Modelling the Risk of Underfunding in ALM Models

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Abstract

Asset and Liability Management (ALM) models have become well established decision tools for pension funds. ALMs are commonly modelled as multi-stage, in which a large terminal wealth is required, while at intermediate time periods, constraints on the funding ratio, that is, the ratio of assets to liabilities, are imposed. Underfunding occurs when the funding ratio is too low; a target value for funding ratios is pre-specified by the decision maker. The risk of underfunding has been usually modelled by employing established risk measures; this controls one single aspect of the funding ratio distributions. For example, controlling the expected shortfall below the target has limited power in controlling shortfall under worst-case scenarios.

We propose ALM models in which the risk of underfunding is modelled based on the concept of Second Order Stochastic Dominance (SSD). This is a criterion of ranking random variables - in our case funding ratios - that takes the entire distributions of interest into account and works under the widely accepted assumptions of decision makers being rational and risk averse. In the proposed SSD models, investment decisions are taken such that the resulting short-term distribution of the funding ratio is non-dominated with respect to SSD, while a constraint is imposed on the expected terminal wealth. This is done by considering progressively larger tails of the funding ratio distribution and considering target levels for them; a target distribution is thus implied. Different target distributions lead to different SSD efficient solutions. Improved distributions of funding ratios may be thus achieved, compared to the existing risk models for ALM. This is the first contribution of this thesis.

Interesting results are obtained in the special case when the target distribution is deterministic, specified by one single outcome. In this case, we can obtain equivalent risk minimisation models, with risk defined as expected shortfall or as worst case loss. This represents the second contribution.

The third contribution is a framework for scenario generation based on the "Birth, Immigration, Death, Emigration" (BIDE) population model and the Empirical copula; the scenarios are used to evaluate the proposed models and their special cases both in-sample and out-of-sample. As an application, we consider the planning problem of a large DB pension fund in Saudi Arabia.

Contents

List of Figures

List of Tables

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List of Abbreviations

ALM	Asset Liability Management
BIDE	(Birth, Immigration, Death, Emigration) Population Model
CCP	Chance Constrained Programming
CVaR	Conditional-value-at-risk
DB	Defined Benefit
DC	Defined Contribution
EUM	Expected Utility Maximisation
FSD	First Order Stochastic Dominance
ICCP	Integrated Chance Constraints Programming
LDI	Liability driven investment
LP	Linear Programming
LPM	Lower Partial Moment
MOO	Multi-Objective Optimisation
RPM	Reference Point Method
SD	Stochastic Dominance
SG	Scenario Generation
SP	Stochastic Programming
SSD	Second Order Stochastic Dominance
VaR	Value-at-Risk

Chapter 1

Introduction and Background

1.1 Decision Making Under Uncertainty

Operations research (OR) is a branch of knowledge that deals with the application of advanced analytical methods to the study and analysis of complex decision-making problems. It involves a wide range of problem-solving methods and techniques which have a computational and statistical nature, and because of that OR has strong ties to mathematics and computer science [64]. The word Operations is derived from the successful applications of OR to military operations in the late 1930s - early 1940s when the first formal activities of OR were initiated in England [64]. After World War II, the ideas that originated in the efforts of military planners were rapidly spread and applied to other fields, such as industrial production, finance, business management, engineering, medicine, and many other fields ([13], [64]).

Optimisation/Mathematical Programming (MP) is a sub-field of OR that is concerned with determining the best decisions among alternatives offering different outcomes. MP captures the essential aspects of a decision problem and represents them mathematically in an algebraic form by a set of relations (referred to as *constraints*) to be satisfied. An *objective function* is used to quantify how good a decision is; the objective function may represent profit, performance (to maximise), loss, or risk (to minimise). An optimal decision is one that leads to the maximum/minimum value of this objective function and satisfies all the constraints.

In many situations, the outcomes or consequences of decisions depend on parameters whose future value is not known with certainty at the decision moment; an example may be a financial decision making situation where the future outcome depends on the future prices of assets. This issue can be addressed in different ways. In this work, we will use the paradigm of Stochastic Programming (SP). Here, these uncertain parameters are described by discrete distributions with a limited number of outcomes. In most applications, the underlying parameters do not come directly in this form; they may have continuous or unknown distributions or be specified by a set of statistical properties.
Hence, to use SP, the model parameters have to be described by a finite discrete set of possible outcomes, called *scenarios*; each scenario occurs with an associated probability of occurrence. SP is then used to find the optimal decisions. Defining optimality is a non-trivial matter; usually an optimal solution is defined as one that optimises a statistic of the distribution of the objective function value and satisfies the imposed constraints under all (or a large proportion of) the possible values of the uncertain parameters.

Applications of SP to decision making problems in finance have been discussed in various contexts; of particular interest are the *asset allocation* or *portfolio selection* and *asset and liability management* problems.

1.2 The Asset Allocation Problem and Decision-Making Models: Mean-Risk Theory, Utility Theory and Stochastic Dominance

Formally, the problem of asset allocation can be stated as follows: how to divide *now* an amount of money W amongst a set of n *assets*, such that, after *a* *specified period of time*, to obtain a return on investment as high as possible? [56]. If the return of each asset is known with certainty, it is easy to decide on the best allocation: that would mean to invest only in the asset with the highest return. However, since the prices of the assets at the end of the investment period are usually unknown at the moment we made the decision, the consequence of choosing a specific allocation is uncertain; so that making the best decision is not straightforward any more.

SP deals with this uncertainty by considering a set of S states of the world or "scenarios" that could occur. If each scenario s ($s = 1, .., S$) has a probability of occurrence π_s, we can define a discrete probability space $\{\Omega, \mathcal{F}, P\}$ with $\Omega = \{1, ..., S\}$, $\mathcal{F}$ is σ-field, and $P(s) = \pi_s$ with $\sum_{s=1}^{S} \pi_s = 1$. The return of asset j ($j = 1, ..., n$), denoted here by R_j, is a random variable on this probability space $\{\Omega, \mathcal{F}, P\}$ with possible outcomes $r_{j1}, ..., r_{jS}$ that occur with probabilities $\pi_1, ..., \pi_S$ respectively.

Let us denote by x_j the fraction of capital W invested in asset j, $j = 1, ..., n$; these are the required investment decisions, also called portfolio weights; the vector $x = (x_1, ..., x_n)$ represents the portfolio. The weights $x_1, ..., x_n$ must satisfy some constraints that define a feasible set of decision vectors $\mathcal{X}$. Assuming short selling is not allowed, the largest feasible set is:

$$\mathcal{X} = \{\ (x_1, ..., x_n) \mid x_1 + ... + x_n = 1,\ x_j \geq 0,\ j = 1...n\}$$

Consider a portfolio x with random return R_x; R_x is the weighted sum (by the portfolio weights) of the random returns of the individual assets: $R_x = x_1 R_1 + ... + x_n R_n$. The realisation of R_x under scenario s, denoted by r_{sx}, depends on, individual asset returns under this specific scenario and on the portfolio weights: $r_{xs} = x_1 r_{1s} + ... + x_n r_{ns}$. Hence, R_x is a random variable with a distribution function that depends on the portfolio mix. Each decision x may be identified with a random variable; the problem of choosing between two different portfolios x and y becomes the problem of choosing between random variables R_x and R_y. In order to find the required decision, a valuation criterion (a preference relation) has to be defined on the set of random variables representing portfolio returns and corresponding to decision vectors. This treatment represents the theoretical basis of a model of choice. The computational part refers to the procedure of identifying the non-dominated random variables and their corresponding decisions to be implemented, by solving an optimisation problem.

The main approaches for modelling choice between random variables are: mean-risk models [40], Expected Utility Maximisation (EUM) [67] and Stochastic Dominance (SD) [68] methodology [56]. The scope of such models is to: (a) formally define a preference relation between random variables/alternative decisions, (b) identify the decisions corresponding to random variables that are non-dominated with respect to the specified preference relation. In the rest of this section we present briefly each of these approaches.

In the mean-risk approach, two scalars are attached to each random variable, in this case representing portfolio return. The first scalar is the expected value (mean) and the second is the value of a risk measure. Largely speaking, a risk measure is a function that associates to each random variable a number which describes its "riskiness". Examples of risk measures include variance, quantiles and expected value of deviation below a target. Lets denote the risk measure by ρ, $\rho : V \to \mathbb{R}$ where V is the set of random variables representing returns of feasible portfolios. Portfolios with higher values of mean and lower values of risk are desirable. Let x and y be two choices with corresponding random returns R_x and R_y. x is preferred to y if $\mathbb{E}(R_x) \geq \mathbb{E}(R_y)$ and $\rho(R_x) \leq \rho(R_y)$ with at least one strict inequality.

In this approach, we select from the universe of the possible decisions those that are *efficient* or *non-dominated*: for a given value of mean, the risk is minimised or, equivalently, for a given value of risk, the mean is maximised. The aim of these models is to find an allocation of assets that achieves an optimal trade-off between risk and return [14]. The exact trade-off chosen, in other words the preferred decision, will depend on objectives and attitude of the investor. In most cases, an optimisation model is solved:

$$\min \rho(R_x)$$

s.t.

$$E(R_x) \geq d$$

$$x \in \mathcal{X}$$

where d is a parameter specified by the decision maker: a target mean return.

The first and most commonly used mean-risk model in asset allocation problems is a methodology called "mean-variance optimisation", pioneered in the 1950's by Markowitz [40]. Here, the risk of a portfolio return is quantified by its variance. A mean-variance efficient portfolio is found as the optimal solution of the following quadratic program (QP):

$$\min \sum_{j=1}^{n} \sum_{k=1}^{n} x_j x_k \sigma_{jk}$$

s.t.

$$\sum_{j=1}^{n} x_j \mu_j \geq d$$

$$x \in \mathcal{X}$$

where σ_{jk} denotes the covariance between R_j and R_k: $\sigma_{jk} = E[(R_j - E(R_j))(R_k - E(R_k))]$ and $\mu_j = E(R_j)$ is the expected return of asset j, $j = 1, \ldots, n$. The first constraint in the above QP imposes a minimum expected value d for the portfolio return.
Since 1950's, many alternative risk measures have been proposed; more on this is developed in Chapter 2.

A utility function is a real-valued function defined on real numbers representing possible wealth values. It quantifies the relative value (*utility*) of outcomes. In EUM [67], a single scalar value (the expected utility) is attached to each random variable in terms of the utility value of its outcomes and the probabilities associated with these outcomes. In this approach, a random variable (or a portfolio) is non-dominated or efficient if and only if its expected utility is maximal.

Definition 1.1. Given a utility function U, the expected utility function of a discrete random variable R with outcomes $r_1, ..., r_s$ and probabilities $p_1, ..., p_s$ is:

$$E[U(R)] = p_1 U(r_1) + ... + p_s U(r_s)$$

A major difficulty in EUM is deciding on the relative value of outcomes; the specification of the utility function is a subjective task. There are well established assumptions about the behavior of investors or decision-makers (DM); the use of utility functions in an economic context is based on an implied assumption that they reflect this behavior. It is widely accepted that financial DM are (i) rational and (ii) risk averse.
The first assumption means that investors prefer "more to less" or higher returns to lower ones (non-satiation attitude); this is expressed by a non-decreasing utility function. Since all investors are assumed to be rational, this is the only non-arguable condition for utility functions on wealth.
The second assumption is about investors' attitudes towards risk. Suppose a DM has to choose between two random variables representing wealth or return. One is a "sure thing": a deterministic random variable with one possible (positive) outcome $r > 0$ occurring with probability 1. The other is a "50/50 bet"; a random variable with two possible outcomes r_1 and r_2 each with 0.5 probability of occurrence. The two random variables have the same (positive) expected value, thus $r = \frac{r_1+r_2}{2}$. A risk averse DM is one who will prefer the first choice (the sure thing) and in turn will often lose out on possible higher rates of return.

Thus, the expected utility of the first choice is greater for a decision maker who is risk averse: $U(\frac{r_1+r_2}{2}) > \frac{1}{2}[U(r_1) + U(r_2)]$. Generalising, we can consider any non-deterministic random variable with possible outcomes r_1 and r_2 occurring with probabilities p and $(1-p)$ and the equivalent (sure thing) with one possible outcome $pr_1 + (1-p)r_2$.

The utility function U in this case is strictly concave; the segment that joint two points $(r_1, U(r_1))$ and $(r_2, U(r_2))$ on the graph of U always lies below the graph as illustrated in Figure 1.1.

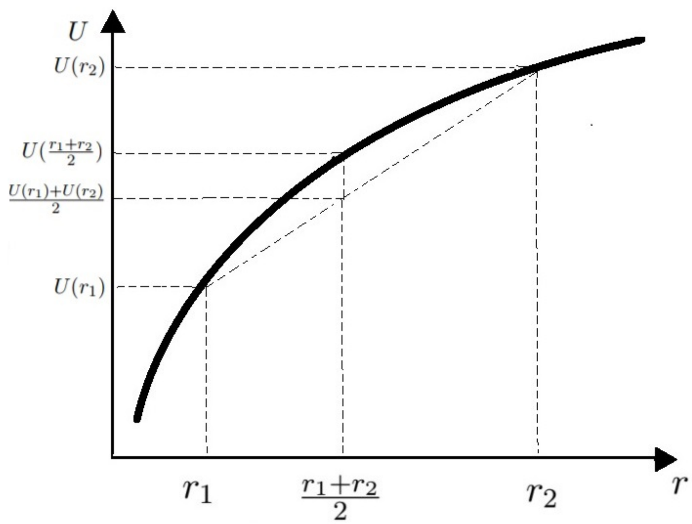

Figure 1.1: Concave utility function and risk-aversion

Thus, a risk averse DM has a non-decreasing and concave utility function: as wealth increases, an additional increment in wealth is less valued than an increment of the same magnitude but at lower levels of wealth, please see Figure 1.2.

