{x}'_k , the combination of Theorem 1 and (11) implies

$$h_k(\mathbf{x}_k + \frac{\lambda}{n_k} \cdot \mathbf{1}) - h_k(\mathbf{x}_k) = h_k(\mathbf{x}'_k + \frac{\lambda}{n_k} \cdot \mathbf{1}) - h_k(\mathbf{x}_k).$$

Hence, the difference in impacts cannot depend on \mathbf{x}_k and \mathbf{x}'_k and must depend only on $\frac{\lambda}{n_k}$, implying that

$$h_k(\mathbf{x}_k + \frac{\lambda}{n_k} \cdot \mathbf{1}) - h_k(\mathbf{x}_k) = \psi_k(\frac{\lambda}{n_k}), \quad (12)$$

where $\psi_k(\cdot)$ is a continuous function because it is the difference of two continuous functions.

By the same token as in (11),

$$\begin{aligned} p_k(\left\{ \mathbf{x}_j + \frac{\lambda}{n_j} \right\}_{j \in \mathbb{K}}) - p_k(\mathbf{x}) &= p_l(\left\{ \mathbf{x}_j + \frac{\lambda}{n_j} \right\}_{j \in \mathbb{K}}) - p_l(\mathbf{x}) \Rightarrow \\ h_k(\mathbf{x}_k + \frac{\lambda}{n_k} \cdot \mathbf{1}) - h_k(\mathbf{x}_k) &= h_l(\mathbf{x}_l + \frac{\lambda}{n_l} \cdot \mathbf{1}) - h_l(\mathbf{x}_l) \\ \psi_k(\frac{\lambda}{n_k}) &= \psi_l(\frac{\lambda}{n_l}), \end{aligned} \quad (13)$$

for any two groups $k, l \in \mathbb{K}^*$.

On the other hand, because (12) holds for any $\lambda > 0$, then one can write

$$h_k(\mathbf{x}_k + \frac{\lambda + \mu}{n_k} \cdot \mathbf{1}) = h_k(\mathbf{x}_k + \frac{\lambda}{n_k} \cdot \mathbf{1}) + \psi_k(\frac{\mu}{n_k}) = h_k(\mathbf{x}_k) + \psi_k(\frac{\lambda}{n_k}) + \psi_k(\frac{\mu}{n_k}),$$

which implies that

$$\psi_k(\frac{\lambda + \mu}{n_k}) = \psi_k(\frac{\lambda}{n_k}) + \psi_k(\frac{\mu}{n_k}).$$

By induction, this property implies that $\psi_k(\lambda) = n_k \psi_k(\frac{\lambda}{n_k})$. Hence, $\psi_k(\lambda + \mu) = \psi_k(\lambda) + \psi_k(\mu)$. This is again the Cauchy functional equation whose solution is $\psi_k(\lambda) = \beta_k \lambda$. This together with (12) implies that

$$\psi_k(\frac{\lambda}{n_k}) = \frac{\psi_k(\lambda)}{n_k} = \frac{\beta_k \lambda}{n_k} = \frac{\beta_l \lambda}{n_l} = \frac{\psi_l(\lambda)}{n_l} = \psi_l(\frac{\lambda}{n_l}),$$

so $\beta_k = \beta n_k$ where β is an arbitrary positive scalar. Therefore, impact functions are translatable of degree βn_k because $h_k(\mathbf{x}_k + \frac{\lambda}{n_k} \cdot \mathbf{1}) = h_k(\mathbf{x}_k) + \beta n_k \lambda$. ■

For an example of a translation invariant difference-form group CSF, consider the following group impact function which we employ in a companion paper (Cubel and Sanchez-Pages, 2014):

$$h_k(\mathbf{x}_k) = n_k^\delta \ln \left(\frac{1}{n_k} \sum_{i=1}^{n_k} e^{-\gamma x_{ik}} \right)^{-\frac{1}{\gamma}} \quad \text{where } \gamma \geq 0 \text{ and } \delta \in \{0, 1\}.$$

This is the natural logarithm of a CES function with exponential efforts. The parameter γ measures the degree of complementarity of members' efforts. This function is linear when $\gamma = 0$. As $\gamma \rightarrow \infty$ it converges to the weakest-link technology (Hirshleifer, 1983). This function satisfies Translation Invariance when $\delta = 0$ and Group Translation Invariance when $\delta = 1$.

