Determination of Economic Systems Behaviour under Uncertainty

A. I. Serguieva

Department of Systems and Control, Technical University-Sofia, 1756 Sofia, BULGARIA

Abstract[*][1]

This paper discuses systems of difference equations with fuzzy parameters and presents some solution procedures with the purpose to study the dynamic behaviour of economic systems in case of uncertainty. The trajectories of the endogenous variables are evaluated firstly at contiguous moments of time, and then, simultaneously. The relations between different solutions are shown. The author also consider essential to provide an algorithm for computing the exact α-cuts of the obtained solutions.

Keywords: *Econometrics; dynamics; systems of difference equations; nonlinear programming.*

1: Introduction

An essential number of economic models, especially macroeconomic models, represent systems of difference equations [9,10,18,19]. They are dynamic models and allow an examination of the economic systems development (in contrast to the static models and comparative static models). An economic model includes endogenous variables, control (policy) variables, and exogenous (uncontrollable) variables. Having the values of the exogenous variables available and giving some values to the control variables in an simultaneous difference equations model we will be able to obtain the trajectories of the endogenous variables and study the economic system behaviour.

It is well known that the macroeconomic models only approximate to the complex interrelations between a countless number of indices characterising the national economy. Within limits, it is also true for microeconomic models. Therefore, it is reasonable to assume that the models are uncertain. There are a lot of methods for uncertainty modelling. In this paper we suggest that the parameters of an economic model are uncertain use real fuzzy numbers with triangular membership functions instead of the crisp parameters or real-valued fuzzy variables with triangular possibility distributions, whose values are the possible parameters. The main result of these methods is that we can determine all trajectories of the endogenous variables using values of the parameters all at the same level of uncertainty. As the fuzzy analysis is essentially plural interval analysis, we believe it will be useful for the economic analists. We also provide a more informative crisp solution (than the ordinary crisp solution) for each level of uncertainty, in case the economic system authorities have no enough time to consider all possibilities.

Consequently, the problem of determination of economic systems behaviour under uncertainty turns into a problem of solving systems of difference equations with fuzzy parameters. Single equations (linear, quadratic or non-linear) with fuzzy parameters are discussed in [2,3,6,11,12,15]. In [7,14,17] single fuzzy differential equations are considered, as well. There have been also proposed some methods for solving systems of linear fuzzy equations [4,6], systems of non-linear fuzzy equations [8], and in [5,6] the author solves an exemplary system of two fuzzy differential equations. In this paper we extend the implementation of both the solution procedure based on the united extension and the solution procedure based on the possibility theory to the case of systems of fuzzy difference equations. First, we evaluate the values of the endogenous variables consecutively (at contiguous moments of time), and then, consider a simultaneous evaluation of the whole trajectory. We also apply the nonlinear programming to compute the exact solutions. In [13] the author uses the nonlinear programming to find the exact interval determinant of an interval matrix. In this paper it is applied with the purpose the exact α-cuts of the solution to be find.

2: Evaluating fuzzy trajectories
2.1: Problem statement

The structural form for an economic model consisting of a system of difference equations can be written as:

$$y_t = A_0 y_t + A_1 y_{t-1} + ... + A_K y_{t-K} + B_1 u_{t-1} + ... + $$
$$+ B_M u_{t-M} + C_1 z_{t-1} + ... + C_L z_{t-L} \quad , \qquad (1)$$

where y_{t-k} are n_1-vectors of endogenous variables appearing with lag $k = \overline{0, K}$, u_{t-m} are r_1-vectors of control variables appearing with lag $m = \overline{1, M}$, and z_{t-l} are s_1-vectors of exogenous variables appearing with lag $l = \overline{1, L}$. A_k are $n_1 \times n_1$ matrices, B_m are $n_1 \times r_1$ matrices, and C_l are $n_1 \times s_1$ matrices. The problem can be formulated as follows. In model (1) we know the crisp trajectories of the exogenous variables on interval [-L,T], the crisp trajectories of the endogenous variables on interval [-K,0], and the trajectories of the control variables on interval [-M,-1]. Giving some values to the control variables at moments t=0,...,T-1, we will consecutively obtain the values of the endogenous variables at moments t=1,...,t=T.