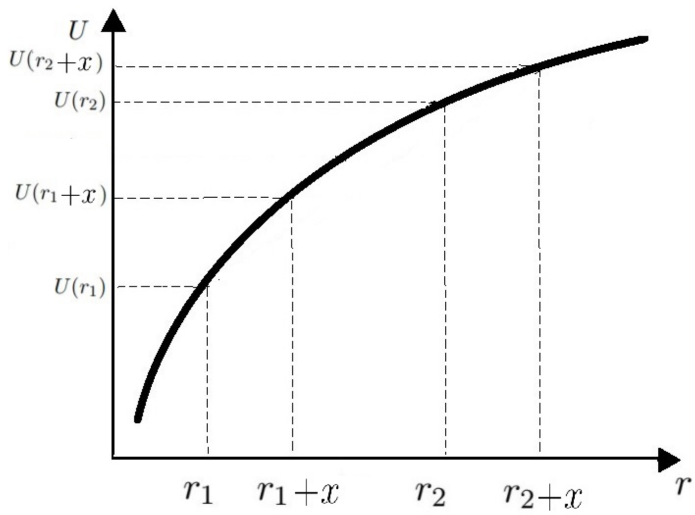

Figure 1.2: Increasing and concave utility function; a surplus of wealth is more valued at low levels

Beside the difficulty in the specification of a utility function, different utility functions of the same type (e.g. non-decreasing and concave) may lead to different ranking of random variables. Stochastic dominance overcomes these issues; here, random variables are compared under general assumptions on classes of utility function which stem from observed economic behavior, without the need to specify a utility function. From a computational point of view, ranking of random variables is made via a point wise comparison of some performance functions constructed from their distribution functions. For a random variable R with a cumulative distribution function H, we define:

$$H^{(1)}(r) = H(r) = P(R \leq r) \quad r \in \mathbb{R};$$
$$H^{(k)}(r) = \int_{-\infty}^{r} H^{(k-1)}(t)dt, \quad r \in \mathbb{R}, \quad k \geq 2$$

Definition 1.2. Let H and H' be the cumulative distribution functions of R and R' respectively and $k \in \mathbb{N}$, $k \geq 1$. R is preferred to R' with respect to the k-th order stochastic dominance (denoted: $R \succ_k R'$) if and only if: $H^{(k)}(r) \leq H'^{(k)}(r)$, $\forall r \in \mathbb{R}$, with at least one strict inequality [68].
A random variable R in a set V is non-dominated with respect to the k-th

order stochastic dominance, if there is no random variable $R' \in V$ such that $R' \succ_k R$.

Intuitively, the stochastically larger random variable has the smaller distribution function; this corresponds to saying that a smaller distribution function describes outcomes that are distributed farther to the right. SD is theoretically attractive and recognised as a valid criterion of choice as it takes into account all the distribution rather than tackling only two of its properties. However, in practice it is computationally challenging, as it involves an infinite number of comparisons. Finding portfolios whose return distribution is non-dominated with respect to SD has been considered an intractable problem until recently. However, SD relations can be much simplified if the random variables under consideration are discrete and even further, if they have equally likely outcomes. This will be developed in Section 3.2.

Of particular importance are First Order Stochastic Dominance (FSD) and Second Order Stochastic Dominance (SSD). That is because there are progressively stronger assumptions about investors' behavior that are used in expected utility theory, leading to first and second order stochastic dominance relations. For example, a random variable is preferred to another with respect to SSD if its expected utility is higher for any utility function U that is non-decreasing and concave [68]. Thus, SSD expresses the preference of a rational and risk averse DM; it is applied to random variables whose outcomes are desired to be high and for whom an increase is more valued if it is at low levels, rather than at high levels; typical such random variables represent return or wealth.

The asset allocation problem is also called "pure investment" problem as there are no obligations involved and the only aim is in maximising the return. It is commonly modeled as a single period problem; decisions are taken "now" and evaluated at the end of the investment period.

1.3 Asset and Liability Management

Many financial organisations such as banks, pension funds, insurance companies, and even some wealthy individuals are underpinned by the balancing of cash flow to match and outperform some future obligations or "liabilities". In such situations, the presence of liabilities gives rise to alternative paradigms of asset allocation [14]. Asset management techniques which take into account the stochastic nature of liabilities are given the generic label: Asset and Liability Management techniques (ALM) which have been recently renamed by some authors as Liability Driven Investment (LDI) [59].
While pure asset allocation problems are usually modeled as single period, in ALM, the presence of liabilities to be paid over a number of future time periods raises the need to adopt a multi-period setting. In such a setting, some decisions are taken "now" while other decisions are taken once the uncertainty around the assets returns and the liabilities is revealed progressively in stages. Due to the long duration of these liabilities, a long-term planning horizon is typically considered, in which, "long-term" wealth increase is sought. Throughout the planning horizon, the trade-off between long term objectives and short term risks should be made carefully during the decision process.

In a multi-period planning setting, since actual values of the problem parameters are gradually revealed in stages, these uncertain parameters are described by stochastic processes rather than distributions. The set of possible scenarios, considered in the static one-period models, is expanded to the so called *scenario tree*. Each path through the tree represents one possible sequence of outcomes of the stochastic elements throughout the planning horizon under consideration. The problem is then formulated as a scenario based optimisation problem which can find a strategy that is optimal in some sense.

A formal study of ALM problems usually starts by defining a planning horizon which specifies the total number of years, or more generally periods, which are considered in the decision making process. The planning horizon, denoted by T, is then divided into a set of decision moments $t \in \{0, 1, ..., T\}$.

Through this planning horizon, plausible sample paths for liabilities and returns of instruments in the portfolio are to be modeled and form a scenario tree. The initial time period $t = 0$ represents the present, when decisions need to be determined "now" with respect to changing the composition of an existing portfolio or investing an initial wealth. At each decision moment $t = 1, ..., T-1$, recourse investment decisions are made, after new information of asset returns, funding levels if any, and liabilities at that time point, are revealed. At the planning horizon, no more decisions will be made but the financial position will be considered over all scenarios.

As with pure investment problems, the main issues faced in the decisions making process are connected to: (a) modelling of the stochastic underlying parameters; that is, the scenario generation process, and (b) modelling paradigms and optimisation models. The multi-stage setting and the presence of liabilities (and possibly additional funding) make the scenario generation problem for ALM more complex and challenging than in the pure investment problems; additional issues (such as inflation) arise. Thus, integrated approaches such as ALM that combine both liability models and asset allocation decisions are needed; several domains of knowledge are involved in this. ALM models have been successfully used for banks, pension funds, insurance companies, and wealthy individuals (see for example [7], [12], [34], [30], [39]).

- **The Pension Fund Problem**

A pension fund's primary goals are to design the plan and to manage the assets and pension surplus so that the obligations to its contributors will be met, the sponsor's contribution over time is minimised and the growth in the plan's surplus is maximised [44]. The management of the investment portfolio is an important aspect of the pension fund. According to [16], for asset allocation, most firms use the well known and probably the most widely used method to solve such problems: the mean-variance analysis. The standard implementation of this model is static (one-period) and thus fails to capture the multi-period nature of the ALM problem [16]. By contrast, the dynamic stochastic programming models hedge portfolio allocations against future uncertainties in asset returns and liabilities over a longer horizon and preview possible future problems [16]. Comparisons between stochastic programming models and static models have been reported in the ALM literature by many authors; see for example [38] in a study about a Dutch pension fund problem, [7] and [25] in the context of ALM for insurance companies.

To solve a pension fund's ALM problem means to determine decisions on one or more among the following: allocation of the assets, contributions rate or level of payments for the participants, which are optimal in some sense, subject to a number of constraints. Some of the most important challenges in solving the problem are caused by the uncertain evolution in both demographic and economic factors which affect the future assets returns, the streams of contributions and liabilities. In pension funds, future liabilities depend on factors like longevity, possible earlier retirement of the members, salary growth rates and inflation. Moreover, those factors are driven also by other factors; for example, longevity is mainly due to the improvement in the living standards and medication.

Taking the example of Saudi Arabia, the demographic predictions of the United Nations shows that population aging in Saudi Arabia is entering a new phase and approaching its highest ever rate. The population predicted for Saudi Arabia is approximately 60 million in 2050, nearly 13% of whom will be aged 60 or more. The percentage in the same age group (60 years and over) was 5.6% in 1950 and 4.8% in 2000 [65]. *The dependency ratio*, that is, the number of the elderly population per 100 members of working age population (15-64 years old), is anticipated to rise to 12.8 percent in 2050 compared with 5.5 percent in 2000 [65]. Thus, the number of people claiming pension benefits will rise and working members will be charged more to pay for the extra pension costs in that period [29]. Although these facts and predictions illustrate the problem size, the exact amount of future contributions and liabilities are still uncertain and needs to be modeled. Figure 1.3 shows the demographics of Saudi Arabia in 1950, 2000 and the projected population of 2050.

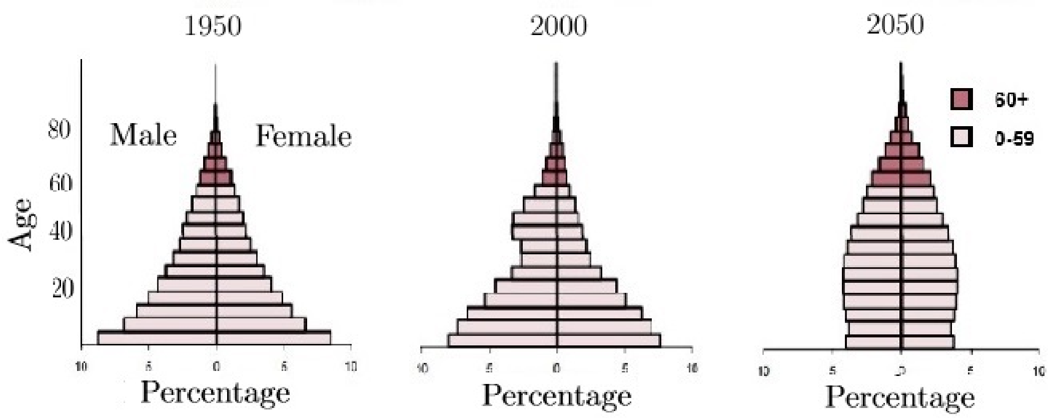

Figure 1.3: Saudi Arabia Demographics. Source: The United Nations [65]

Other uncertain parameters relate to future assets returns; since the prices of the investment vehicles throughout the planning horizon are unknown. The uncertainty around these returns is a source of risk, in the broadest sense, of an unfavorable outcome. When the uncertainty is described by a set of possible outcomes/scenarios, risk can be measured in a variety of ways, depending on which is considered to be the distribution of interest and what unfavorable aspect of it needs to be penalised or controlled. One of the greatest concerns of the board of a pension fund is the risk of underfunding. Roughly speaking, this is the risk that the liability values will be higher than the total asset values. In order to address this, ALM for pension funds should involve making decisions that guarantee, with high certainty, that the firm will be sufficiently solvent during the planning horizon. The solvency at a certain time moment is quantified using the funding ratio (the ratio of assets value over liabilities value) [26]. As a result of the long term commitments to meet the liabilities in pension funds, the planning horizon is typically long and the fund may aim to increase the growth in the plan's surplus while complying with solvency requirements throughout the process.

In a fluctuating demographic and economic climate, mathematical programming models that take into account uncertainty and risk have become increasingly important tools for pension funds. The paradigm of stochastic programming [5] is well suited for these problems and has already been applied in this context as shown in ([43], [74] and references therein). Some of the advantages of these models are that they could take into account complex real world constraints, allow the use of long term scenarios, give the flexibility for many decision variables and objectives which could tackle the consideration of many parties' interests [18].

In this research, we develop a family of stochastic programming ALM models that employ the SSD methodology in order to model the risk of underfunding. More specifically, we model the distribution of the funding ratio to be non-dominated with respect to SSD relations, while imposing a specific level of growth on the plan's wealth.
In what follows, we present general background information on pension funds in general with some concentration on the system of the General Organisation for Social Insurance (GOSI) which is a large pension fund in Saudi Arabia. The main aspects that we consider and the assumptions that we made in order to create datasets for the numerical experiments are summarised at the end of the next section. In order to create the data sets for the optimisation models, we use historical data on (1) rate of returns of assets available on Saudi Arabian market (2) population inflow and outflow, salaries, contributions levels and level of payments drawn from the GOSI database.

1.4 A Case Study: The General Organisation for Social Insurance (GOSI)

There are two types of pension schemes: Defined Benefit (DB) and Defined Contributions (DC). In both plans, the employers and/or employees pay a percentage of the employee's salary - a contribution - each month. In the DB plan (also known as final salary scheme) the benefit formula is pre-specified by contractual agreement, with benefits determined in terms of the final salary at retirement (or average salary per all, or some of the years worked) and the length of service. In contrast, in the DC plane (also called money purchase) the contributions to the fund are known (defined), while the benefits are unknown until retirement. The contributions are invested by the pension plan and the benefits are determined according to the accumulated contributions during the years the pension has been built up and the return of the investments. The benefits are paid to contributors either in the form of a lump sum at retirement or annuities [60].

The General Organisation for Social Insurance (GOSI) is a large DB pension fund in Saudi Arabia. There are two different DB pension funds in Saudi Arabia. One of them is the public pension agency for civil sector employees and the other is the GOSI which covers the workers in the private sector and a group of workers in public sectors. In what follow, we summarise the main features of the GOSI; for more informations please see [1].

The GOSI was established to implement the provisions of the Social Insurance Law which was issued under the Royal Decree No. M/22 in 15/11/1969 and was amended later to follow-up the process of achieving the compulsory insurance coverage, collecting contributions from employers, and paying benefits for the eligible beneficiaries or their family members. It is a semi-state body that enjoys administrative and financial independence and is supervised by the minister of labour, as the chairman [1].

The Social Insurance Law is an aspect of social cooperation and solidarity provided by the society for citizens. It provides the contributors and their families with a decent life after leaving work by the following insurance branches:

1. *Occupational Hazards Branch*, which provides benefits in cases of employment injuries. It is compulsorily applied to all workers without any discrimination as to sex, nationality or age.

2. *Annuities Branch*, which provides benefits in cases of non-occupational disability, old-age, and death. It is compulsorily applied to all Saudi

workers.

The revenues of the organisation consist of the following:

1. The contributions of the employers and employees; each branch has a different contribution rate:

   (a) The contributions of the occupational hazards branch are fixed at 2% of the wage of each participant and are paid by the employer.

   (b) The contributions of the annuities branch are fixed at 18% of the basic monthly wage received by the contributor, of which 9% is paid by the employer and 9% by the contributor.

2. The returns of investment.

3. The state annual subsidy allocated in the State general budget, as needed.

The liabilities that the GOSI needs to meet can be separated into two main categories. Firstly, there are the liabilities under the occupational hazards branch such as medical care and daily allowances for temporary work disability if, by reason of the injury, the contributor becomes temporarily unable to work. Monthly benefit or lump sum permanent, for total or partial disability, is to be paid for the injured participant or their family members. A grant should be paid to the family of the injured person or recipient of the benefit in the event of death.