4.2 Aggregation

A distinctive feature of group contests is that impact functions must aggregate members' efforts. Because of that, the group contest versions of Theorems 2 and 3 are less tight than for individual contests.⁹ Further assumptions on the aggregation of efforts are necessary in order to obtain sharper characterizations of the group impact functions.

Consider the following axiom.

Axiom 7 (Intragroup Aggregation) *For any $k \in \mathbb{K}$ consider two effort vectors \mathbf{x}_k and \mathbf{x}'_k such that $\sum_{i=1}^{n_k} x_{ik} = \sum_{i=1}^{n_k} x'_{ik}$. Then it must be that*

$$p_k(\mathbf{x}_k, \mathbf{x}_{-k}) = p_k(\mathbf{x}'_k, \mathbf{x}_{-k}).$$

This axiom was introduced by Münster (2009) who calls it Summation. We changed its name in order to distinguish it better from the axiom we discuss below. Intragroup Aggregation implies that winning probabilities should remain invariant to changes in the distribution of efforts within groups which leave total group effort unchanged. In the context of lobbying or rent-seeking, where efforts are monetary, such property seems granted. Underlying this axiom is the assumption that efforts within groups are perfect substitutes, so the marginal productivity of individual effort does not depend on the effort made by other group members.

Let us now apply this property to our characterization.

Proposition 1 *If a group CSF satisfies axioms A1-A4 and A7, then it is of the form (3) and the group impact functions $h_k(\mathbf{x}_k)$ satisfy*

$$h_k(\mathbf{x}_k) = \alpha_k + \beta \ln\left(\frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}\right), \quad (14)$$

where α_k and $\beta > 0$ are parameters.

⁹In Theorem 2, it can be proven that group impact functions must satisfy $h_k(\mathbf{x}_k) = \alpha_k + \beta \ln g(\mathbf{x}_k)$ where $g(\mathbf{x}_k) : \mathbb{R}_+^{n_k} \rightarrow \mathbb{R}_+$ is increasing, homogeneous of degree one and satisfies $g(\mathbf{0}) = 0$ and $g(\mathbf{1}) = 1$. In the group contest version of Theorem 3, each $h_k(\mathbf{x}_k)$ must be translatable of the same degree $\beta > 0$.

Proof. Following the same steps as in Theorem 2, one can show that

$$h_k(\mathbf{x}_k) = \alpha_k + \beta \ln g(\mathbf{x}_k).$$

At that point we only need to prove that $g(\mathbf{x}_k) = \frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}$. By A7, we know that the group impact function can be expressed just as a function of the average group effort

$$h_k(\mathbf{x}_k) = h_k\left(\frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}, \dots, \frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}\right).$$

This together with expression (9), implies that it is possible to write $g(\mathbf{x}_k) = \phi\left(\frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}\right)$. Since by Theorem 2, $\phi\left(\frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}\right)$ must be homogeneous of degree one, and because it is a function of one variable, it must be a multiple of a power function. Hence,

$$\phi\left(\frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}\right) = c \frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik},$$

where by Theorem 2 again, $c = \phi(1) = g(\mathbf{1}) = 1$. This leads to the functional form (14). Note that this function $g(\mathbf{x}_k)$ satisfies also $g(\mathbf{0}) = 0$. ■

The addition of Intragroup Aggregation to our set of axioms produces a tighter characterization of the group impact function. Proposition 1 highlights once more that the difference-form CSF can be scale invariant when the function mapping members' efforts into impact is logarithmic.

Let us now turn our attention to the case of translation invariant CSFs:

Proposition 2 *If a CSF satisfies axioms A1-A3, A5 and A7, then it is of the form (3) and the impact functions $h_k(\mathbf{x}_k)$ satisfy*

$$h_k(\mathbf{x}_k) = \alpha_k + \beta_k \sum_{i=1}^{n_k} x_{ik}, \quad (15)$$

where α_k and $\beta_k > 0$ are parameters, and $\beta_k = \frac{\beta}{n_k}$ for all k . If A5 is replaced by A6, then $\beta_k = \beta$.