We can re-specify model (1) as a quasi first-order system.[10,16] This means defining new variables to replace those variables that appeared in the model with lags greater than one period and adding their definitional equations to the model. The new vectors are:

$$y1_t = y_{t-1}, \ y2_t = y_{t-2}, ..., \ y(K-1)_t = y_{t-K+1}, \ u1_t = u_{t-1}, ..., $$
$$u(M-1)_t = u_{t-M+1}, \ z1_t = z_{t-1}, ..., \ z(L-1)_t = z_{t-L+1} \quad . \qquad (2)$$

Now the model can be re-written as:

$$
\begin{bmatrix} y_t \\ y1_t \\ \vdots \\ y(K-1)_t \\ u1_t \\ \vdots \\ u(M-1)_t \\ z1_t \\ \vdots \\ z(L-1)_t \end{bmatrix}
=
\begin{bmatrix} A_0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & & \\ \vdots & \vdots & \cdots & \vdots \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}
\begin{bmatrix} y_t \\ y1_t \\ \vdots \\ y(K-1)_t \\ u1_t \\ \vdots \\ u(M-1)_t \\ z1_t \\ \vdots \\ z(L-1)_t \end{bmatrix}
+
$$

$$
+
\begin{bmatrix}
A_1 & A_2 & \cdots & A_{K-1} & A_K & B_2 & \cdots & B_{M-1} & B_M & C_2 & \cdots & C_{L-1} & C_L \\
I & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & I & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I & 0
\end{bmatrix}
\begin{bmatrix} y_{t-1} \\ y1_{t-1} \\ y2_{t-1} \\ \vdots \\ y(K-1)_{t-1} \\ u1_{t-1} \\ u2_{t-1} \\ \vdots \\ u(M-1)_{t-1} \\ z1_{t-1} \\ z2_{t-2} \\ \vdots \\ z(L-1)_{t-1} \end{bmatrix}
+
\begin{bmatrix} B_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_{t-1}
+
\begin{bmatrix} C_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix} z_{t-1}
\qquad (3)
$$

If we define:

$$x_t = [y_t ... z(L-1)_t]' \qquad (4)$$

and label appropriately the matrices in (3), we will obtain the following equation:

$$x_t = D_0 x_t + D_1 x_{t-1} + D_2 u_{t-1} + D_3 z_{t-1} \quad . \qquad (5)$$

Supposing $(I - D_0)^{-1}$ exists, the (5) can be re-written as:

$$x_t = (I - D_0)^{-1} D_1 x_{t-1} + (I - D_0)^{-1} D_2 u_{t-1} + (I - D_0)^{-1} D_3 z_{t-1} = $$
$$= A x_{t-1} + B u_{t-1} + C z_{t-1} \quad , \qquad (6)$$

where the dimension of vector x_t is $n = K n_1 + (M-1) r_1 + (L-1) s_1$, the dimension of matrix A is $n \times n$, B is an $n \times r_1$ matrix, and C is an $n \times s_1$ matrix. Practically, $n << K n_1 + (M-1) r_1 + (L-1) s_1$, because some of the endogenous variables might appear with maximum lags less than K, some of the control variables might appear with maximum lags less than M, and some of the exogenous variables with maximum lags less than L. The solution of the quasi first-order system (6) is:

$$x_t = A^t x_0 + \sum_{i=0}^{t-1} A^{t-i-1}(Bu_i + Cz_i) . \qquad (7)$$

Now, using model (7), we can give an equivalent formulation of the problem. The initial value of vector x_t, x_0, and the trajectory of vector z_t on interval $[0, T-1]$ are given, and the trajectory of vector x_t on interval $[1, T]$ is to be find. Various economic policies (trajectories of the control variables) can be examine to learn more about the dynamic behaviour of the system. An advantage of model (7) is that it allows a simultaneous evaluation of the whole trajectory of the endogenous variables as well as determination of a separate value of the endogenous vector, when we are interested only in a forecast at a definite future moment.

2.2: Introduction of the necessary notations

We want to present and solve the problem in case of uncertainty, which requires firstly to introduce some notations. Let $r = [r_1 ... r_p]'$ be a p-vector of the non-zero elements of matrices $A_0, ..., A_K, B_1, ..., B_M, C_1, ..., C_L$. We will denote the real fuzzy numbers to be substituted for the parameters r_i of the model as $\tilde{R}_i$, $i = \overline{1, p}$, and $\tilde{S}_{tj}$, $j = \overline{1, n_t}$, $t = \overline{1, T}$, will represent the obtained real fuzzy numbers for solutions. By analogy, the real-valued fuzzy variables whose values are to be parameters will be denoted as $\breve{R}_i$, and $\breve{S}_{tj}$ will represent the real fuzzy variables whose values are possible solutions. For simplicity, we employ triangular membership functions for fuzzy numbers $\tilde{R}_i$ and triangular possible distributions for fuzzy variables $\breve{R}_i$. We also choose

$$\text{Poss}[\breve{R}_i = r_i] = \mu(r_i | \tilde{R}_i) = (r_{i1}/r_{i2}/r_{i3}) , \qquad (8)$$