Secondly, there are the liabilities under the annuities branch which involve four different sub categories:

1. **Pension payments**; to be paid for the following categories:

   (a) Retirees; contributors who attain retirement age and have completed a minimum period of contribution of 120 months, ceasing to be engaged in any activity subject to this Law, or contributors who have not attained retirement age but have completed a minimum period of contribution of 300 months. The retirement age is 60 years for male, and 55 years for female.

(b) Contributors who are afflicted with a non-occupational disability shall be entitled to a pension computed in accordance with the retirement pension. The disability pension may be increased by 50% as an allowance if the disabled is in need of the help of others in the performance of their everyday life activities.

(c) In the event of the death of a recipient of a non-occupational disability pension, or a recipient of a retirement pension, or a contributor in an insurable employment who had a period of contribution of not less than three consecutive months, each of the deceased family members shall be entitled to a share of the pension. The benefit shall be paid to the eligible family members on equal basis at a rate of 100% for three members or more, 75% for two members, and 50% for one member. The term "eligible family members" means the following members:

- The widow or widower of the deceased.
- The daughters until they marry.
- The sons who are under twenty-one years of age and this period could be extended until they complete twenty six full years if they are continuing their studies in educational or vocational institution, and no age limit is set so long as they are unable to engage in any occupation by reason of chronic disease or disability.
- The grandsons and granddaughters whose father died during the lifetime of the contributor and were supported by the contributor, subject to the same conditions as prescribed in respect of the sons and daughters.
- The parents of the deceased contributor who were supported by the deceased at the time of his death, provided that the father is unable to work, or is over sixty years of age and not working.
- The grandfather and grandmother, subject to the same conditions required in respect of the parents.
- The brothers and sisters of the contributor provided they were

supported by the deceased at the time of death, subject to the same conditions referred to in respect of sons and daughters.

In case of cancellation of a share of a member of the family, his share shall be repaid to the other eligible family members.

In all the above cases, pensions are computed in the same way. The amount of monthly pension payment is obtained by multiplying one-fortieth of the "average monthly wage" for the last two years by the number of contribution years and months. "Average monthly wage" means one twenty-fourth of the total wages received throughout the last twenty four months of contribution period.

2. **Lump sum payments**; are to be paid in the following cases:

   (a) A contributor who leaves the job and he was not eligible to receive pension because of not satisfying either age or contribution's period requirements.

   (b) The family members of a deceased contributor who is not entitled to receive a pension.

   (c) A contributor who transferred to another job and his own subscription period will not be taken into account when determining his rights in the new scheme s/he joins.

3. **Marriage grant**; a grant shall be paid to the widow, daughter, sister or granddaughter who is eligible for monthly benefit when she gets married. This grant equals eighteen times the monthly benefit she was receiving, and accordingly, payment of such benefit shall be discontinued.

4. **Death grant**; a grant equivalent to the deceased contributor's pension or benefit for three months to be paid to the family members in case of death of a contributor or a recipient of a pension.

Another source of revenues and liabilities is caused by the issuance of "The Law for Portability of Benefit Rights between the Civil and Military Retirement Schemes and the Social Insurance Scheme". This law was issued pursuant to the Royal Decree No. 53/M dated 23/07/1424 H and allows citizens to take advantage in as much as possible of their service periods in both schemes; they receive a pension by aggregating their services rather than receiving a lump sum compensation. However, period's aggregation is important, as it allows receiving a higher pension amount, if possible. According to this law, if a contributor transfers between jobs governed by different schemes, the former fund should transfer the aggregated contributions to the new scheme and accordingly the service period to be considered by the new scheme when determining the pension [1].

The Social Insurance Law of the Kingdom of Saudi Arabia has become compulsorily applied to all Saudi workers in the private sector. Any other Gulf Cooperation Council (GCC) Member State citizen enjoys all the benefits provided by the annuities branch of the Social Insurance Scheme, in the same way as a Saudi worker in the Kingdom of Saudi Arabia. Also, the benefits of annuities branch could be voluntarily applied to the Saudi citizens who are engaged in liberal professions. In this case, the contribution will be fixed at 18% of the assumed wage chosen by the contributor [1].

The organisation enjoys administrative and financial independence and is guaranteed and controlled by the State. Each of the branches of insurance has accounts of its own. The Board of Directors allocates to each branch its share in the administrative expenses and lays down the rules governing the distribution among the various branches of the revenues that do not belong to any particular branch. The GOSI is exempted from all taxes and fees [1].

The annuity branch of the organisation acts as a pension fund to the participants while the occupational hazards branch acts mainly as an insurance against the occupational injuries. In this research, we will consider the annuity branch (only) of the GOSI for creating a data set that is used in the numerical experiments.

Summary of the GOSI characteristics and the ALM framework

In what follows, we present a summary of the assumptions that we consider in order to create datasets for the numerical experiment based on theGOSI's characteristics.

1. Funding is from three sources:
   (a) Contributions from active participants which are fixed at 18% of participants' salaries.
   (b) Money transferred into the fund's account by new participants coming into the GOSI from another pension fund's scheme.
   (c) Investment revenues.
2. The liabilities involve the following:
   (a) Pension payments for retirees, non-occupational disabled, and to next of kin if death occurs.
   (b) Lump sum payments, for cases described earlier in this section.
   (c) Money transferred out of the fund due to participants who join another pension fund's scheme.
   (d) Death grants to next of kin upon death.
3. The GOSI invests locally only and in two types of assets: the money market instruments and the Saudi Stocks market represented by 15 sectors.
4. There is a restriction (upper bound) for investment in each of the sectors.

1.5 Historical Background

Different approaches for modelling risk in the context of ALM can be found in the literature. They mainly stem from the single period asset allocation modelling framework, where the most common approach is to find investment decisions which result in a return distribution with a high expected value and low value of "risk". Expected value can be maximised with a constraint on risk; alternatively, risk can be minimised with a constraint on expected value. This modelling approach has been extended to the case of multi-period setting, liability driven investment, usually by maximising the expected terminal wealth while for intermediate time periods, the risk factor is incorporated into constraints, or considered by additional terms in the objective function assigned for "risk" penalties. We start by reviewing some of the well known approaches for modelling risk related to the mean-risk paradigm, often employed in asset allocation, and their related modelling approaches in ALM.

In a single stage context, Markowitz [40] proposed variance as a risk measure. He measured the risk of a portfolio using the covariance matrix associated with individual asset returns. Variance as a risk measure has been criticized, mostly for its symmetric nature, since it penalises favorable (upside) deviations in the same way it penalises unfavorable (downside) deviations. Since then, alternative asymmetric, or downside, risk measures have been proposed. Fishburn [24], and Bawa [4] introduced a family of risk measures called Lower Partial Moments (LPMs): LPM with target τ and order n of a random variable R (e.g. representing future return) is by definition $E[\max\{\tau - R, 0\}^n]$. In the particular case $n \to 0$, the LPM measures the probability of falling below target τ, which had been used in asset allocation models by Roy [57].
An important step was the introduction of risk measures concerned only with quantifying extremely unfavorable results, in other words, the left tail of a return-type distributions (the right tail of a "loss-type" distributions). Among the most important risk measures of this type is Conditional Value-at-Risk (CVaR) proposed by Rockafellar and Uryasev [53] in the context of single stage asset allocation. Consider a random variable representing loss [1] and a confidence level $\alpha \in (0, 1)$; for example, $\alpha = 0.95$ or $\alpha = 0.99$. In this case, $(1 - \alpha)$ represents a percentage of worst case scenarios. CVaR at confidence

[1]Such a random variable could be defined for example as $T - R$, where R is the return distribution and T is a fixed number. A common loss distribution is $-R$, where R is the return.

level α measures, largely speaking, the average of losses in the worst $(1-\alpha)$ of cases.

Arguably, describing and ranking distributions by just two parameters, for example mean and risk, involves loss of information. As introduced in Section 1.2, Second Order Stochastic Dominance is another criterion for ranking random variables and it takes into account the whole of the distributions involved. SSD expresses the preference of risk averse decision makers, i.e. those whose utility function is increasing and concave. SSD is treated extensively in Chapter 3. The conceptual advantages of using SSD, together with the difficulty to apply it in practice have been long recognised [68]. Recently, computationally tractable portfolio models that employ the concept of SSD have been proposed for single stage portfolio optimisation - for example, [21], [22], [52], [55]. In [55], the portfolio resulting in SSD efficient distributions are found via a multi-objective model in which the objective functions are unscaled tails of the return distributions; this approach has been further extended in [22], where the scaled tails of return distributions are considered. Particular solutions are found by setting aspiration (targets) for each of the objective functions and optimising an overall achievement function, more details are in Section 3.4.

When modelling risk in ALM problems, the distribution of interest is not necessarily that of wealth or asset return. As the relationship to liabilities is crucial, the distribution of interest is the (or related to the) funding ratio, that is, the ratio of assets to liabilities. Commonly, a target funding ratio is set: a number $\lambda \geq 1$ below which the value of the funding ratio is desired not to fall. Risk constraints usually are employed in order to limit the probability and/or the magnitude of the funding ratio distribution falling below λ.

The basic concepts of ALM models under uncertainty were developed by Kallberg, White and Ziemba [30] and Kusy and Ziemba [39]; afterwards, large scale applications were developed (please also see [43] and references within). In [39], a multi-period stochastic linear programming model was developed for the Vancouver City Savings Credit Union. The objective was to maximise the

expected bank profits minus expected penalty costs for constraints violation. One of the first commercial applications of an asset-liability model reported in the literature is the Russel-Yasuda Kasai financial planning model, which has been developed by Carino et al. [7] and Carino and Ziemba [8] for the second largest Japanese insurance company. The modelling approach was a multi-stage stochastic linear programming model tailored to Yasuda's asset/liability management problem. The objective function maximises the final time period expected wealth minus penalty costs for underfunding. The result shows the advantages of this model over the previous technology used by Yasuda Kasai: static mean-variance analysis recomputed in each period with a rolling one-period horizon.
Other successful commercial applications include the Towers Perrin Tillinghast ALM system of Mulvey et al. [44] and the InnoALM system which has been developed at Innovest for the largest corporate DC pension plan in Austria by Geyer [27]. The latter model uses a multi-period stochastic linear programming framework; the objective function is to maximise the expected discounted value of terminal wealth minus the expected discounted penalty costs of shortfalls from a wealth target.
A generic computer-aided asset/liability management model (CALM) was developed by Consigli and Dempster [12]. It is a stochastic programming model that maximises terminal wealth at the end of a time horizon of 10 years.

In order to model risk constraints in ALM models, Dert [17] used chance constrained programming (CCP): an SP paradigm first proposed by Charnes and Cooper [9]. Omitting the time index, denote by A the distribution of asset value and by L the distribution of liabilities. The constraint $A \geq \lambda L$ under all scenarios is relaxed by allowing a small percentage of scenarios $B\%$ under which underfunding may happen. Formally, $\text{Prob}(A/L < \lambda) \leq (1-\beta)$ or equivalently $\text{Prob}(A - \lambda L < 0) \leq (1-\beta)$, where $\beta = 1 - B\%$ is the reliability level.
Klein Haneveld [33] argued that chance constraints are based on a qualitative risk concept; they control the probability of constraint violation but do not account for the amount by which it is violated. In addition, they require bi-

nary variables in the formulation of the optimisation model, that count the number of times when the constraint is violated, thus increasing computational complexity. Klein Haneveld [33] and Klein Haneveld and Van der Vlerk [35] proposed an alternative SP approach: the Integrated Chance Constrained Programming (ICCP) which has been used in the context of ALM by Klein Haneveld et al. [34] for a Dutch pension fund. With the above notations, an ICCP constraint requires that $E[\max\{\lambda L - A, 0\}] \leq \theta$, where θ is the maximum amount of average underfunding that a decision maker accepts.

Lower partial moments of order two have been adopted by Kouwenberg [38] to control risk in an ALM model for a defined benefit Dutch pension fund. The LPM with target λ and order 2 of the random variable A/L (the funding ratio) is constrained to not exceed a user specified value θ: $E[\max\{\lambda - A/L, 0\}^2] \leq \theta$. In the context of multi-stage ALM, CVaR has been used by Bogentoft et al. [6]; they considered the loss random variable $\lambda L - A$ and imposed an upper limit on its CVaR, in order to control the risk in a DB pension funds ALM problem in the Netherlands.

Schwaiger et al. [59] and Sheikh Hussin et al. [60] implemented decision models using Linear Programming (LP), Two Stage Stochastic Programming, Chance Constrained Programming and Integrated Chance Constrained Programming for pension funds. Schwaiger et al. [59] consider a portfolio comprised of UK bonds that matches the liabilities while minimising the cost of the portfolio, this model is applied to a DB pension fund. They consider a model that has two objective functions to minimise: the initial cash which has to be injected to achieve a (feasible) matching between assets and liabilities and the total deviations of assets and liabilities. Sheikh Hussin et al. [60] apply the models to the Employees Provident Fund (EPF) of Malaysia which is a DC pension fund; the objective is to maximise terminal wealth.
Dupačová and Polívka [18] proposed a multi-stage stochastic programming model for a Czech DC pension fund. The objective is to maximise the present value of the expected terminal wealth while penalising the discounted expected shortfalls.

De Oliveira et al. [15] developed a multistage stochastic programming ALM model for a Brazilian pension fund. The model maximises the expected terminal value of the fund. Using chance constraints, they enforce the funding ratio not to fall below a specific threshold with high probability.

In the context of insurance companies, Flent et al. [25], compare two different approaches in asset liability management. The first approach is a multistage stochastic programming, while the other is a static approach based on the so-called constant rebalancing or fixed mix. This comparison and tests were applied to a Norwegian mutual life insurance company. They found that a dynamic stochastic approach dominates a fixed mix approach, but that the degree of domination is much smaller when the models are compared out-of-sample than when they are compared in-sample. The objective is to minimise risk subject to a minimum target expected portfolio return. Risk is measured by the expected accumulated shortfalls of different types relative to legal requirements. The decision maker can weight the relative different importance to each shortfall type in the objective function.

Using term-life insurance along with the traditional asset classes (stocks and bonds) as a hedging tool against longevity risk, Kim and Mulvey [32] proposed an ALM model for wealthy individuals with a focus on the optimum investment strategy after retirement.