Proof. As established in the proof of Proposition 1, by A7 it is possible to write

$$h_k(\mathbf{x}_k + \lambda \cdot \mathbf{1}) = h_k\left(\frac{\sum_{i=1}^{n_k} x_{ik}}{n_k} + \lambda, \dots, \frac{\sum_{i=1}^{n_k} x_{ik}}{n_k} + \lambda\right) = \phi\left(\frac{\sum_{i=1}^{n_k} x_{ik}}{n_k} + \lambda\right).$$

By Theorem 3 and A5, we know that function $\phi_k(\cdot)$ must be translatable of degree β , that is, it must satisfy

$$\phi_k\left(\frac{\sum_{i=1}^{n_k} x_{ik}}{n_k} + \lambda\right) = \phi_k\left(\frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}\right) + \beta\lambda.$$

Now define $H_k(t) = \exp\{\phi_k(t)\}$. Then

$$H_k\left(\frac{\sum_{i=1}^{n_k} x_{ik}}{n_k} + \lambda\right) = \exp\{\beta\lambda\} H_k\left(\frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}\right).$$

Let $\mathbf{x}_k = \mathbf{0}$. In that case, $H_k(\lambda) = \exp\{\beta\lambda\} H_k(0)$. Substituting λ by t shows that it must be that

$$H_k(t) = a_k \exp\{\beta t\},$$

where $a_k = H_k(0) = \exp\{\phi_k(0)\}$.

Summarizing all the steps made so far,

$$h_k(\mathbf{x}_k) = \phi_k\left(\frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}\right) = \ln H_k\left(\frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}\right) = \ln a_k + \frac{\beta}{n_k} \sum_{i=1}^{n_k} x_{ik} = \alpha_k + \frac{\beta}{n_k} \sum_{i=1}^{n_k} x_{ik}.$$

If we employ A6 instead of A5, A7 implies that it is possible to write

$$h_k(\mathbf{x}_k \cdot \mathbf{1}) = h_k\left(\frac{\sum_{i=1}^{n_k} x_{ik}}{n_k} + \frac{\lambda}{n_k}, \dots, \frac{\sum_{i=1}^{n_k} x_{ik}}{n_k} + \frac{\lambda}{n_k}\right) = \phi\left(\frac{\sum_{i=1}^{n_k} x_{ik}}{n_k} + \frac{\lambda}{n_k}\right).$$

Recall that we know from Theorem 4 that the function $\phi_k(\cdot)$ must be translatable of degree βn_k , that is

$$\phi_k\left(\frac{\sum_{i=1}^{n_k} x_{ik}}{n_k} + \frac{\lambda}{n_k}\right) = \phi_k\left(\frac{\sum_{i=1}^{n_k} x_{ik}}{n_k}\right) + \beta\lambda.$$

Now define again $H_k(t) = \exp\{\phi_k(t)\}$ so

$$H_k\left(\frac{\lambda + \sum_{i=1}^{n_k} x_{ik}}{n_k}\right) = \exp\{\beta\lambda\} H_k\left(\frac{\sum_{i=1}^{n_k} x_{ik}}{n_k}\right).$$

Applying the same procedure when $\mathbf{x}_k = \mathbf{0}$, it must be the case that $H_k\left(\frac{\lambda}{n_k}\right) = \exp\{\beta\lambda\} H_k(0)$, so substituting $\frac{\lambda}{n_k}$ by t

$$H_k(t) = a_k \exp\{\beta n_k t\}$$

yields

$$h_k(\mathbf{x}_k) = \phi_k\left(\frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}\right) = \ln H_k\left(\frac{1}{n_k} \sum_{i=1}^{n_k} x_{ik}\right) = \ln a_k + \beta \sum_{i=1}^{n_k} x_{ik} = \alpha_k + \beta \sum_{i=1}^{n_k} x_{ik}.$$

■

Intragroup Aggregation plus Translation Invariance imply that impact functions must be linear and the resulting difference-form group CSF becomes fully separable.

One potentially undesirable consequence of the Intragroup Aggregation axiom is that the resulting CSFs can admit biases. Take for instance the linear impact in (15) for the case of two-group contests, i.e. $K = 2$. In that case the translation invariant difference-form CSF characterized in Proposition 2 boils down to

$$p_k(\mathbf{x}) = \min \left\{ \max \left\{ \frac{1}{2} + \frac{\alpha_k - \alpha_l}{2} + \frac{\beta}{2}(\bar{x}_k - \bar{x}_l), 0 \right\}, 1 \right\},$$

where \bar{x}_k denotes the average effort in group k . Note that group k has a head-start (handicap) whenever $\alpha_k > (<) \alpha_l$. The reason why the CSF admits this type of biases is because the Intragroup Aggregation axiom remains silent on the relative success of different groups with the same total effort. One possibility is to modify the axiom in order to account for this.