The $\tilde{R}_i(\alpha)$ will be called an α-cut of $\tilde{R}_i$ if $\tilde{R}_i(\alpha) = [\underline{r}_i(\alpha), \overline{r}_i(\alpha)] = \{r_i | \mu(r_i | \tilde{R}_i) \geq \alpha\}$ for $0 < \alpha \leq 1$ and $\tilde{R}_i(0)$ is the closure of the union of $\tilde{R}_i(\alpha)$, $0 < \alpha \leq 1$. If interrelation (8) holds, then

$$\tilde{R}_i(\alpha) = [\underline{r_i(\alpha)}, \overline{r_i(\alpha)}] = \{r_i | \text{Poss}[\breve{R}_i = r_i] \geq \alpha\}, 0 \leq \alpha \leq 1. \quad (9)$$

We adopt the definition of a real fuzzy number given in [1] and definition of a fuzzy vector given in [8]. If $\tilde{S}_{tj}$, $j = \overline{1, n_t}$, are real fuzzy numbers, then $\tilde{S}_t$, whose membership function is $\mu(y_t | \tilde{S}_t) = \min_{1 \leq j \leq n_t} \{\mu(y_{tj} | \tilde{S}_{tj})\}$, is a fuzzy vector. In contrast, if $\tilde{S}_t$ is a fuzzy vector, then its marginals $\tilde{S}_{tj}$, defined by their membership functions

$$\mu(y_{tj} | \tilde{S}_{tj}) = \sup_{\substack{y_{tq} \\ 1 \leq q \leq n_t, q \neq j}} \{\mu(y_t | \tilde{S}_t) | y_t = (y_{t1}, ..., y_{tj}, ..., y_{tn_t})\}, \quad j = \overline{1, n_t},$$

are real fuzzy numbers.

Finally, we will introduce the Cartesian product

$$\tilde{R}(\alpha) = \prod_{i=1}^{p} \tilde{R}_i(\alpha) , \quad 0 \leq \alpha \leq 1 . \qquad (10a)$$

and the joint possibility distribution

$$\pi_r = \min_{1 \leq i \leq p} \{\mu(r_i | \tilde{R}_i)\} . \qquad (10b)$$

In the next subsection we will find $\tilde{S}_t(\alpha)$ using $\tilde{R}(\alpha)$, which means to find all solutions for y_t using all the values of r_i at the same level of uncertainty, as the last ones belong to the sets $\{r_i | \text{Poss}[\breve{R}_i = r_i] \geq \alpha\}$.

2.3: Solutions

Supposing that both $(I - D_0)^{-1}$ and $(I - A_0)^{-1}$ exist for all $r_i \in \tilde{R}_i(0)$, $i = \overline{1, p_0}$, $p_0 < p$, where $(r_1, ..., r_{i p_0})$ are elements of A_0, we give eight solutions for the fuzzy trajectories of the endogenous variables and compare them. Then, for each level of uncertainty we propose a crisp solution which contains part of the information about the fuzzy solution, and consequently is more possible than the ordinary crisp solution.

First, we will substitute fuzzy numbers $\tilde{R}_i$ for r_i in models (1) and (7). In the fuzzified version of equation (1), the solution at moment $t \geq 2$ depends on both the fuzzy parameters and the solutions at moments $t-K, ..., t-1$ for $t > K$ (solutions at moments $1, ..., t-1$ for $t \leq K$). Therefore, we can write

$$y_{tj} = \begin{cases} f_{tj}(r), j = \overline{1, n_1}, t = 1 \\ f_{tj}(r, y_1, ..., y_{t-1}), j = \overline{1, n_1}, t = \overline{2, K} \\ f_{tj}(r, y_{t-K}, ..., y_{t-1}), j = \overline{1, n_1}, t = \overline{K+1, T} \end{cases}, \quad (11a)$$

$$y_t = \begin{cases} F_t(r), F_t = [f_{t1} ... f_{tn_t}]', t = 1 \\ F_t(r, y_1, ..., y_{t-1}), F_t = [f_{t1} ... f_{tn_t}]', t = \overline{2, K} \\ F_t(r, y_{t-K}, ..., y_{t-1}), F_t = [f_{t1} ... f_{tn_t}]', t = \overline{K+1, T} \end{cases} .(11b)$$