The approaches used in the models presented above are related to the mean-risk paradigms; usually the expected value of terminal wealth is maximised with constraints on risk (measured on random variables usually related to funding ratio) at intermediate time periods. In imposing such risk constraints, one single aspect of the funding ratio, or more generally of the distribution of interest, is controlled. For example, a limit on the expected shortfall does not guarantee manageable worst case realisations and hence does not exclude the possibility of catastrophic losses or massive shortfall below target. Similarly, a CVaR upper limit may guarantee manageable outcomes under worst case scenarios, but it may leave open the possibility of under-achievement in the

rest of the distribution.
Moreover, deciding on a meaningful right hand side on a risk constraint is often a challenging task; a fine tuning is necessary in order to avoid infeasibility or under-achievement, that is, solutions that could be improved upon. Klein Haneveld [34] stressed that a prior selection of the model parameter is difficult in practice and a reasonable parameter could be found by numerical experiments. Fabian and Veszprémi [23] construct an approximation of the efficient frontier to help the decision maker in calibrating the right hand side parameter in CVaR constraints as a first step in formulating their model.

Previously, a stochastic dominance concept has been used in the ALM context by Yang et al. [72]. They formulate an LP model in which the objective is to maximise expected terminal wealth minus penalties for underfunding, while controlling the market risk and the risk of underfunding via SSD and interval second order stochastic dominance (ISSD) constraints. More specifically, they require that the distribution of asset value dominates a benchmark distribution with respect to SSD, and they also impose an ISSD constraint on the funding ratio limiting the probability that the asset value falls below the liabilities.
More recently, Kopa et al. [37] apply first and second order stochastic dominance constraints in a multistage SP model. In collaboration with a commercial Italian bank, they propose a model for individual optimal pension allocation. The objective is to minimise the Average Value at Risk Deviation measure while satisfying a wealth target; the optimum portfolio is constrained to dominate a benchmark with respect to FSD and SSD relations.

Another major research area in ALM concerns scenario generation for the future values of assets returns, contributions and liabilities.
Scenario generation for asset prices has been extensively researched, mainly in the context of pure investing/asset allocation. For an overview of scenario generation methods applied in finance and economic decision making, see [66]. Commonly used methods include sampling or bootstrapping of historical data [19] and Vector Autoregression model (VAR) introduced by Sims [61]. Sce-

nario generation using VAR in the area of ALM has been used in [17], [38] and [60].
Another established scenario generating method is the moment matching approach [28]; it has been used in financial applications, including ALM, see for example [18], [25] and [38]. In this approach, the decision maker specifies a set of statistical properties (e.g. moments of order up to four). The scenario set is constructed in such a way that these statistical properties are matched. This is done by using non-linear optimisation in which the objective is to minimise the difference between the specified statistical properties and the statistical properties of the generated data.
More recently, an alternative moment matching scenario generation method was proposed by Ponomoreva et al. [51]. Their method produces scenarios and corresponding probability weights that match exactly the given mean, the covariance matrix, the average of the marginal skewness, and the average of the marginal kurtosis of each individual component of a random vector without employing optimisation in the scenario generation process.
With all the above scenario generation methods, a major difficulty is the multivariate nature of data. One way to overcome this is to separately model the univariate marginal distributions and the dependencies between random variables via a "*copula*" [62]. Different copulas are used in order to satisfy specific assumptions on data dependency. Kaut and Wallace [31] propose an scenario generation method in which an *empirical copula* is used.

Unlike in pure investment problems, the scenario generation process does not stop with generating future possible values for asset returns; in ALM, future vales for liabilities and contributions are also part of the model parameters. A main underlying source of uncertainty is the number of members (paying contributions) and past members/ retirees to whom liabilities are to be paid; population models are thus necessary in order to generate scenarios for future liabilities and contributions. The plan's demographic dynamics could be analysed either in a closed system without staff turnover or in an open system, which allows for joining new employees. Markov processes are broadly used to describe the population dynamics as they allow for a flexible representation of life contingencies. In an ALM context, Mettler [42], described the population dynamics in a closed and open systems for a DB pension plan using Markov processes. In [60], the flow and the status of the members in a DC pension fund system are simulated using an open system Markov population model too.

1.6 Thesis Outline and Contributions

This research is concerned with modelling the risk of underfunding in ALM models. In our ALM model, one stochastic process of interest is the wealth of the fund or asset value, for which we want high values in the long term. Another one is the funding ratio, for which we want high values shorter term; at each point in time, the funding ratio is a random variable. Target values, below which the funding ratio should not fall, are commonly specified. Imposing a hard constraint requiring that the target funding ratio is achieved under all scenarios may be in most cases unfeasible or may limit the set of decisions to some unfavorable ones, for the long-term fund wealth.

As presented in the previous section, a commonly used modelling approach has been to maximise the expected value of the terminal fund wealth with penalties for not achieving the target funding ratio at intermediate times. Since the expected terminal wealth and the value of the shortfall are likely to be not in the same scale, setting the weights for the penalties in the objective function is not a straightforward task; this approach has limited power in modelling the distributions of the funding ratios.

More recent approaches consider risk measures for the distribution of the funding ratio. As essentially the left part of this distribution is mostly of interest (more precisely, outcomes below the target), asymmetric risk measures have been employed, mainly lower partial moments or CVaR. Constraints on LPM of order 0 and 1 have been used in ALM under the SP paradigm of chance constrains (CCs) and integrated chance constraints (ICCs). In Chapter 2, we present a review of these methods, within a general SP framework for ALM.

As pointed out in Section 1.5, risk constraints offer limited control on the shape of the funding ratio distribution, leaving open the possibility of a distribution either with unacceptably low worst case outcomes or a distribution with too many outcomes below the target.

In Chapter 3, we propose ALM models in which the risk of underfunding is modelled using an SSD concept. We adapt SSD-based models for single period asset allocation proposed by Roman et al. [55] and [22] to the multi-period setting and we apply the SSD comparisons to distributions of funding ratio, rather than return or wealth. The motivation is to improve on the modelling framework in the sense of more control on the shape of the funding ratio distribution. This represents the first contribution of the thesis.
As exposed in Section 1.2, SSD is a framework for comparing random variables where higher outcomes are preferred but where an increase in value is less and less appreciated as the level gets higher. Distributions of funding ratio seem ideally suited to this framework for ranking and comparing, as the left tail is mostly of concern, while an increase in the right tail (more specifically, above a target) has relatively little value.
The purpose of the models proposed in Chapter 3 is to find investment decisions such that the distribution of funding ratio is non-dominated with respect to SSD, while a constraint on the level of long-term expected growth in the fund wealth is imposed.
The set of funding ratio distributions that are non-dominated with respect to SSD is large; it is the set of optimal solutions of a multi-objective model, in which the objective functions are tails of the funding ratio distributions at various confidence levels. An additional criterion of selection is employed, using *reference* (or *aspiration* or target) points, chosen by the decision maker, for the tails of the distributions. These aspiration points for tails imply target levels for the outcomes of the funding ratio distribution; thus, they define a target distribution. A decision maker can modify the target levels depending on what s/he wants to achieve or what part of the funding ratio distribution is undesirable and can be improved.
The "tails" can be *unscaled*, equivalent to the sum of progressively larger number of worst outcomes, or *scaled*, equivalent to the average of progressively larger number of worst outcomes. The largest difference between a tail and

its reference point is maximised; the solution obtained has thus a distribution that, in addition to being SSD non-dominated, is close to a reference vector. Depending on whether we consider scaled or unscaled tails, we obtain two different models. Both models provide different SSD efficient distributions of funding ratio, irrespective of the reference distribution chosen. The unscaled model penalises more the accumulation of many outcomes below their desired value, while the scaled model penalises more higher magnitudes of shortfalls below the desired values.
The models are extended by introducing reservation levels, in addition to aspiration levels, for even more modelling power.

The models proposed in Chapter 3 differ from other ALM models that apply SSD concepts not only because SSD relations are applied to funding ratio distributions, taking thus into account the relationship between asset value and liabilities. It is also the fact that the SSD criterion is employed in the objective rather than a constraint. This overcomes some of the undesirable situations that might occur, depending on the benchmark distribution chosen by the decision maker. For example, if the outcomes of the benchmark distribution are too high and as a result this distribution cannot be dominated or attained, a stochastic dominance constraint would result in infeasibility. In the opposite situation when the benchmark distribution is SSD dominated, the optimal solution is guaranteed to improve on the benchmark but not necessarily to be SSD non-dominated.

With the models proposed in this thesis, there are three possible cases. Firstly, if the benchmark represents a funding ratio distribution that is dominated with respect to SSD, the optimal solution results in a funding ratio distribution which is "better than target": it improves on the benchmark until SSD efficiency is attained. Secondly, if the benchmark is SSD efficient, the optimal solution of the model has a funding ratio distribution that exactly matches the benchmark. Finally, if the target is not attainable (in the sense that no feasible solution could match or improve on it), the optimal solution has a funding ratio distribution which is SSD efficient and comes as close as possible, in a well defined sense, to the target.

A particular and important case is when the reference distribution is deterministic, represented by one single possible outcome $\lambda \geq 1$; naturally, λ can be set as the target funding ratio specified by the fund managers. In this case, the SSD-based models present interesting connection with established risk minimisation models.
The SSD scaled model is equivalent (in the sense of providing the same optimal solutions) to a Maxmin model: maximisation of the worst outcome of the funding ratio. The SSD unscaled model is equivalent, under mild conditions, to a model in which the expected shortfall (lower partial moment of order 1) with target λ is minimised. This in turn is equivalent to an ICCP model, in which there is a constraint on the expected shortfall with respect to λ.
The relationships developed here between SSD based models and established risk models or SP paradigms represent the second contribution of this thesis.

An important aspect in ALM models is the scenario generation part, both with respect to future returns of financial assets and to values of liabilities and contributions; Chapter 4 concerns this. In sample scenarios for asset returns are created by bootstrapping. For liabilities and contributions, we employ a population model based on the "Birth, Immigration, Death, Emigration" (BIDE) population model [45]; this is used for in and out-of-sample scenarios. Out-of-sample scenarios for asset returns are created by employing a historical copula and sampling from the marginals. Historical samples are fitted to univariate distributions; new samples are generated and then combined using a historical copula. One can repeat the procedure and generate as many scenario sets as desired.
To the best of our knowledge, the use of BIDE population models in ALM, as well as the framework for out-of-sample testing based on univariate distribution fitting and the empirical copula have not been reported in the literature; this represents the third contribution of the thesis.

Chapter 5 presents the numerical experiments. The SSD-based models, formulated in Chapter 3 as linear programming models of large size, are implemented in AMPL and solved using CPLEX 12.5.1.0. They are compared, using a data set drawn from the GOSI, the largest pension fund in Saudi Arabia, against well established models such as Maxmin (maximisation of worst outcome) and ICCP. Finally, Chapter 6 concludes the thesis and contains prospective research directions.

Chapter 2

Stochastic Programming Models for ALM

In this chapter, we present a family of optimisation models that are applied to the pension fund's ALM problem. The first model is a scenario based two-stage stochastic program where the uncertainty around the asset returns, contributions and liabilities is modeled via a scenario tree in the form of a fan. Extensions to this SP model, to take into account a pension funds prospective underfunding situations, leads to Chance Constrained Programming (CCP) and Integrated Chance Constrained Programming (ICCP) models presented in Section 2.2. Section 2.3 contains a summary of the formulation of the two-stage SP model, CCP and ICCP models. In Section 2.4, we introduce a review on some of the well known risk measures used in the ALM context followed by concluding remarks in the last section. In the next section, we set a basic framework for an ALM decision problem within an SP model.

2.1 ALM Problem and SP Setting

Pension funds typically consider a long-term planning horizon due to the long duration of their liabilities. We consider a pension fund problem in which the planning horizon is split into T sub-periods. At the beginning of each of these

periods (i.e. at each time point $t \in \{0, ..., T\}$), liabilities are to be paid out and contributions are to be paid in. Decisions on investing the contributions and rebalancing the portfolio are taken at times $t \in \{0, ..., T-1\}$, with the short-term objective that, at the next time period, the asset value (uncertain) is in some desirable proportion to the liability value (also uncertain at decision moment). The returns of the available assets, as well as value of liabilities and contributions are uncertain and observed at times $t \in \{1, ..., T\}$.
Similarly to [6], we consider a two-stage SP and a scenario tree in the form of a fan as illustrated in Figure 2.1; the root node represents the present ($t = 0$). The uncertainty is represented by a number of S scenarios. Each path from $t = 0$ to $t = T$ represents one scenario, that is, one possible sequence of outcomes of the stochastic elements throughout the time horizon T. Each scenario has an associated probability of occurrence π_s, $s \in \{1, ..., S\}$, where $\pi_s > 0$ and $\sum_{s=1}^{S} \pi_s = 1$.

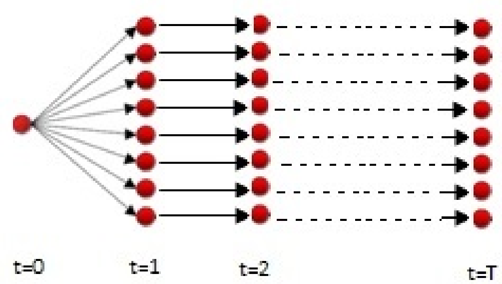

Figure 2.1: Scenario tree in the form of a fan

At time $t = 0$, first stage investment decisions need to be taken. The returns of the assets between $t = 0$ and $t = 1$, as well as the value of liabilities to be paid (and also contributions to be cashed in) at time $t = 1$ are unknown when first stage decisions are taken, thus, the value of the funding ratio is unknown. Recourse decisions about rebalancing the portfolio are taken at times $t = 1, \ldots, T-1$; these are scenario dependent.
At any time $t = 1, \ldots, T$, the liabilities value is a discrete random variable; so is the contributions value, while the returns of the assets form a random vector. For any portfolio decisions taken at time $t = 0, \ldots T-1$, the funding ratio

at time $(t+1)$ is a random variable whose values depends on the decisions, the returns of the component assets between t and $(t+1)$ and the liabilities and contributions at time $(t+1)$.

At the planning horizon, no more decisions will be determined but the fund wealth is evaluated with respect to all scenarios. In executing a trading strategy, we assume that each trade has an associated transaction cost, short selling is not allowed, and there is an upper bound in investing in each asset class.