Axiom 8 (Intergroup Aggregation) *For any two groups $k, l \in \mathbb{K}$ such that $\sum_{i=1}^{n_k} x_{ik} = \sum_{i=1}^{n_l} x_{il}$ it must be that $p_k(\mathbf{x}) = p_l(\mathbf{x})$.*

This axiom is a stronger version of Intragroup Aggregation; it is actually a combination of Summation and the Between-Group Anonymity axiom in Münster (2009). It requires that two groups with the same total effort must have the same winning probability regardless of their size. Again, this property can make sense when efforts are monetary units, but less so when efforts represent time or when group size matters. For instance, the impact of a group of 10 people demonstrating for 100 hours may not be the same as the impact of a group of 1000 people demonstrating for an hour.

The following Proposition shows that when Intergroup Aggregation replaces Intragroup Aggregation, the bias described above vanishes.

Proposition 3 *If A7 is replaced by A8, then the impact functions characterized in Propositions 1 and 2 must satisfy $\alpha_k = \alpha$ for all $k \in \mathbb{K}$.*

Proof. It suffices to show that when A8 holds, impact functions, whatever their functional form, should be identical across groups. To see this note that

$$h_k(\mathbf{x}_k) = h_k\left(\frac{\sum_{i=1}^{n_k} x_{ik}}{n_k}, \dots, \frac{\sum_{i=1}^{n_k} x_{ik}}{n_k}\right),$$

because A8 also applies to changes in the distribution of efforts within groups which maintain total effort constant. Hence, for any vector \mathbf{x}_k it is possible to write the group impact as a function of its total effort, i.e. $h_k(\mathbf{x}_k) = \phi_k(\sum_{i=1}^{n_k} x_{ik})$. Similarly for group l , i.e. $h_l(\mathbf{x}_k) = \phi_l(\sum_{i=1}^{n_l} x_{il})$. From this it is clear that ϕ_k and ϕ_l are identical functions since by A8 they yield the same value whenever they are applied to the same argument. Hence, impact functions (14) and (15) must not differ across groups and $\alpha_k = \alpha$ for all $k \in \mathbb{K}$. ■

Intergroup Aggregation eliminates biases in favor of certain groups. Such biases can be desirable in some instances. For instance, when they are the result of affirmative action policies aimed at fostering the participation of disadvantaged groups (Franke, 2012). In other contests, such as when a social planner seeks to commit to a fair and impartial sharing rule (Corchon and Dahm, 2011), these biases should be removed.

A particularly interesting CSF emerges when the Intergroup Aggregation axiom and Scale Invariance are imposed: Denote by X_k the sum of efforts in group k . Then, the CSF characterized by Proposition 3 must be

$$p_k(\mathbf{x}) = \begin{cases} \frac{1}{K^*} + \beta \ln \frac{X_k}{G_X} & \text{for } k \in \mathbb{K}^* \\ 0 & \text{otherwise} \end{cases}$$

where $G_X = \left(\prod_{l \in \mathbb{K}^*} X_l\right)^{\frac{1}{K^*}}$ is the geometric mean of the sums of efforts of groups in \mathbb{K}^* .

5 Conclusion

In this paper, we have offered a normative study of contests where winning probabilities depend on the difference between contestants' effective efforts. Our axiomatic characterization rests on a Pairwise Comparison axiom, which relates winning probabilities of any pair of participants to their winning probabilities in their bilateral contest. One advantage of this axiom is that it does not bound contestants to have a zero probability of defeating any other participant when they have zero chances of winning the grand contest. Other difference-form CSFs, such as the logistic function (Hirshleifer, 1989, 1991) or the Baik (1998) CSF do not satisfy this axiom.

The difference-form CSF resulting from our characterization generalizes the functional form introduced by Che and Gale (2000) and later employed by Rohner (2006), Besley and Persson (2008, 2009) and Gartzke and Rohner (2011). This functional form has three distinctive features: 1) Impacts across contestants are separable, 2) it can award a sure victory to a contender who overpowers its rivals by a large enough margin and 3) it allows contenders to enjoy a positive winning probability when their impact is zero provided that other contenders are not too strong.