From equation (7) it follows that

$$x_{tj} = g_{tj}(r), j = \overline{1, n} , \qquad (12a)$$

$$x_t = G_t(r), G_t = [g_{t1}, ..., g_{tn}]' . \qquad (12b)$$

As the first n_t elements of vector x_t correspond to vector y_t, it is true that

$$y_{tj} = g_{tj}(r), j = \overline{1, n_t} , \qquad (13a)$$

$$y_t = G_{yt}(r), G_{yt} = [g_{t1}, ..., g_{t n_t}]' . \qquad (13b)$$

Let $\tilde{S}1_{tj}$, $\tilde{S}2_t$, $\tilde{S}3_{tj}$ and $\tilde{S}4_t$ be the united extension solutions of fizzified equations (11a), (11b), (13a) and (13b), respectively. Form Cartesian products:

$$\Theta 1_t(\alpha) = \tilde{R}(\alpha) \times \tilde{S}1_{t-K,1}(\alpha) \times \tilde{S}1_{t-K,n_1}(\alpha) \times ... \times \tilde{S}1_{t-1,n_1}(\alpha), (14a)$$

$$\Theta 2_t(\alpha) = \tilde{R}(\alpha) \times \tilde{S}2_{t-K}(\alpha) \times ... \times \tilde{S}2_{t-1}(\alpha) , \qquad (14b)$$

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$0 \le \alpha \le 1$, where $\tilde{S}1_{ij}(\alpha)=y_{ti}$ and $\tilde{S}2_t(\alpha)=y_t$ for $t<1$, and define solutions $\tilde{S}1_{ij}$, $\tilde{S}2_t$, $\tilde{S}3_{ij}$ and $\tilde{S}4_t$ by their membership functions

$$\mu(y_{ij}|\tilde{S}1_{ij})=\sup\{\alpha|y_{ij}\in \tilde{S}1_{ij}(\alpha)\}, \quad j=\overline{1,n_1}, \text{ where}$$

$$\tilde{S}1_{ij}(\alpha)=\{f_{ij}|(r,y_{t-1,1},...,y_{t-K,n_1})\in \Theta 1_t(\alpha)\},0\le\alpha\le1,j=\overline{1,n_1}, \quad (15a)$$

$$\mu(y_t|\tilde{S}2_t)=\sup\{\alpha|y_t\in \tilde{S}2_t(\alpha)\}, \text{ where}$$

$$\tilde{S}2_t(\alpha)=\{F_t|(r,y_{t-1},...,y_{t-K})\in \Theta 2_t(\alpha)\}, \ 0\le\alpha\le1, \quad (15b)$$

$$\mu(y_{ij}|\tilde{S}3_{ij})=\sup\{\alpha|y_{ij}\in \tilde{S}3_{ij}(\alpha)\}, \quad j=\overline{1,n_1}, \text{ where}$$

$$\tilde{S}3_{ij}(\alpha)=\{g_{ij}(r)|r\in \tilde{R}(\alpha)\}, 0\le\alpha\le1, j=\overline{1,n_1}, \quad (15c)$$

$$\mu(y_t|\tilde{S}4_t)=\sup\{\alpha|y_t\in \tilde{S}4_t(\alpha)\}, \text{ where}$$

$$\tilde{S}4_t(\alpha)=\{G_{yt}(r)|r\in \tilde{R}(\alpha)\}, 0\le\alpha\le1. \quad (15d)$$

As the function G_{yt} is continuous and $\tilde{R}_j$ are triangular fuzzy numbers, it can be easily proved that $\tilde{S}4_t$ is a fuzzy vector. It is also true that $\tilde{S}4_{ij}$ is a fuzzy number with membership function

$$\mu(y_{ij}|\tilde{S}4_{ij})= \sup_{\substack{y_{tq} \\ 1\le q\le n_1, q\ne j}} \{\mu(y_t|\tilde{S}4_t)|y_t=(y_{t1},...,y_{tj},...,y_{t n_1}\}, \quad (16)$$

$j=\overline{1,n_1}$. It is obvious that $\tilde{S}3_{ij}$ is a real fuzzy number and $\tilde{S}3_t$ is a fuzzy vector with membership function

$$\mu(y_t|\tilde{S}3_t)= \min_{1\le j\le n_1} \{\mu(y_{ij}|\tilde{S}3_{ij})\}. \quad (17)$$