In what follows, we present the basic modelling framework. We use the following notations:

I = The number of financial assets available for investment

T = The number of time periods

S = The number of scenarios

The parameters of the model are denoted by:

OP_i = The amount of money held in asset i at the initial time period $t = 0$; $i = 1 \ldots I$

L_0 = Aggregated liability payments to be made "now" $(t = 0)$

C_0 = The funding contributions received "now" $(t = 0)$

$L_{t,s}$ = Liability value for time period t under scenario s; $t = 1 \ldots T$, $s = 1 \ldots S$

$C_{t,s}$ = The contributions paid into the fund at time period t under scenario s; $t = 1 \ldots T$, $s = 1 \ldots S$

$R_{i,t,s}$ = The return of asset i at time period t under scenario s; $i = 1 \ldots I$, $t = 1 \ldots T$, $s = 1 \ldots S$

u_i = The upper bound imposed on the investment in asset i; $i = 1 \dots I$

ψ = The transaction cost expressed as a percentage of the value of each trade

π_s = The probability of scenario s occurring; $s = 1 \dots S$.

Let us denote **the first stage decision variables** by:

$B_{i,0}$ = The monetary value of asset i to buy at the beginning of the planning horizon ($t = 0$); $i = 1 \dots I$

$S_{i,0}$ = The monetary value of asset i to sell at $t = 0$; $i = 1 \dots I$

$H_{i,0}$ = The monetary value of asset i to hold at $t = 0$; $i = 1 \dots I$.

with $H_{i,0} = OP_i + B_{i,0} - S_{i,0} \quad , \qquad i = 1 \dots I.$

Let us denote **the recourse decision variables** by:

$B_{i,t,s}$ = The monetary value of asset i to buy at time t under scenario s; $i = 1 \dots I, \quad t = 1 \dots T-1, \quad s = 1 \dots S$

$S_{i,t,s}$ = The monetary value of asset i to sell at time t under scenario s; $i = 1 \dots I, \quad t = 1 \dots T-1, \quad s = 1 \dots S$

$H_{i,t,s}$ = The monetary value of asset i to hold at time t under scenario s; $i = 1 \dots I, \quad t = 1 \dots T, \quad s = 1 \dots S$

$A_{t,s}$ = The total asset value at time t under scenario s, **before** portfolio rebalancing; $t = 1, ..., T, \quad s = 1, .., S$.

A common approach encountered in the literature is to maximise the expected value of the terminal asset value A_T while imposing risk constraints on short or medium term.

The Objective Function

The objective is to maximise the expected terminal wealth:

$$\max \sum_{s=1}^{S} \pi_s A_{T,s} \tag{2.1}$$

- **Asset Value Constraints**

These express the relationship between asset value and holdings in assets.

$$A_{1,s} = \sum_{i=1}^{I} H_{i,0} R_{i,1,s} \ , \qquad s = 1 \dots S \tag{2.2}$$

$$A_{t,s} = \sum_{i=1}^{I} H_{i,t-1,s} R_{i,t,s} \ , \qquad t = 2 \dots T, \ s = 1 \dots S \tag{2.3}$$

- **Asset Holding Constraints**

The amount of each individual asset class held in the portfolio at each time period is computed with respect to the amount held of that asset class during the previous time period, the return over the corresponding time period and any change in the composition. For the last time period there will be no buying or selling variables.

$$H_{i,0} = OP_i + B_{i,0} - S_{i,0} \ , \qquad i = 1 \dots I \tag{2.4}$$

$$H_{i,1,s} = H_{i,0} R_{i,1,s} + B_{i,1,s} - S_{i,1,s} \ , \qquad i = 1 \dots I, \ s = 1 \dots S \tag{2.5}$$

$$H_{i,t,s} = H_{i,t-1,s} R_{i,t,s} + B_{i,t,s} - S_{i,t,s}, \ i = 1 \dots I, \ t = 2 \dots T-1, \ s = 1 \dots S \tag{2.6}$$

$$H_{i,T,s} = H_{i,T-1,s} R_{i,T,s} \ , \qquad i = 1 \dots I, \ s = 1 \dots S \tag{2.7}$$

- **Fund Balance Constraints**

These are cash flow contributions, assuming that no other funding occurs during the time horizon.

$$\sum_{i=1}^{I} B_{i,0}(1+\psi) + L_0 = \sum_{i=1}^{I} S_{i,0}(1-\psi) + C_0 \tag{2.8}$$

$$\sum_{i=1}^{I} B_{i,t,s}(1+\psi)+L_{t,s} = \sum_{i=1}^{I} S_{i,t,s}(1-\psi)+C_{t,s} \ , \ t = 1 \ldots T-1, \ s = 1 \ldots S \tag{2.9}$$

- **Short-Selling Constraints**

To short sale, or shorting, means to sell an asset that is not owned. Under no-short selling assumption we can add the following constraints:

$$S_{i,0} \leq OP_i \ , \qquad i = 1 \ldots I \tag{2.10}$$

$$S_{i,t,s} \leq H_{i,t-1,s} \ , \qquad i = 1 \ldots I, \ t = 1 \ldots T-1, \ s = 1 \ldots S \tag{2.11}$$

- **Bound Constraints**

To insure that the portfolio held at each time period will be diversified, we impose an upper bound on the proportions of each asset class by adding the following constraints:

$$H_{i,t,s} \leq u_i \sum_{i=1}^{I} H_{i,t,s} \ , \qquad i = 1 \ldots I, \ t = 1 \ldots T, \ s = 1 \ldots S \tag{2.12}$$

2.2 Modelling Risk

Omitting the time index, let us denote the funding ratio by F; it is defined as the ratio of assets over liabilities A/L [26]. It is an important measure in determining the financial soundness of a fund at a certain time moment. It is commonly used to model risk constraints in ALM, although there are variations in the way these are modelled. Common practice is to use a *target* funding ratio, a number $\lambda \geq 1$ below which the value of the funding ratio should not fall. Values of $\lambda > 1$ are often used to add some extra safety margin. Consider decisions taken "now"; the future value of the funding ratio is not known with certainty at decision time. It is modelled as a discrete random variable, described by possible outcomes under the S scenarios.

One may require that the funding ratio (at a specific time t) is greater than or equal to the target λ with probability 1; that is

$$F_{t,s} \geq \lambda \qquad \text{or} \qquad A_{t,s} \geq \lambda L_{t,s} \qquad \forall\ s = 1 \dots S \tag{2.13}$$

This however may be infeasible or very costly; such a conservative approach may lead to a considerable decrease of performance, in the sense of not achieving high returns.

A common approach is to allow violation of these constraints and penalise them in the objective function. This approach has benefits in the sense that it does not result in infeasibility. It however offers limited scope for modelling. Important classes of SP models can be employed to replace constraints (2.13) by a "risk constraint" on the funding ratio. Such a risk constraint is connected to the probability of not achieving the target ratio or/and the magnitude of shortfall below the target ratio.

2.2.1 Chance Constrained Programming (CCP)

An important class of stochastic programming models, introduced by Charens and Cooper [9], is the Chance Constrained Programming (CCP). The concept behind the *probabilistic* or *chance constraints* is that instead of satisfying a constraint with probability equal to 1, the constraint is satisfied with given (high) probability.

In an ALM approach, chance constraints serve as tools to control the probability of underfunding; instead of (2.13) we impose $A_t \geq \lambda L_t$ with high probability; or equivalently, we allow this constraint to be violated only under a small percentage of scenarios:

$$P(F_t < \lambda) \leq (1-\beta) \quad \text{or} \quad P(A_t < \lambda L_t) \leq (1-\beta)$$

Where $\beta \in [0,1]$ is a user pre-specified parameter indicating the "reliability level", typically $0.95 \leq \beta < 1$. Thus, there is a small proportion $(1-\beta)$ of scenarios under which we might have the funding ratio below λ.

The inclusion of chance constraints in multi-stage recourse model for pension funds was used by Dert [17]. The model formulation includes binary variables that count the number of times when the constraint is violated. Following [17], for a specific time $t = 1, \ldots, T$, we formulate the chance constraint $P(A_t < \lambda L_t) \leq (1-\beta)$ by introducing S additional binary decision variables $\delta_{t,s},\ s = 1..S$ that count the number of scenarios under which there is underfunding; i.e. when $\lambda L_t - A_t > 0$. We add the following constraints to (2.2)-(2.12) in order to restrict the probability of underfunding:

$$M\delta_{t,s} \geq \lambda L_{t,s} - A_{t,s}\ , \qquad t = 1 \ldots T,\ s = 1 \ldots S \tag{2.14}$$

$$M(1-\delta_{t,s}) - \frac{1}{M} \geq (A_{t,s} - \lambda L_{t,s})\ , \qquad t = 1 \ldots T,\ s = 1 \ldots S \tag{2.15}$$

$$\sum_{s=1}^{S} \pi_s \delta_{t,s} \leq 1 - \beta\ , \qquad t = 1 \ldots T \tag{2.16}$$

$$\delta_{t,s} \in \{0,1\}\ , \qquad t = 1 \ldots T,\ s = 1 \ldots S$$

The additional parameters for the CCP models are:

λ = User pre-specified target funding ratio; a number ≥ 1;

β = Reliability level: a parameter that indicates the probability of satisfying the constraint; $\beta \in (0, 1)$ with values close to 1;

M = Sufficiently large number.

For each time $t = 1 \ldots T$ when a chance constrain is imposed, there are S additional binary variables $\delta_{t,s}$, $s = 1 \ldots S$ defined as follows. If there is underfunding at time t under scenario s, that is, if $\lambda L_{t,s} - A_{t,s} > 0$, then $\delta_{t,s} = 1$ due to equation (2.14). Equation (2.15) becomes $A_{t,s} - \lambda L_{t,s} \leq -\frac{1}{M}$, which is satisfied for M large enough. If there is no underfunding at time t under scenario s, that is, if $\lambda L_t - A_t \leq 0$, then $\delta_{t,s} = 0$ due to equation (2.15). To summarise,

$$\delta_{t,s} = \begin{cases} 1 & \text{if} \quad \text{underfunding at time } t \text{ under scenario } s \text{ occures} \\ 0 & \quad\quad \text{elsewhere} \end{cases} \tag{2.17}$$

If we want a strict correspondence between the value of $\delta_{t,s}$ and the funding situation (i.e. $\delta_{t,s}$=1 is equivalent to an underfunding situation at time t under scenario s and $\delta_{t,s}$=0 is equivalent to no underfunding) then (2.15) is necessary.

Chance constraints control the likelihood of a shortfall, but have no control on the amount of shortfall; there is the possibility of the amount of underfunding (occurring under low probability) being unacceptably large. In addition, including binary variables increases the computational complexity. These disadvantages are not shared by the closely related *Integrated Chance Constrained programming* models which are introduced in the next section.

2.2.2 Integrated Chance Constrained Programming (ICCP)

Klein Haneveld ([33]; [35]) introduced an alternative to CCP called *Integrated Chance Constrained programming* (ICCP) and applied it in [34] for modelling short term risk ($t = 1$) in an ALM model for Dutch pension funds.

With an ICC, the amount of expected shortfall is controlled, rather than the probability of shortfall: $E[\max\{\lambda L_t - A_t, 0\}] \leq \theta$, where θ is the maximum amount of average underfunding that a decision maker accepts. This could be equivalently formulated as $E[\max\{\lambda - F_t, 0\}]$ does not exceed a pre-specified level.

Modelling such a constraint does not require additional binary variables but only continuous ones, as formulated below:

$$A_{t,s} - \lambda L_{t,s} + Sh_{t,s} \geq 0 \ , \qquad t = 1 \dots T, \ s = 1 \dots S \tag{2.18}$$

$$\sum_{s=1}^{S} \pi_s Sh_{t,s} \leq \theta \ , \qquad t = 1 \dots T \tag{2.19}$$

$$Sh_{t,s} \geq 0 \ , \qquad t = 1 \dots T, \ s = 1 \dots S$$

Where $Sh_t = \max\{\lambda L_t - A_t, 0\}$ is a random variable that measure the shortage at time t, $t = 1 \dots T$; its realisations $Sh_{t,s}$ under each scenario s, $s = 1 \dots S$ take the value 0 if the asset value $A_{t,s}$ is above $\lambda L_{t,s}$ and otherwise take a value equal to the shortfall $\lambda L_{t,s} - A_{t,s}$. Equation (2.19) ensures that the expected value of shortage will not exceed the user's defined maximum allowed expected shortfall θ. θ is a parameter determined by the decision maker. Choosing this parameter in a meaningful way is a non-trivial task, as pointed out by Klein Haneveld et al. in [34].

2.3 Summary of the Formulations of Stochastic Programming Models for ALM

The Objective Function

$$\mathbf{Max} \sum_{s=1}^{S} \pi_s A_{T,s}$$

Subject to:

- **Asset Value Constraints**

$$A_{1,s} = \sum_{i=1}^{I} H_{i,0} R_{i,1,s} \ , \qquad s = 1 \ldots S$$

$$A_{t,s} = \sum_{i=1}^{I} H_{i,t-1,s} R_{i,t,s} \ , \qquad t = 2 \ldots T, \ s = 1 \ldots S$$

- **Asset Holding Constraints**

$$H_{i,0} = OP_i + B_{i,0} - S_{i,0} \ , \qquad i = 1 \ldots I$$

$$H_{i,1,s} = H_{i,0} R_{i,1,s} + B_{i,1,s} - S_{i,1,s} \ , \qquad i = 1 \ldots I, \ s = 1 \ldots S$$

$$H_{i,t,s} = H_{i,t-1,s} R_{i,t,s} + B_{i,t,s} - S_{i,t,s} \ , \qquad i = 1 \ldots I, \ t = 2 \ldots T-1, \ s = 1 \ldots S$$

$$H_{i,T,s} = H_{i,T-1,s} R_{i,T,s} \ , \qquad i = 1 \ldots I, \ s = 1 \ldots S$$

- **Fund Balance Constraints**

$$\sum_{i=1}^{I} B_{i,0}(1+\psi) + L_0 = \sum_{i=1}^{I} S_{i,0}(1-\psi) + C_0$$

$$\sum_{i=1}^{I} B_{i,t,s}(1+\psi) + L_{t,s} = \sum_{i=1}^{I} S_{i,t,s}(1-\psi) + C_{t,s} \ , \qquad t = 1 \ldots T-1, \ s = 1 \ldots S$$

- **Short-Selling Constraints**

$$S_{i,0} \leq OP_i \ , \qquad i = 1 \ldots I$$

$$S_{i,t,s} \leq H_{i,t-1,s} \ , \qquad i = 1 \ldots I, \ t = 1 \ldots T-1$$

- **Bound Constraints**

$$H_{i,t,s} \leq u_i \sum_{i=1}^{I} H_{i,t,s} \ , \qquad i = 1 \ldots I, \ t = 1 \ldots T, \ S = 1 \ldots S$$

- **The Chance Constraint**

$$M\delta_{t,s} \geq \lambda L_{t,s} - A_{t,s} \ , \qquad t = 1 \ldots T, \ S = 1 \ldots S$$

$$M(1 - \delta_{t,s}) - 1/M \geq (A_{t,s} - \lambda L_{t,s}) \ , \qquad t = 1 \ldots T, \ S = 1 \ldots S$$

$$\sum_{s=1}^{S} \pi_s \delta_{t,s} \leq 1 - \beta \ , \qquad t = 1 \ldots T$$

$$\delta_{t,s} \in \{0, 1\} \ , \qquad t = 1 \ldots T$$

- **The Integrated Chance Constraint**

$$A_{t,s} - \lambda L_{t,s} + Sh_{t,s} \geq 0 \ , \qquad t = 1 \ldots T, \ s = 1 \ldots S$$

$$\sum_{s=1}^{S} \pi_s Sh_{t,s} \leq \theta \ , \qquad t = 1 \ldots T$$

$$Sh_{t,s} \geq 0 \ , \qquad t = 1 \ldots T, \ s = 1 \ldots S$$

The corresponding AMPL code for SP and ICCP models can be found in Appendix **C**.