We showed that, contrary to other difference-form CSFs in the literature, our functional form can be homogeneous of degree zero, and that it does not force differences in winning probabilities to remain invariant when absolute differences in raw efforts remain constant, i.e. it can have a positive elasticity of augmentation.

In the last part of the paper, we explored group contests. We first flagged-up that the Translation Invariance property builds in an implicit bias against big groups which should be corrected. We then explored possible technologies of aggregation of efforts within groups. This helped us to sharpen our characterization of admissible group impact functions. We also showed that a modified version of the Summation axiom in Münster (2009) can unbiased the CSF, a desirable property in contexts where impartiality has a value.

The family of difference-form CSFs we characterized here has not been employed in the contest literature as often as other functional forms. We hope that, by clarifying its properties, our axiomatization can persuade researchers in the area to include this family of CSFs in their toolkit. Of course, our characterization is normative and leaves out strategic interactions. Che and Gale (2000) showed that their linear difference-form CSF often leads to mixed-strategy equilibria and that any equilibrium in pure-strategies involves at most one contender exerting positive effort. One possible next step would be to explore whether the equilibria of contests under the generalized difference-form CSF we axiomatized still presents such features. In addition, this form leads to dominant strategy equilibria in individual contests and in group contests when group impact functions are linear. We explore these issues in a companion paper (Cubel and Sanchez-Pages, 2014). Other avenues of further research might be the generalization of our characterization to the case of multiple-prizes and to provide microfoundations from the perspective of noise performance rankings as in Fu and Lu (2012).

References

- [1] Aczél, J. 1966. *Lectures on functional equations and their applications*. New York: Academic Press.
- [2] Alcalde, J. and Dahm, M. 2007. Tullock and Hirshleifer: a meeting of the minds. *Review of Economic Design*, 11(2), 101-124.
- [3] Arbatskaya, M. and Mialon, H.M. 2010. Multi-activity contests. *Economic Theory*, 43(1), 23-43.
- [4] Atkinson, A.B. 1970. On the measurement of inequality. *Journal of Economic Theory*, 2, 244-263.
- [5] Baik, K.H. 1998. Difference-form contest success functions and effort levels in contests. *European Journal of Political Economy*, 14(4), 685-701.
- [6] Besley, T.L. and Persson, T. 2008. The incidence of civil war: Theory and Evidence. NBER Working Paper No. 14585.
- [7] Besley, T.L. and Persson, T. 2009. Repression or civil war? *The American Economic Review: Papers and Proceedings*, 99(2), 292-297.
- [8] Beviá, C. and Corchón, L. 2015. Dominant strategy implementation in contests. Mimeo.
- [9] Blackorby, C. and Donaldson, D. 1980. A theoretical treatment of indices of absolute inequality. *International Economic Review*, 21, 107-136.
- [10] Blavatskyy, P.R. 2010. Contest success function with the possibility of a draw: axiomatization. *Journal of Mathematical Economics*, 46 (2), 267-276.
- [11] Bozbay, I. and Vesperoni, A. 2014. A contest success function for networks. Siegen University Working Paper 2014/05.
- [12] Bradley, R.A. and Terry, M.E. 1952. Rank analysis of incomplete block designs. I. The method of paired comparisons. *Biometrika*, 39, 324-345.
- [13] Chakravarty, S.R. and Maharaj, B. 2014. New contest success functions. ECINEQ Working paper 340.
- [14] Che, Y.-K. and Gale, I. 2000. Difference-form contests and the robustness of all-pay auctions. *Games and Economic Behavior*, 30, 22-43.