If we evaluate $\tilde{S}1_{ij}$ ($\tilde{S}2_t$) considering $\tilde{S}1_{t-K,j},...,\tilde{S}1_{t-1,j}$ ($\tilde{S}2_{t-K},...,\tilde{S}2_{t-K}$) given, then it can be proved that $\tilde{S}1_{ij}$ is a fuzzy number ($\tilde{S}2_t$ is a fuzzy vector). Also, the $\tilde{S}1_t$ is a fuzzy vector with membership function

$$\mu(y_t|\tilde{S}1_t)= \min_{1\le j\le n_1} \{\mu(y_{ij}|\tilde{S}1_{ij})\}. \quad (18)$$

and $\tilde{S}2_{ij}$, $j=\overline{1,n_1}$, are fuzzy numbers with membership functions

$$\mu(y_{ij}|\tilde{S}2_{ij})= \sup_{\substack{y_{tq} \\ 1\le q\le n_1, q\ne j}} \{\mu(y_t|\tilde{S}2_t)|y_t=(y_{t1},...,y_{tj},...,y_{t n_1}\}. \quad (19)$$

As $\tilde{S}1_{ij}$ and $\tilde{S}2_t$ are evaluated consecutively, and $\tilde{S}3_{ij}$ and $\tilde{S}4_t$ are evaluated simultaneously, it is logical to obtain that $\tilde{S}1_{ij}\ge\tilde{S}3_{ij}$, $\tilde{S}1_t\ge\tilde{S}3_t$, $\tilde{S}2_t\ge\tilde{S}4_t$, and $\tilde{S}2_{ij}\ge\tilde{S}4_{ij}$. Furthermore, as $\tilde{S}1_{ij}$ ($\tilde{S}3_{ij}$) are usually interactive, the relation $\tilde{S}1_t\ge\tilde{S}2_t$ ($\tilde{S}3_t\ge\tilde{S}4_t$) holds.

Now model the uncertainty by real-valued fuzzy variables $\tilde{R}_j$ whose values are to be parameters r_j. Let $\tilde{S}5_{ij}$, $\tilde{S}6_t$, $\tilde{S}7_{ij}$, and $\tilde{S}8_t$ be fuzzy variables whose values are possible solutions for y_{ij} to equation (11a), for y_t to equation (11b), y_{ij} to equation (13a), and y_t to equation (13b), respectively. Then, form the joint possibility distributions

$$\pi_{S_5,t-K}= \min_{1\le j\le n_1} \{\mu(y_{t-K,j}|\tilde{S}5_{t-K,j})\},..,\pi_{S_5,t-1}= \min_{1\le j\le n_1} \{\mu(y_{t-1,j}|\tilde{S}5_{t-1,j})\},$$

$$\pi_{S_5}=\min\{\pi_t,\pi_{S_5,t-K},...,\pi_{S_5,t-1}\}, \quad (20a)$$

$$\pi_{S_6}=\min\{\pi_t,\mu(y_{t-K}|\tilde{S}6_{t-K}),...,\mu(y_{t-1}|\tilde{S}6_{t-1})\}, \quad (20b)$$

where

$$\mu(y_{t-K,j}|\tilde{S}5_{t-K,j})=Poss[\breve{S}5_{t-K,j}=y_{t-K,j}],...,$$

$$\mu(y_{t-1,j}|\tilde{S}5_{t-1,j})=Poss[\breve{S}5_{t-1,j}=y_{t-1,j}], j=\overline{1,n_1}, \quad (21a)$$

$$\mu(y_{t-K}|\tilde{S}6_{t-K})=Poss[\breve{S}6_{t-K}=y_{t-K}],...,$$

$$\mu(y_{t-1}|\tilde{S}6_{t-1})=Poss[\breve{S}6_{t-1}=y_{t-1}], \quad (21b)$$

and define

$$Poss[\breve{S}5_{ij}=y_{ij}]=\sup\{\pi_{S_5}|(11a) \text{ hold}\}, \quad (22a)$$