2.4 Connection with Risk Measures

In general, risk measures used in a financial context could be classified into two main groups.

Risk measures of the **first kind** consider the magnitude of the deviation from a target [50]. The target could be fixed (e.g. a minimal acceptable return), distribution dependent (e.g. the expected value) or even stochastic (e.g. an index). The values of these risk measures can only be positive. Moreover, these risk measures can be further divided into: *symmetric* (two-sided) and *asymmetric* (one-sided, downside, shortfall) risk measures. Symmetric risk, such as variance [40], penalise any deviation from the target either upside or downside. Asymmetric risk measures penalise only deviations below a specified target; any realisation above the target does not count in the risk quantification. This makes asymmetric risk measures to be more in accordance with the intuitive idea about risk, as an undesirable result. Among asymmetric risk measures, Lower Partial Moments (LPM) introduced by Fishburn [24] and Bawa [4] in the context of single stage asset allocation are of great importance.

Risk measures in the **second category** consider only a pre-specified percentage of the lower tail of a distribution. They take into account only a certain number of worst outcomes of the distribution, depending on a confidence level α. In a financial context, commonly used risk measures in this category are Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) [53]. The values of these risk measures can be both positive and negative.
In this section, we review some of the well known risk measures used in ALM and illustrate the connection between these risk measures and the use of chance constraints and integrated chance constraints in an ALM context.

2.4.1 Lower Partial Moments (LPMs)

The LPM with target τ and order n of a random variable R (e.g. representing future return) is by definition

$$E[\max\{\tau - R, 0\}^n].$$

The most commonly used below- target risk measures could be formulated as lower partial moment as follows:

- **Safety First:** In the particular case $n \to 0$,

  $$SF(R) = LPM_{n\to 0}(\tau, R) = E[\max\{\tau - R, 0\}^{n\to 0}] = P(R < \tau)$$

In this case, the LPM measures the probability of falling below target τ; this had been introduced in asset allocation models by Roy [57].
As per Section 2.2.1, a chance constraint in ALM models requires that, for a specified time period t, $P(F_t < \lambda) \leq (1-\beta)$ or $P(A_t < \lambda L_t) \leq (1-\beta)$, where β is the reliability level. This is the same as imposing an upper limit $(1-\beta)$ on the lower partial moment with target λ and order 0 of the random variable $F_t = A_t/L_t$ representing the funding ratio at time t.

- **Target Shortfall:** In the particular case n = 1,

$$LPM_1(\tau, R) = E[\max\{\tau - R, 0\}]$$

As per Section 2.2.2, an ICCP constraint require that $E[\max\{\lambda L - A, 0\}] \leq \theta$, where θ is the maximum amount of average underfunding that a decision maker accepts. This is the same with imposing an upper limit θ on the lower partial moment with target 0 and order 1 of the random variable $A - \lambda L$.

- **Target Semi-Variance:** In the particular case n = 2,

$$LPM_2(\tau, R) = E[\max\{\tau - R, 0\}^2]$$

Risk control in ALM models via lower partial moments of order two of the funding ratio has been adopted by Kouwenberg [38] to measure and control both the probability and the level of deficits: $E([\max\{\lambda - F, 0\}]^2) \leq \theta$.

2.4.2 Conditional Value-at-Risk (CVaR)

Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) are widely used risk measures that consider only a pre-specified percentage of worst case scenarios. They have been defined either in the context of a return distribution R (where the left tail is of interest) or of a loss distribution L (where the right tail is of interest).
For a generic loss distribution L, VaR at confidence level α (where $\alpha = A\% \in (0, 1)$) is an α-quantile, which is, loosely speaking, an outcome of L such that

$A\%$ of the outcomes of L are better (i.e. lower) and $(1 - A\%)$ are worse. Formally, for the loss distribution L (where $L = -R$) with a cumulative distribution function H, VaR at confidence level α (denoted by VaR_α) is the value

$$\mathrm{VaR}_\alpha = \min\{t \in \mathbb{R} \mid H(t) \geq \alpha\}$$

where $H(t) = P(L \leq t)$ [53].

CVaR at the same confidence level α can be defined as, largely speaking, the average of losses in the worst $(1 - A\%)$ of cases. Mathematically, in our setting, CVaR_α can be given as

$$\mathrm{CVaR}_\alpha = (1 - \alpha)^{-1} \int_{L \geq \mathrm{VaR}_\alpha} L \, dH$$

In this formula, if the $P(L \geq \mathrm{VaR}_\alpha) = 1 - \alpha$, then, CVaR_α is the conditional expectation of the losses exceeding or equal VaR_α [53].

In the context of multi-stage ALM, CVaR was used by Bogentoft et al. [6]; they considered a loss random variable defined by $\lambda L - A$ and imposed an upper limit on its CVaR to model short term risk (that is, for time $t = 1$). If CVaR is considered at confidence level $A\% = \alpha \in (0, 1)$ (e.g. $\alpha = 0.95$), such a constraint can be expressed as follows: the average of the highest $(1 - A\%)$ of the losses is not higher than a level pre-specified by the decision maker.

Just as with ICC's, a CVaR constraint is modeled by introducing additional (continuous) variables and linear constraints; the reader is referred to [53] and [6]. We can summarise the difference between an ICC constraint and a CVaR constraint as follows. With ICCP, all the outcomes of the funding ratio are considered, in which the target funding ratio is not met. CVaR constraints are at the opposite spectrum of modelling the funding ratio distribution, as they look only at a pre-specified percentage of worst case outcomes; these worst case outcomes may or may not include all the cases in which the target funding ratio is not met and have no control on the rest of the distribution. It can be argued that ICCP provides a better modelling approach since all scenarios when underfunding occurs are considered. But on the other hand, there is less control in worst case scenarios; although the average underfunding may seem acceptable, the possibility exists for a heavy left tailed loss distribution.

2.5 Concluding Remarks

In this chapter, we have formulated a two-stage SP model for ALM. The risk of underfunding, that is, the risk of funding ratio being below a target λ, can be modelled, for example via a chance constraint (CC) or an integrated chance constraint (ICC). The former restricts the number of underfunding events while the latter restricts the expected amount of underfunding. We have formulated both the CC and ICC within the two-stage SP.
We have also reviewed some of the well known risk measures, traditionally used in single stage asset allocation; we concentrated in particular on LPMs and CVaR, because they have been also used in ALM models in connection with the risk of underfunding. CVaR has been considered for loss distributions related to the funding ratio failing to achieve target λ, more precisely $\lambda L - A$. CCs and ICCs can be formulated as constraints requiring that LPM of target λ and order 0 and 1 respectively are not higher that a pre-specified amount. The LPMs are considered for random variables representing the funding ratios.

Chapter 3

Optimisation Models for ALM Based on Stochastic Dominance

3.1 Introduction and Motivation

As pointed out throughout the previous chapters, the ALM problem for pension funds is inherently a dynamic decision problem under uncertainty; it involves making decisions on changes in the composition of the asset portfolio at each decision moment aiming to keep the firm solvent through the planning horizon. The risk of underfunding has been commonly modelled by using the expected shortfall of the funding ratio below its target value, either in a constraint or in the form of a penalty term in the objective function. This approach does non exclude the possibility of a funding ratio distribution with unacceptably low values under a very small percentage of worst case scenarios. For example, a distribution might have a small percentage of very low outcomes and considerably higher outcomes in the rest; in this case, an integrated chance constraints may be satisfied but the worst case realisation could be unacceptably low. Similarly, imposing an upper limit in a CVaR constraint may guarantee manageable outcomes even under worst case scenarios, but ignore the rest of the distribution. For instance, a distribution may be "flat" in that the worst case realisations are not very low, but with little improvement in the rest of the distribution, including possibly too many scenarios in which the target funding ratio is not met. Indeed, in imposing risk constraints, only one single aspect of the distribution of interest is controlled; on the other hand, they are convenient from a computational point of view.

In this research, we propose an alternative risk modelling framework based on the concept of Second Order Stochastic Dominance (SSD). This is a criterion of ranking random variables that takes the entire distribution of outcomes into account. It is applied to random variables whose outcomes are desired to be high and for whom an increase is more valued if it is at low levels, rather than at high levels; typical such random variables represent return or wealth. By definition, a random variable is preferred to another with respect to SSD if its expected utility is higher, for any non-decreasing and concave utility function [68]. SSD eliminates the need to specify a utility function, which is difficult to elicit, but works under the general and widely accepted assumptions of decision makers being rational (utility function is non-decreasing) and risk averse (utility function is concave). Hence, the conceptual advantages of using SSD as a choice criterion can be clearly seen: it expresses the preference of rational and risk-averse decision makers. It is obviously desirable to eliminate the random variables that are dominated and make a choice among the SSD non-dominated ones, possibly employing another criterion to help in the final selection. However, stochastic dominance is demanding from a computational point of view, we elaborate on this later in this chapter.

In the ALM optimisation models proposed in this chapter, SSD is used as a choice criterion for random variables representing funding ratios. The approaches developed in [55] and [22] are extended and adapted to the ALM multi-period case, taking into account the relationship between asset value and liabilities. An optimal solution has a corresponding funding ratio distribution that is SSD non-dominated; in addition, it comes close, in well defined sense, to a target distribution of funding ratio, whose outcomes are specified by the decision maker. A constraint on the expected terminal wealth is imposed, by considering a minimum acceptable compounded return. Different target distributions lead to different SSD efficient solutions, target distributions can be chosen by the decision maker in an interactive way, by analysing the funding ratio distribution obtained, modifying the target distribution accordingly and re-running the optimisation model with the new target distribution. Improved distributions of funding ratios may be thus achieved, compared to the existing risk models for ALM, which is a contribution of this work. Another contribution is of theoretical nature; interesting results, connecting the proposed models to well established risk models and well established classes of SP models are derived for the particular case when the target distribution is deterministic, specified by one single outcome.

The advantage of using SSD models over previous approaches of imposing a risk constraint lies not only in better modelling of the (entire) funding ratio distribution. With a risk constraint on the funding ratio distribution, the decision maker has to set a right hand side, which is not a straightforward task in this context. Klein Haneveld et al. [34] stressed that a priori selection of the model parameter is difficult in practice and a reasonable parameter could be found by setting a numerical experiments. Fabian and Veszprémi [23] construct an approximation of the efficient frontier to help the decision maker in calibrating the right hand side parameter in CVaR constraints as a first step in formulating their model.
The chosen maximum acceptable level of deficit can lead to infeasibility, or, in the opposite case, it may be under-restrictive. These issues are not encountered in our approach.

Previous work in using SSD within ALM models employed the concept in the constraints; this, may result in infeasibility or under-achievement depends on the target chosen by the decision maker. In solving the models proposed in this chapter, even if the target distribution has too high / unachievable values, the model is not infeasible; its optimal solution represents an investment decision with an SSD efficient funding ratio distribution, that comes "as close as possible" to the target. In the opposite case, if the target distribution has not "high" enough outcomes, the resulting distribution of funding ratio will be "better than target", that is, not just attain it, but improve on it until SSD efficiency is obtained. More precisely, there are three possible cases. Firstly, if the reference distribution is dominated with respect to SSD, the model provides a solution which comes close to this reference distribution, but is SSD efficient (non-dominated). In this case, the achieved funding ratio distribution is "better than target". In the second case, if the reference distribution is non-dominated with respect to SSD, the model provides a portfolio, whose resulting funding ratio distribution is exactly the reference distribution. Finally, the reference distribution may not be attainable (in the sense that some of its outcomes are too high such that there are no feasible portfolio weights that could produce such a funding ratio distribution). In this case, the model provides a SSD efficient portfolio whose associated funding ratio distribution comes close to the reference distribution.

In Section 3.2, we present basic definitions of stochastic dominance and discuss the particular case in which the random variables under consideration have equally likely, or equi-probable, outcomes. We show that the SSD non-dominated solutions are Pareto optimal solutions of a multi-objective model, in which the objective functions represent tails of the funding ratio, at different confidence levels. An introduction to multi-objective optimisation is presented in Section 3.3. In Section 3.4, two versions of the SSD model are formulated: (SSD-Scaled) and (SSD-Unscaled). With the former model, the objective functions (to which we set targets) are scaled tails, or conditional expectations of progressively higher percentages of worst case scenarios; with the latter, the objective functions represent unscaled tails.
Interesting particular cases for these models are obtained when the target distribution is deterministic: having a single outcome that occurs with probability one. The simpler equivalent formulation of these models and the connection with risk minimisation is discussed in Section 3.5. To widen the range for modelling and controlling the funding ratio distributions, the SSD models has been extended to include "reservation" levels (that should "pre-empt" aspiration levels, that is, have priority in being achieved) in addition to aspiration levels. This model is presented in Section 3.6. Finally, concluding remarks are presented in Section 3.7.