- [15] Clark, D.J. and Riis, C. 1998. Contest success functions: an extension. *Economic Theory*, 11, 201–204.
- [16] Corchón, L. 2007. The theory of contests: a survey. *Review of Economic Design*, 11, 69–100.
- [17] Corchón, L. and Dahm, M. 2010. Foundations for contest success functions. *Economic Theory*, 43(1), 81-98.
- [18] Corchón, L. and Dahm, M. 2011. Welfare maximizing contest success functions when the planner cannot commit. *Journal of Mathematical Economics*, 47(3), 309-317.
- [19] Cubel, M. and Sanchez-Pages, S. 2014. Difference-form group contests. IEB Discussion Paper 2014/13.
- [20] Franke, J. 2012. Affirmative action in contest games. *European Journal of Political Economy*, 28(1), 105-118.
- [21] Fu, Q. and Lu, J. 2012. Micro foundations for generalized multi-prize contests: A noisy ranking Perspective. *Social Choice and Welfare*, 38(3), 497-517.
- [22] Gartzke, E. and Rohner, D. 2011. The political economy of imperialism, decolonization and development. *British Journal of Political Science*, 41(3), 525-556.
- [23] Gersbach, H. and Haller, H. 2009. The affectionate society: does competition for partners promote friendliness? *Economic Theory*, 40(3), 389-403.
- [24] Hirshleifer, J. 1983. From weakest-link to best-shot: The voluntary provision of public goods. *Public Choice*, 41, 371-86.
- [25] Hirshleifer, J. 1989. Conflict and rent-seeking success functions: ratio vs. difference models of relative success. *Public Choice*, 63, 101–112.
- [26] Hirshleifer, J. 1991. The technology of conflict as an economic activity. *American Economic Review*, 81(2), 130-134.
- [27] Hirshleifer, J. 2000. The macrotechnology of conflict. *Journal of Conflict Resolution*, 44(6), 773-792.
- [28] Hwang, S.-H. 2012. Technology of military conflict, military spending, and war. *Journal of Public Economics*, 96(1–2), 226-236.

- [29] Jia, H. and Skaperdas, S. 2011. Technologies of conflict. In: Garfinkel, M.R., Skaperdas, S., eds., *Oxford Handbook of the Economics of Peace and Conflict*. Oxford University Press: Oxford.
- [30] Jia, H., Skaperdas, S., and Vaidya, S. 2013. Contest functions: theoretical foundations and issues in estimation. *International Journal of Industrial Organization*, 31(3), 211-222.
- [31] Kolm, S.-C. 1976a. Unequal inequalities I. *Journal of Economic Theory*, 12, 416-442.
- [32] Kolm, S.-C. 1976b. Unequal inequalities II. *Journal of Economic Theory*, 13, 82-111.
- [33] Konrad, K.A. 2009. *Strategy and dynamics in contests*. Oxford University Press: New York.
- [34] Levine, P. and Smith, R. 1995. The arms trade and arms control. *The Economic Journal*, 105, 471-484.
- [35] Lu, J. and Wang, Z. 2014. Axiomatizing multi-prize contests: a perspective from complete ranking of players. *Journal of Economic Behavior and Organization*, 116, 127-141.
- [36] Luce, R.D. 1959. *Individual choice behavior: A theoretical analysis*. Wiley: New York.
- [37] Münster, J. 2009. Group contest success functions. *Economic Theory*, 41(2), 345-357.
- [38] Polishchuk, L. and Tonis, A. 2013. Endogenous contest success functions: a mechanism design approach. *Economic Theory*, 52(1), 271-297.
- [39] Rai, B. and Sarin, R., 2009. Generalized contest success functions. *Economic Theory*, 40, 139-149.
- [40] Rohner, D. 2006. Beach holiday in Bali or East Timor? Why conflict can lead to under- and overexploitation of natural resources. *Economics Letters*, 92, 113-17.
- [41] Sham, P.C. and Curtis, D. 1995. An extended transmission/disequilibrium test (TDT) for multi-allele marker loci. *Annals of Human Genetics*, 59(3), 323-336.

- [42] Skaperdas, S. 1996. Contest success functions. *Economic Theory*, 7(2), 283-290.
- [43] Skaperdas, S. and Vaidya, S. 2012. Persuasion as a contest. *Economic Theory*, 51, 465–486.
- [44] Stigler, S.M. 1994. Citation patterns in the journals of statistics and probability. *Statistical Science*, 9, 94–108.
- [45] Tullock, G. 1967. The welfare costs of tariffs, monopolies and theft. *Western Economic Journal*, 5, 224-232.
- [46] Tullock, G. 1980. Efficient rent seeking. In: Buchanan, J.M., Tollison, R.D., and Tullock, G., eds., *Toward a Theory of Rent Seeking Society*, Texas A&M University Press.
- [47] Vesperoni, A. 2013. A contest success function for rankings. NEPS Working Paper 8/2013.
- [48] Whiting, M.J., Stuart-Fox, D.M., O'Connor, D., Firth, D., Bennett, N.C., and Blomberg, S.P. 2006. Ultraviolet signals ultra-aggression in a lizard. *Animal Behaviour*, 72, 353-363.