$$Poss[\breve{S}6_t=y_t]=\sup\{\pi_{S_6}|(11b) \text{ hold}\}, \quad (22b)$$

$$Poss[\breve{S}7_{ij}=y_{ij}]=\sup\{\pi_t|(13a) \text{ hold}\}, \quad (22c)$$

$$Poss[\breve{S}8_t=y_t]=\sup\{\pi_t|(13b) \text{ hold}\}. \quad (22d)$$

It is not difficult to prove that

$$Poss[\breve{S}5_{ij}=y_{ij}]=\mu(y_{ij}|\tilde{S}1_{ij}), \quad (23a)$$

$$Poss[\breve{S}6_t=y_t]=\mu(y_t|\tilde{S}2_t), \quad (23b)$$

$$Poss[\breve{S}7_{ij}=y_{ij}]=\mu(y_{ij}|\tilde{S}3_{ij}), \quad (23c)$$

$$Poss[\breve{S}8_t=y_t]=\mu(y_t|\tilde{S}4_t). \quad (23d)$$

In this paper we do not consider solutions based on the extension principle, because the extension principle solution to equation (1) may not exist, and the extension principle solutions to equations (11a), (11b), (13a) and (13b) will be equal to $\tilde{S}1_{ij}$, $\tilde{S}2_t$, $\tilde{S}3_{ij}$ and $\tilde{S}4_t$, respectively. We also do not present solutions based on α-cuts and interval arithmetic, because they are essentially only a way for approximate computation.

Consequently, we can give the following four solutions for the fuzzy trajectories of the endogenous variables:

$$\tilde{S}1=\bigcup_{t=1}^{T} \{t x \tilde{S}1_t\}, \ \tilde{S}2=\bigcup_{t=1}^{T} \{t x \tilde{S}2_t\},$$

$$\tilde{S}3=\bigcup_{t=1}^{T} \{t x \tilde{S}3_t\}, \ \tilde{S}4=\bigcup_{t=1}^{T} \{t x \tilde{S}4_t\}. \quad (24)$$

These solutions are very informative and helpful to economics analysts, but the authorities taking economic decisions usually prefer forecasts to be presented in a simpler form. The problem is to propose a simple prognosis which still provides enough information. Let S_α be this informative solution. If $\tilde{S}$ is one of the solutions given in (24) with α-cuts

$$\tilde{S}(\alpha)=[\underline{y}(\alpha),\overline{y(\alpha)}]=\bigcup_{t=1}^{T} \{t x \tilde{S}_t(\alpha)\}=$$

$$=\bigcup_{t=1}^{T} \{t x [\underline{y_t}(\alpha),\overline{y_t(\alpha)}], 0\le\alpha\le1,$$

then define

$$S_\alpha=H(\tilde{S}(\alpha)), 0\le\alpha\le1, \text{ where}$$

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$$H: \underbrace{I(R)^{n_1} x...xI(R)^{n_1}}_{T} \rightarrow \underbrace{R^{n_1} x...xR^{n_1}}_{T} \quad , \qquad (25)$$

(I(R) is the set of all real interval numbers). Now for each level of uncertainty there is a crisp solution. We must say that when α increases, the level of uncertainty decreases. The H can takes different forms. For example, S_α can be defined by:

$$S_\alpha = \bigcup_{t=1}^{T} \{txS_{\alpha t}\}, \ 0 \le \alpha \le 1, \text{ where}$$

$$S_{\alpha t} = \frac{\alpha}{2\alpha+1} \underline{y_t(\alpha)} + \frac{1}{2\alpha+1} y_t(1) + \frac{\alpha}{2\alpha+1} \overline{y_t(\alpha)}. \quad (26)$$

2.4: Evaluating the exact solutions

Now the exact interval $\tilde{S}3_{ij}(\alpha) = [\underline{y_{ij}(\alpha)}, \overline{y_{ij}(\alpha)}]$ will be computed. For each $\alpha \in [0,1]$, we consider a nonlinear programme

Minimize (Maximize) $y_{ij} = g_{ij}(r_1,...,r_p)$

subject to $v_i(r_1,...,r_p) = r_i - \underline{r_i(\alpha)} \ge 0, \ i = \overline{1,p}$

and $v_{p+i}(r_1,...,r_p) = r_i - \overline{r_i(\alpha)} \le 0, i = \overline{1,p}$.

The Kuhn-Tucker conditions for this problem takes form

$\lambda_i \ge 0 \quad (\lambda_i \le 0), \quad i = \overline{1,p}$,

$\lambda_{p+i} \le 0 \quad (\lambda_{p+i} \ge 0), \quad i = \overline{1,p}$,

$\dfrac{\partial g_{ij}(r_1,...,r_p)}{\partial r_i} - \sum_{w=1}^{2p} \lambda_w \dfrac{\partial v_w(r_1,...,r_p)}{\partial r_i} = 0, \quad i = \overline{1,p}$,

$\lambda_w v_w(r_1,...,r_p) = 0 , \quad w = \overline{1,2p}$,

$v_i(r_1,...,r_p) \ge 0 , \quad i = \overline{1,p}$,

$v_{p+i}(r_1,...,r_p) \le 0, \quad i = \overline{1,p}$,

where $\lambda_i \ge 0$ and $\lambda_{p+i} \le 0$ ($\lambda_i \le 0$ and $\lambda_{p+i} \ge 0$) correspond to the minimization (maximization problem). As the feasible region is a convex set and all the constraints are linear, the constraint qualification are invariably met and the Kuhn-Tucker conditions [9] are necessary conditions for an extremum. They will be also sufficient for a minimum (maximum) if the objective function $g_{ij}(r_1,...,r_p)$ is convex (concave) in the feasible region.