3.2 Second Order Stochastic Dominance (SSD) and the Case of Equally Likely Scenarios

Stochastic dominance ranks random variables under assumptions about general characteristics of utility functions, drawn from observed economic behavior. In practice, two random variables - representing, for example, portfolio returns, asset values, or funding ratios - are compared by point wise comparison of some performance functions constructed from their distribution functions. The usual definition of stochastic dominance uses cumulative distribution functions, please refer to Section 1.2. SSD is formally defined using the second performance function:

$$H^{(2)}(r) = \int_{-\infty}^{r} H(t)dt, \quad r \in \mathbb{R},$$

where $H(r) = P(R \leq r) \quad \forall r \in \mathbb{R}$. For the random variables R and R' with cumulative distribution functions H and H' respectively, we say that R dominates R' with respect to second order stochastic dominance (denoted: $R \succ_2 R'$) if and only if: $H^{(2)}(r) \leq H'^{(2)}(r)$, $\forall r \in \mathbb{R}$, with at least one strict inequality [68].

SSD is of particular importance in investments because of its connection with non-decreasing and concave utility functions, which represent the preference of risk averse investors (see Section 1.2). In what follows, we state four equivalent definitions of SSD.

Proposition 3.1. For random variables R and R', the following conditions are equivalent:

(a) $E(U(R)) \geq E(U(R'))$ holds for any utility function U that has the properties of non-satiation (it is non-decreasing, first derivative is positive) and risk aversion (it is concave, second derivative is negative) and for which these expected values exist and are finite.

(b) $E([\tau - R]^+) \leq E([\tau - R']^+)$ holds for each target $\tau \in \mathbb{R}$. In other words, the expected shortfall with respect to any target is always lower for the first random variable.

(c) $\text{Tail}_\alpha(R) \geq \text{Tail}_\alpha(R')$ holds for each $0 < \alpha \leq 1$, where $\text{Tail}_\alpha(R)$ denotes the unconditional expectation of the least $\alpha * 100\%$ of the outcomes of R.

(d) $\text{ScaledTail}_\alpha(R) \geq \text{ScaledTail}_\alpha(R')$ holds for each $0 < \alpha \leq 1$, where $\text{ScaledTail}_\alpha(R)$ denotes the conditional expectation of the least $\alpha * 100\%$ of the outcomes of R; $\text{ScaledTail}_\alpha(R) = \frac{1}{\alpha}\text{Tail}_\alpha(R)$.

If any of the relations above hold, the random variable R is said to dominate the random variable R' with respect to SSD. The proof for the equivalence of SSD to (a), (b) and (c) was proved in [68], [48] and [49] respectively.

For the rest of this work, we consider the special case where the random variables to compare are discrete with equally probable outcomes. This is the usual situation when scenarios are generated via sampling from historical data. In this case, SD relations can be greatly simplified as illustrated in this section.

Definition 3.1. *The random variables R and R' defined on a discrete probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ are equal in distribution, written $R =^d R'$, if the distribution functions of R and R' are identical* [68].
Random variables equal in distribution need not be identical [68].

Now, suppose $\widetilde{R}$ and $\widetilde{R}'$ are defined on $\{\Omega, \mathcal{F}, P\}$, with $\Omega = \{1, ..., S\}$, $\mathcal{F}$ is a σ-field and $P(s) = 1/S$, $s = 1, ..., S$, and have outcomes α_1,...,α_S and β_1,...,β_S respectively. Without loss of generality, we denote by R and R' the random variables equal in distribution to $\widetilde{R}$ and $\widetilde{R}'$, respectively, whose outcomes are in ascending order: $\alpha_1 \leq .... \leq \alpha_S$ and $\beta_1 \leq ... \leq \beta_S$. In this case, first order stochastic dominance and second order stochastic dominance relations may be written as follows:

Proposition 3.2. *In the case of discrete random variables R and R' having equally likely outcomes $\alpha_1 \leq .... \leq \alpha_S$ and $\beta_1 \leq ... \leq \beta_S$ respectively, $R \succ_1 R'$ if and only if $\alpha_k \geq \beta_k$, $\forall k \in \{1, ..., S\}$ with at least one strict inequality* [54].

Proposition 3.3. *In the case of discrete random variables R and R' having equally likely outcomes $\alpha_1 \leq .... \leq \alpha_S$ and $\beta_1 \leq ... \leq \beta_S$ respectively, $R \succ_2 R'$ if and only if $\sum_{i=1}^{k}\alpha_i \geq \sum_{i=1}^{k}\beta_i$, $\forall k \in \{1, ..., S\}$ with at least one strict inequality* [54].

Proofs of Propositions 3.2 and 3.3 could be found in [54].

Definition 3.2. For a set Q of random variables, a variable $R \in Q$ is called *SSD-efficient* (or *FSD-efficient*) in Q if there is no $R' \in Q$ such that $R' \succ_2 R$ (or $R' \succ_1 R$).

Obviously, dominance in the larger class necessarily implies dominance in the smaller class (but not conversely). Thus, $R \succ_1 R'$ implies $R \succ_2 R'$ [68].

Remark 3.1. SSD is stronger than FSD in the sense that it is able to order more pairs of random variables. We could have indifference between R and R' with respect to FSD but prefer R or R' with respect to SSD. Consider the following example, let R have outcomes 1, 1, 4, and 4 (all with probability 1/4) and R' has outcomes 3, 4, 0, 2 (all with probability 1/4). By ordering the outcomes of R and R' we obtain two 4-dimensional vectors: (1, 1, 4, 4) and (0, 2, 3, 4) respectively. Note that there is indifference between R and R' with respect to FSD. However, if we cumulate the outcomes of these vectors, we obtain the vectors: (1, 2, 6, 10) and (0, 2, 5, 9) respectively. Hence, R dominates R' with respect to SSD.
As a final comment, the set of SSD non-dominated random variables is a subset of the larger set of FSD non-dominated random variables.

Remark 3.2. Note that in the example in the previous Remark, we have indifference between R and R' according to the generic comparison framework of a mean-variance model, since, although R has a higher expected value, it also has higher risk. If in addition the DM has a risk-aversion parameter ρ, then there will be a clear preference between R and R' depending on the value of $E(R) - \rho\,\sigma^2(R)$ (higher values preferred).
Note also that any rational and risk averse DM will prefer R irrespective to the utility values that s/he will attach to each of the outcomes (as long as these values are drawn from a non-decreasing and concave function; the values will increase as the outcomes increase and will assign a value that shows less appreciation for each additional increment as the outcomes increase).

Remark 3.3. *If R is a discrete random variable with equally likely outcomes $\alpha_1 \leq .... \leq \alpha_S$, we have:*

$$\mathrm{Tail}_{k/S}(R) = (\sum_{i=1}^{k} \alpha_i)/S$$

and

$$\mathrm{ScaledTail}_{k/S}(R) = (\sum_{i=1}^{k} \alpha_i)/k$$

In the case of random variables with equally likely outcomes, the comparisons in Proposition 3.1 dramatically reduce to a finite number (S comparisons). As in [36], [55] also used in [21], [22], it is enough in this case to compare tails of random variables R and R' only for confidence levels k/S with $k = 1...S$.
In this situation, R dominates R' with respect to SSD if and only if

$$\mathrm{Tail}_{k/S}(R) \geq \mathrm{Tail}_{k/S}(R') , \qquad k = 1 \ldots S$$

or equivalently,

$$\mathrm{ScaledTail}_{k/S}(R) \geq \mathrm{ScaledTail}_{k/S}(R') , \qquad k = 1 \ldots S$$

with at least one strict inequality.

Thus, finding the SSD efficient solutions could be seen as finding the optimal solutions of a problem with S objectives to consider: the tails (or scaled tails) of the random variable of interest, for different confidence levels. In the next section, we go through a few useful definitions and results on *Multi-Objective Optimisation (MOO)*.

3.3 Multi-Objective Optimisation and the Reference Point Method

Let us consider a generic form of a multi-objective optimisation problem with S objective functions to maximise:

$$\mathbf{Max}\{Obj(x) = (Obj_1(x), ..., Obj_S(x)) : x \in \mathcal{X}\} \tag{3.1}$$

Where $Obj_k(x)$ is the k-th objective function or the criteria, $k \in \{1, .., S\}$, $x \in \mathbb{R}^n$ is a decision vector, and $\mathcal{X}$ is the feasible decision space or the constraints set.

$y \in \mathbb{R}^S$ is called an *achievement vector* if there exists a feasible solution $x \in \mathcal{X}$ such that $y = Obj(x)$ and the set $Y = \{y = Obj(x) | x \in \mathcal{X}\}$ is called the feasible criterion space or the attainable set [41].

Usually, in MOO two achievement vectors are compared using the Pareto preference relation, defined as follows:

Definition 3.3. A feasible solution $x_1 \in \mathcal{X}$ *Pareto dominates* another feasible solution $x_2 \in \mathcal{X}$ if: $Obj_k(x_1) \geq Obj_k(x_2), \forall k \in \{1, ..., S\}$ with at least one strict inequality [56].

A *Pareto optimal*, or *non-dominated*, or *efficient* solution is one such that no other feasible solution Pareto dominates it. Thus, a Pareto efficient solution is a feasible solution such that, in order to improve upon one objective function, at least one other objective function must assume a worse value.

Pareto-optimality does not tell which decisions to choose; it only tells which decisions to avoid. Clearly, it is reasonable to restrict attention on the Pareto optimal solutions, but the problem remains how to select one of them. There are various methods of obtaining a Pareto optimal solution of a multi-objective optimisation problem (see for example [11] , [20], and [41]). A good control on obtaining a specific solution is given by the *Reference Point Method (RPM)* [71].
In the RPM, reference points (or aspiration points) are set by the DM for the values of the objective functions: they are desired values for the objective functions. Then, the multi-objective optimisation is transformed into a single objective optimisation by maximising an *achievement function*: a scalar function constructed depending on the reference points, such that, when optimised, generates a Pareto optimal solution of the original multi-objective problem; the reader is referred to [71] and [70] for a detailed treatment of the subject.

Consider the general case of S real-valued objective functions $Obj_1, \dots Obj_S$ defined on a set $\mathcal{X} \in \mathbb{R}^n$ representing a feasible set of decision vectors and consider the multi-objective model: $\mathbf{Max}\{(Obj_1(x), \dots Obj_S(x))\ \mathbf{s.t.}\ x \in \mathcal{X}\}$. For each of the functions Obj_k, a target point asp_k is set by the decision maker; let $asp = (asp_1, ..., asp_S)$ be the vector of targets/aspiration levels. We can measure the actual achievement of the k-th objective function with respect to its corresponding aspiration level asp_k by the so called *partial achievement functions*, defined for any feasible point. Various functions provide a wide modelling environment for measuring individual achievements. The simplest form of partial achievement functions is the one that measures the difference between the values of the objective functions and their targets: $Obj_k(x) - asp_k$. This could be replaced with more complicated functions depending on $Obj_k(x)$ and asp_k, which must satisfy certain properties: being monotonically increasing functions with respect to $Obj_k(x)$ and taking value 0 if $Obj_k(x) = asp_k$ (see for example [71] and [55]).

A commonly used type of scalarising achievement function (to be optimised), that we also use in our approach, is one that considers the worst deviation of an objective function from its reference point:

$$\delta(x) = \min_{k=1\ldots S}\{Obj_k(x) - asp_k\}$$

It was shown in [69] that maximisation of such an achievement function leads to a Pareto optimal solution of the multi-objective model. In the case of multi optimal solutions, at least one of them will be Pareto optimal, but not necessary all of them. To guarantee Pareto efficiency in the general case, a regularisation term is added to the worst partial achievement: $\epsilon \sum_{k=1}^{S}(Obj_k(x) - asp_k)$, where $\epsilon > 0$ is an arbitrary small parameter.

Remark 3.4. Different values of ϵ can lead to different Pareto optimal solutions. It was shown in [69] that for all $\epsilon > 0$, the maximisation of $\delta(x) + \epsilon \sum_{k=1}^{S}(Obj_k(x) - asp_k)$ results in a Pareto optimal solution for $\mathbf{Max}\{(Obj_1(x), \ldots, Obj_S(x))\ \mathbf{s.t.}\ x \in \mathcal{X}\}$. However, a small enough value of ϵ should be chosen (possibly on a trial and error basis) to ensure that optimisation of the worst partial achievement is achieved.

The Reference Point Method has been considered as a generalisation of Goal Programming approach as the objective function values do not necessary attain their reference points [71]. In RPM, the main advantage of goal programming is preserved: the appealing idea that we set a goal in the objective space and try to come close to it, without the danger of infeasibility. However, it does more than this, because the meaning of "coming close" is not the traditional one (a distance minimisation), but "coming close or better" [70]. This sense of coming close is deeply related to how in reality people make decisions. The Pareto efficiency of the solution is guaranteed in this method: when the reference point is not a Pareto efficient achievement vector, the method improves on it (i.e. find a decision and thus attainable achievement vector that improves at least one of its outcomes without deteriorating another one), resulting still in a Pareto optimal solution. As opposed to goal programming, optimisation continues even after the goal has been reached [71].

In this method, the obtained Pareto efficient solution results in an achievement vector that is:

- Better than the reference point, in the case if the chosen reference point is not a Pareto efficient achievement vector of the multi-objective problem under consideration.

- Exactly the reference point, if the reference point is a Pareto efficient achievement vector of the multi-objective problem.

- Close to the reference point, in the sense of maximising the worst difference between the cumulative outcomes of the obtained solution and of the reference point, if the reference is not attainable [71]. An alternative formulation, that works both for scaled and unscaled cases, is the worst difference between the (scaled) tails of the obtained solution and those of the reference point.

3.4 Models for ALM Based on SSD

We propose ALM models in which the first-stage investment decisions are such that the resulting funding ratio distribution is non-dominated with respect to SSD, or in other words, SSD efficient. Similarly to [34], we consider the funding ratio at time $t = 1$, thus modelling short-term risk; the approach can be extended for more time periods.

Following [55], we assume that the probabilities of scenarios are equal and thus the short term funding ratio is a discrete random variable with equally likely outcomes. As explained in Section 3.2, the comparison between two random variables with respect to SSD can be greatly simplified in this case; it is enough to compare tails only for confidence levels $\frac{k}{S}$ with $k = 1 \dots S$.

