Recognising a Partitionable Simplicial Complex is in $\mathcal{NP}$.

S. D. Noble*
Mathematical Institute
24–29 St. Giles
Oxford OX1 3LB.

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Abstract

We show that the problem of recognising a partitionable simplicial complex is a member of the complexity class $\mathcal{NP}$, thus answering a question raised in [1].

1 Introduction and Definitions

A *simplicial complex*, or just *complex*, consists of a pair $(E, \mathcal{F})$ where E is a finite set and $\mathcal{F}$ is a family of subsets of E such that if $V \in \mathcal{F}$ and $U \subseteq V$ then $U \in \mathcal{F}$. Inclusionwise maximal members of $\mathcal{F}$ are called *facets*. The *dimension* of a complex is the cardinality of its largest facet. A complex with facets $F_1, \ldots, F_n$ is said to be *partitionable* if there exists a sequence, $\phi(F_1), \ldots, \phi(F_n)$, of subsets of E such that for any $U \in \mathcal{F}$ there is a unique facet F satisfying $\phi(F) \subseteq U \subseteq F$. In [1] the problem of determining the complexity of recognising a partitionable complex was discussed. It is easy to see that for any fixed d, the language of partitionable complexes of dimension at most d is a member of the complexity class $\mathcal{NP}$. Our result removes the d-dimensional restriction, that is it shows that the language of partitionable complexes is in $\mathcal{NP}$, and thus answers a question raised in [1].

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In general membership in $\mathcal{P}$ remains open. It is easy to see that partitionable complexes of dimension at most two can be recognised in polynomial time. However the language of partitionable complexes of dimension at most d is not known to be in $\mathcal{P}$ for any fixed $d \geq 3$ and is conjectured in [1] to be $\mathcal{NP}$-complete for some d.

2 Results

Theorem 1 *Given any simplicial complex $(E, \mathcal{F})$ with facets $F_1, \ldots, F_n$ satisfying $|F_i| \leq d$ and a sequence, $\phi(F_1), \ldots \phi(F_n)$, of subsets of E, we can verify that the intervals $[\phi(F_i), F_i]$ form a partition of $\mathcal{F}$, in time $O(n^2 d)$.*

Corollary 1 *The language of simplicial complexes which are partitionable is a member of $\mathcal{NP}$.*

Proof of corollary: Given $F_1, \ldots, F_n$, the sequence $\phi(F_1), \ldots, \phi(F_n)$ is a suitable witness which we can verify in polynomial time.

$\square$

Proof of theorem: The following is an algorithm to verify that the intervals $[\phi(F_i), F_i]$ form a partition of $\mathcal{F}$.

For $i = 1$ **to** n **do**
    If $\phi(F_i) \not\subseteq F_i$ **then return** NO
For $i = 1$ **to** n **do**
    For $j = 1$ **to** n **do**
        If $i \neq j$ **and** $\phi(F_i) \cup \phi(F_j) \subseteq F_i \cap F_j$
            then return NO
For $i = 1$ **to** n **do**
    If $2^{|F_i|} \neq \sum_{j:\phi(F_j) \subseteq F_i} 2^{|(F_i \cap F_j) - \phi(F_j)|}$
        then return NO
return YES

We now show that this algorithm does what we claim. It is clear that the intervals $[\phi(F_i), F_i]$ form an interval partition of $\mathcal{F}$ if and only if the following three conditions hold:

1. $\phi(F_i) \subseteq F_i$ for all i;

2. $[\phi(F_i), F_i] \cap [\phi(F_j), F_j] = \emptyset$ for all i and j with $i \neq j$;

3. Any subset of a facet is contained in some interval.

Condition (2) is equivalent to

$$\phi(F_i) \cup \phi(F_j) \not\subseteq F_i \cap F_j$$

for all $i \neq j$.

The intersection of $[\emptyset, F_i]$ and $[\phi(F_j), F_j]$ is $[\phi(F_j), F_i \cap F_j]$ which is empty if $\phi(F_j) \not\subseteq F_i$ and otherwise has cardinality $2^{|(F_i \cap F_j) - \phi(F_j)|}$. So if condition (2) holds then condition (3) is equivalent to

$$2^{|F_i|} = \sum_{j:\phi(F_j) \subseteq F_i} 2^{|(F_i \cap F_j) - \phi(F_j)|}$$

for all i.

It is easy to see that given any $\{F_i\}$ and $\{\phi(F_i)\}$ the algorithm runs in time $O(n^2 d)$.

□

References

[1] P. Kleinschmidt and S. Onn. *Oriented matroid polytopes and polyhedral fans are signable*, Proceedings of Fourth Conference on Integer Programming and Combinatorial Optimisation (IPCO IV), Lecture Notes in Computer Science **920**, Springer (1995), 198–211.