Therefore, we can proceed in the following way:

1. Evaluate $\dfrac{\partial^2 g_{ij}(r_1,...,r_p)}{\partial r_i}$, $i = \overline{1,p}$.

2. If $\dfrac{\partial^2 g_{ij}(r_1,...,r_p)}{\partial r_i^2} \ge 0$ for all $r \in \tilde{R}(\alpha)$, $i = \overline{1,p}$,.then

   2.1. find $\underline{y_{ij}(\alpha)}$ by solving the minimization problem.

   2.2. Evaluate $g_{ij}(r_1,...,r_p)$ in both point $(\underline{r_1(\alpha)},..,\underline{r_p(\alpha)})$ and point $(\overline{r_1(\alpha)},...,\overline{r_p(\alpha)})$.

2.2.1. If $g_{ij}(\underline{r_1(\alpha)},..,\underline{r_p(\alpha)}) \ge g_{ij}(\overline{r_1(\alpha)},...,\overline{r_p(\alpha)})$, then $\overline{y_{ij}(\alpha)} = g_{ij}(\underline{r_1(\alpha)},..,\underline{r_p(\alpha)})$.

2.2.2. If $g_{ij}(\underline{r_1(\alpha)},..,\underline{r_p(\alpha)}) \le g_{ij}(\overline{r_1(\alpha)},...,\overline{r_p(\alpha)})$, then $\overline{y_{ij}(\alpha)} = g_{ij}(\overline{r_1(\alpha)},...,\overline{r_p(\alpha)})$.

3. If $\dfrac{\partial^2 g_{ij}(r_1,...,r_p)}{\partial r_i^2} \le 0$ for all $r \in \tilde{R}(\alpha)$, $i = \overline{1,p}$, then

   3.1. solve the maximization problem. Its solution is equal to $\overline{y_{ij}(\alpha)}$.

   3.2. Evaluate $g_{ij}(r_1,...,r_p)$ in both point $(\underline{r_1(\alpha)},..,\underline{r_p(\alpha)})$ and point $(\overline{r_1(\alpha)},...,\overline{r_p(\alpha)})$.

3.2.1. If $g_{ij}(\underline{r_1(\alpha)},..,\underline{r_p(\alpha)}) \le g_{ij}(\overline{r_1(\alpha)},...,\overline{r_p(\alpha)})$, then $\underline{y_{ij}(\alpha)} = g_{ij}(\underline{r_1(\alpha)},..,\underline{r_p(\alpha)})$.

3.2.2. If $g_{ij}(\underline{r_1(\alpha)},..,\underline{r_p(\alpha)}) \ge g_{ij}(\overline{r_1(\alpha)},...,\overline{r_p(\alpha)})$, then $\underline{y_{ij}(\alpha)} = g_{ij}(\overline{r_1(\alpha)},...,\overline{r_p(\alpha)})$.

4. If the sign of $\dfrac{\partial^2 g_{ij}(r_1,...,r_p)}{\partial r_i^2}$ changes in the feasible region, then the exact interval $\tilde{S}3_{ij}(\alpha)$ can not be find. We have to apply interval arithmetic and compute an approximate solution.

Example. We will consider the simple economic model below.

$C_t = r_1 C_{t-1} + r_2 Y_t + r_3 Y_{t-1}$

$I_t = r_4 I_{t-1} + r_5 Y_{t-2}$

$Y_t = C_t + I_t + G_{t-2}$,

where consumption C, total investment I, and national income Y are endogenous variables and government expenditures G is a control variable. We need to define only two new variables, $Y1_t = Y_{t-1}$ and $G1_t = G_{t-1}$, to present this model as a quasi first-order system:

$$\begin{bmatrix} C_t \\ I_t \\ Y_t \\ Y1_t \\ G1_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & -r_2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} r_1 & 0 & r_3 & 0 & 0 \\ 0 & r_4 & 0 & r_5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{t-1} \\ I_{t-1} \\ Y_{t-1} \\ Y1_{t-1} \\ G1_{t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 1 & 1-r_2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ G_{t-1} \end{bmatrix}$$

The initial data are

$G_{-1} = 52.5$, $G_0 = 52.5$, $G_1 = 52.8$, $G_2 = 53.1$, $C_0 = 286.7$,

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$I_0=47.7$, $Y_{-1}=385.8$, $Y_0=386.6$.

Let us now evaluate the exact fuzzy trajectory of C_t. First define functions $g_{t1}(r_1,..,r_5)$, $t=\overline{1,3}$:

$g_{11}=(286.7r_1+52.5r_2+386.6r_3+47.7r_2r_4+385.8r_2r_5)/(1-r_2)$,

$g_{21}=(52.5r_2+47.7r_2r_4^2+386.6r_2r_5+385.8r_2r_4r_5)/(1-r_2)+$
$\quad +(286.7r_1^2+52.5r_1r_2+52.5r_3+673.3r_1r_3+386.6r_3^2+$
$\quad +47.7r_1r_2r_4+47.7r_3r_4+385.8r_1r_2r_5+385.8r_3r_5)/(1-r_2)^2$,

$g_{31}=(52.8r_2+47.7r_2r_4^3+386.6r_2r_4r_5+385.8r_2r_4^2r_5)/(1-r_2)+$
$\quad +(52.5r_1r_2+52.5r_3+47.7r_3^2r_4+47.7r_1r_2r_4^2+47.7r_3r_4^2+$
$\quad +52.5r_2r_5+673.3r_1r_2r_5+386.6r_3r_5+386.6r_2r_3r_5+47.7r_2r_4r_5+$
$\quad +385.8r_1r_2r_4r_5+385.8r_3r_4r_5+385.8r_2r_5^2)/(1-r_2)^2+(286.7r_1^3+$
$\quad +52.5r_1^2r_2+52.5r_1r_3+960r_1^2r_3+52.5r_1r_2r_3+52.5r_3^2+$
$\quad +1059.9r_1r_3^2+386.6r_3^3+47.7r_1^2r_2r_4+47.7r_1r_3r_4+47.7r_1r_2r_3r_4$
$\quad +47.7r_3^2r_4+385.8r_1^2r_2r_5+385.8r_1r_3r_5+385.8r_1r_2r_3r_5+$
$\quad +385.8r_3^2r_5)/(1-r_2)^3$.

One may check that $\dfrac{\partial^2 g_{t1}(r_1,...,r_5)}{\partial r_i^2}\geq 0$, $i=\overline{1,5}$, $t=\overline{1,3}$, if $r_1\geq 0$, $1\geq r_2\geq 0$, $r_3\geq 0$, $r_4\geq 0$, and $r_5\geq 0$, which does not contradict to their economic meaning.

Let $\tilde{R}_1=(0.7752/0.9690/1.3566)$,
$\quad \tilde{R}_2=(0.01896/0.0237/0.03318)$,
$\quad \tilde{R}_3=(0.00512/0.0064/0.00896)$,
$\quad \tilde{R}_4=(0.81208/1.0151/1.42114)$,
$\quad \tilde{R}_5=(0.00008/0.0001/0.00014)$.

The algorithm given above will produce

α	0	0.3	0.6	1
$\underline{C_1(\alpha)}$	230.327	248.021	265.775	289.541
$\overline{C_1(\alpha)}$	409.998	373.574	337.397	289.541
$C_{1\alpha}$	289.541	297.512	315.912	289.541
$\underline{C_2(\alpha)}$	185.302	214.759	265.775	292.406
$\overline{C_2(\alpha)}$	585.318	486.136	337.397	292.406
$C_{2\alpha}$	292.406	314.171	297.413	292.406
$\underline{C_3(\alpha)}$	149.344	186.159	228.743	295.301
$\overline{C_3(\alpha)}$	834.624	631.996	466.185	295.301
$C_{3\alpha}$	295.301	361.237	323.754	295.301

The informative crisp solution is also shown in the table. We have used the computing environment MATHEMATICA and the Optimization Toolbox of package MATLAB WITH SIMULINK to realize the algorithm.

By analogy, a computing procedure for $\tilde{S}1_{tj}$ can be derived.

3: Conclusions

Solving systems of fuzzy difference equations, one can study economic systems dynamics in case of uncertainty. Some methods for evaluating the trajectories of economic variables are presented in the paper. An algorithm for computing the exact α-cuts of the derived solutions is proposed. Further investigations to practical applications will be provided. An extension of the algorithm to solutions $\tilde{S}4_t$ and $\tilde{S}2_t$. is also under consideration.

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