With the ALM setting described in Section 2.1, let us consider the feasible set of solutions $\{B_{i,0},\ S_{i,0},\ H_{i,0},\ B_{i,t,s},\ S_{i,t,s},\ H_{i,t,s},\ H_{i,T,s}\ ,\quad i = 1 \dots I,\ t = 1 \dots T-1,\ s = 1 \dots S\}$ satisfying equations (2.2) to (2.12), that define the feasibility conditions for our generic ALM model, and in addition a constraint on the expected terminal wealth requirement:

$$\sum_{s=1}^{S} \pi_s A_{T,s} \geq \sum_{i=1}^{I} OP_i(1+d) \tag{3.2}$$

where $A_{T,s} = \sum_{i=1}^{I} H_{i,T-1,s} R_{i,T,s}$; $R_{i,T,s}$ is the rate of return of asset i at the final time period T under scenario s.

Let us consider two sets of feasible first stage decisions $(H_{i,0},\ B_{i,0},\ S_{i,0},\ i = 1 \ldots I)$ and $(H'_{i,0},\ B'_{i,0},\ S'_{i,0},\ i = 1 \ldots I)$ with corresponding funding ratios at time $(t = 1)$ F and F' respectively, described by their possible outcomes:

$$F_s = \sum_{i=1}^{I} H_{i,0} R_{i,1,s} / L_{1,s}\ , \qquad s = 1 \ldots S$$

and

$$F'_s = \sum_{i=1}^{I} H'_{i,0} R_{i,1,s} / L_{1,s}\ , \qquad s = 1 \ldots S$$

respectively; each of these outcomes occurs with probability $1/S$.

Let us order $F_1, \ldots F_S$ and $F'_1, \ldots F'_S$ and lets us denote by $\alpha_1 \leq \ldots \leq \alpha_S$ to the outcomes of F in ascending order and $\beta_1 \leq \ldots \leq \beta_S$ to the outcomes of F' in ascending order too.

With these notations here, the relationships developed in [55] can be written as:

F dominates F' with respect to SSD if and only if

$$\mathrm{Tail}_{k/S}(F) \geq \mathrm{Tail}_{k/S}(F')\ , \qquad k = 1 \ldots S$$

or equivalently,

$$\mathrm{ScaledTail}_{k/S}(F) \geq \mathrm{ScaledTail}_{k/S}(F')\ , \qquad k = 1 \ldots S$$

with at least one strict inequality; please also see [36].

In other words, in the case of equally likely scenarios, the vector of tails

$[\text{Tail}_{1/S}(F), \ldots, \text{Tail}_{S/S}(F)]$ Pareto dominates $[\text{Tail}_{1/S}(F'), \ldots, \text{Tail}_{S/S}(F')]$.

Thus, the solutions that would result in a funding ratio distribution that is non-dominated with respect to SSD (we refer to them as the SSD efficient solutions) can be obtained as the Pareto non-dominated solutions of multi-objective problems in which the objective functions are tails (or scaled tails) at confidence levels $\frac{k}{S}$, with $k = 1 \ldots S$. We consider progressively larger left tails of the funding ratio distribution as multiple objective functions, either scaled (equivalent to averages of a progressively higher number of worst case values), or unscaled (equivalent to sums of a progressively higher number of worst case values). Thus, the problem can be written as:

$$\textbf{Max } (\text{ScaledTail}_{1/S}(F),\ \text{ScaledTail}_{2/S}(F), \ldots,\ \text{ScaledTail}_{S/S}(F)) \qquad (3.3)$$

or we can use the "unscaled" version:

$$\textbf{Max } (\text{Tail}_{1/S}(F),\ \text{Tail}_{2/S}(F), \ldots,\ \text{Tail}_{S/S}(F)) \qquad (3.4)$$

subject to equations (2.2) to (2.12) and (3.2).

Problems (3.3) and (3.4) have an infinite number of efficient solutions; thus, in order to select a specific solution, an additional criterion is needed.
In order to obtain Pareto optimal solutions of (3.3) and (3.4), we use the Reference Point Method, similarly to [55]. In our case, the objective functions represent tails or scaled tails, at different confidence levels, of the funding ratio distribution. By specifying a target of the funding ratio under each scenario, we obtain a target "distribution " of the funding ratio, which result in target points for the tails/scaled tails at different confidence levels $\frac{k}{S}$ $(k = 1 \ldots S)$, and hence an aspiration levels for each objective function.

Let us consider a target distribution of funding ratio, with (equally probable) outcomes λ_k, $k = 1 \ldots S$; without loss of generality, let us consider

$\lambda_1 \leq \ldots \leq \lambda_S$. Let asp_k denote the scaled k-th cumulative outcome,

$$asp_k = \frac{1}{k}\sum_{i=1}^{k}\lambda_i \ , \qquad k = 1 \ldots S$$

and asp'_k denote the unscaled k-th cumulative outcome,

$$asp'_k = \frac{1}{S}\sum_{i=1}^{k}\lambda_i \ , \qquad k = 1 \ldots S$$

The target point for the k-th objective function in (3.3) is asp_k, while asp'_k represents the target point for the k-th objective function in (3.4). Following [69], the multi-objective model (3.3) is transformed into a single objective model by maximising the following achievement function :

$$\min_{k=1\ldots S}(\mathrm{ScaledTail}_{k/S}(F) - asp_k) + \epsilon\sum_{k=1}^{S}(\mathrm{ScaledTail}_{k/S}(F) - asp_k) \qquad (3.5)$$

If the unscaled model is used, the objective function to maximise is:

$$\min_{k=1\ldots S}(\mathrm{Tail}_{k/S}(F) - asp'_k) + \epsilon\sum_{k=1}^{S}(\mathrm{Tail}_{k/S}(F) - asp'_k) \qquad (3.6)$$

In order to express the tails and scaled tails of the funding ratio distribution as functions of the decision variables, we use the following proposition. It expresses a cumulative outcome or a tail, at a specified confidence level, of a random variable as the optimal value of an LP model:

Proposition 3.4. *For every $k \in \{1, \ldots, S\}$, the mean of the worst k outcomes of a random variable $\boldsymbol{y}$ with equally likely finite outcomes $y_1, \ldots, y_S$ is the optimal value of the objective function in the following LP problem:*

$$\mathbf{Max}\ (T_k - \frac{1}{k}\sum_{i=1}^{S}d_{k,i})$$

Subject to:

$$T_k - y_s \leq d_{k,s} \ , \qquad s = 1 \ldots S$$

$$d_{k,s} \geq 0\ , \qquad s = 1 \dots S$$

T_k is a free variable representing the $k-th$ worst outcome of the random variable y. For each $s \in \{1, \dots, S\}$, $d_{k,s} = [T_k - y_s]^+$ that is

$$d_{k,s} = \begin{cases} 0, & \text{if } \ \mathrm{y}_s \geq T_k \\ T_k - y_s, & \text{otherwise} \end{cases} \tag{3.7}$$

For the proof of this proposition, the reader is referred to [47].

Thus, for any feasible decision $\{B_{i,0},\ S_{i,0},\ H_{i,0},\ B_{i,t,s},\ S_{i,t,s},\ H_{i,t,s},\ H_{i,T,s}\ ,\ i = 1 \dots I,\ t = 1 \dots T-1,\ s = 1 \dots S\}$, the scaled tail at confidence level $\frac{k}{S}$ ($k = 1 \dots S$) of the first stage funding ratio F, with outcomes $F_s = \sum_{i=1}^{I} H_{i,0} R_{i,1,s} / L_{1,s}$, $s = 1 \dots S$, is:

$$\text{ScaledTail}_{k/S}(F) = \textbf{Max}\ (T_k - \frac{1}{k}\sum_{i=1}^{S} d_{k,i})$$

Subject to:

$$T_k - F_s \leq d_{k,s}\ , \qquad s = 1 \dots S$$

$$d_{k,s} \geq 0\ , \qquad s = 1 \dots S$$

$$F_s = \sum_{i=1}^{I} H_{i,0} R_{i,1,s} / L_{1,s}\ , \qquad s = 1 \dots S$$

and also subject to equations (2.2) to (2.12) and (3.2).

Hence, we can express the SSD efficient solution of (3.3) as the Pareto efficient solutions of the following multi-objective optimisation problem:

$$\textbf{Max}\ (T_1 - \sum_{i=1}^{S} d_{1,i},\ T_2 - \frac{1}{2}\sum_{i=1}^{S} d_{2,i}, ...,\ T_S - \frac{1}{S}\sum_{i=1}^{S} d_{S,i}) \tag{3.8}$$

Subject to:

$$T_k - F_s \leq d_{k,s}\ , \qquad k, s = 1 \dots S$$

$$d_{k,s} \geq 0\ , \qquad k, s = 1 \dots S$$

and subject to equations (2.2) to (2.12) and (3.2).

Similarly, (3.4) becomes:

$$\textbf{Max } (T_1 - \sum_{i=1}^{S} d_{1,i},\ 2T_2 - \sum_{i=1}^{S} d_{2,i}, ...,\ ST_S - \sum_{i=1}^{S} d_{S,i}) \tag{3.9}$$

Subject to:

$$T_k - F_s \leq d_{k,s}\ , \qquad k, s = 1 \ldots S$$

$$d_{k,s} \geq 0\ , \qquad k, s = 1 \ldots S$$

and subject to equations (2.2) to (2.12) and (3.2).

The objective functions in (3.8) and (3.9) do represent tails and scaled tails of the funding ratio distribution at different confidence levels, $\frac{k}{S}$, $k = 1 \ldots S$. To prove this, consider a Pareto optimal solution of (3.8) $(T_k^*)_{k=1..S}$, $(d_{ks}^*)_{k,s=1..S}$, and $(H_{i0}^*)_{i=1..I}$ with the corresponding first-stage funding ratio distribution F^* with possible outcomes $F_s^* = \sum_{i=1}^{I} H_{i,0}^* R_{i,1,s}/L_{1,s},\ s = 1 \ldots S$. Suppose that there exists $k \in \{1, ..., S\}$ such that $(T_k^* - \frac{1}{k}\sum_{s=1}^{S} d_{ks}^*)$ - that is, the k-th objective function in (3.8)- is not the mean of the worst k outcomes of F^*. We solve the optimisation problem:

$$\textbf{Max} \quad [T_k - \frac{1}{k}\sum_{s=1}^{S} d_{k,s}]$$

Subject to:

$$T_k - F_s^* \leq d_{k,s}\ , \qquad s = 1 \ldots S$$

$$d_{k,s} \geq 0\ , \qquad s = 1 \ldots S$$

and denote by $(T_k')_{k=1..S}$ and $(d_{ks}')_{k,s=1..S}$ the optimal solution. As per Proposition 3.4, T_k' represents the k-th worst outcome of F^*, while the optimal value of the objective function $T_k' - \frac{1}{k}\sum_{s=1}^{S} d_{ks}'$ is the mean of the worst k outcomes (the k-th scaled tail of F^*).

$$T_k' - \frac{1}{k}\sum_{s=1}^{S} d_{ks}' \geq T_k^* - \frac{1}{k}\sum_{s=1}^{S} d_{ks}^*$$

Thus, we have obtained another feasible solution of (3.8) $(T_1^*, \dots T_{k-1}^*, T_k', T_{k+1}^* \dots T_S^*,\ d_{11}^*, \dots d_{1S}^*, \dots, d_{k1}' \dots, d_{kS}', \dots d_{S1}^*, \dots d_{SS}^*, H_{i0}^*)$ that has the same objective function values, apart from the k-th one, where it strictly improves. This is a contradiction with $(T_k^*)_{k=1..S}$, $(d_{ks}^*)_{k,s=1..S}$, and $(H_{i0}^*)_{i=1..I}$ being Pareto optimal. Hence, for a Pareto efficient solution of (3.8), the objective function values represent scaled tails. $\square$

In this section, we formulate two SSD models that result in an SSD efficient solution, a scaled model and an unscaled model. In addition to the decision variables for the ALM model presented in Section 2.1, both problem (3.8) and (3.9) have S free variables $T_1, ..., T_S$ with T_k being the k-th worst outcome of the funding ratio distribution $(F_1, ..., F_S)$. Besides these variables, the problem has also S^2 non-negative variables $(d_{k,s})_{k,\ s=1,...,S}$ with the interpretations given in Proposition 3.4; $d_{k,s} = [T_k - F_s]^+$, $s = 1 \dots S$. In the next sections, we reformulate these MOO as single objective problems using the Reference Point Method.

3.4.1 The SSD Scaled Model Formulation

By considering scaled tails in the multi-objective optimisation problem as per (3.8) we obtain the SSD scaled model. The MOO problem (3.8) is transformed into a single objective optimisation problem by maximising the objective function (3.5), representing the worst difference between a scaled tail and its reference, with the addition of a regularisation term. We refer to this model as (**SSD-Scaled**) optimisation model:

$$\textbf{Max} \quad \delta + \epsilon\left(\sum_{k=1}^{S} Z_k - \sum_{k=1}^{S} asp_k\right)$$

Subject to:

$$Z_k = T_k - \frac{1}{k}\sum_{s=1}^{S} d_{k,s}\ , \qquad k = 1 \dots S \tag{3.10}$$

$$Z_k - asp_k \geq \delta \ , \qquad k = 1 \dots S \tag{3.11}$$

$$T_k - F_s \leq d_{k,s} \ , \qquad k, s = 1 \dots S \tag{3.12}$$

$$F_s = \sum_{i=1}^{I} H_{i,0} R_{i,1,s} / L_{1,s} \ , \qquad s = 1 \dots S \tag{3.13}$$

$$d_{k,s} \geq 0 \ , \qquad k, s = 1 \dots S \tag{3.14}$$

$$\frac{1}{S} \sum_{s=1}^{S} A_{T,s} \geq \sum_{i=1}^{I} OP_i (1 + d) \tag{3.15}$$

and also subject to equations (2.2) to (2.12), representing asset holding constraints, fund balance constraints, bound constraints, and short sales constraint. OP_i is the monetary value of asset i ($i = 1 \dots I$) in the original portfolio and d is a minimum accepted rate of return over T years, specified by the decision maker.

In addition to the decision variables $H_{i,0}, B_{i,0}, S_{i,0}, H_{i,t,s}, B_{i,t,s}, S_{i,t,s}$ representing investment decisions, we have additional decision variables whose nature is discussed below:

F_s = The funding ratio under scenario s at time t=1; (F_s=$A_{1,s}/L_{1,s}$), $s = 1 \dots S$;

T_k= The k-th worst outcome of the funding ratio at time 1, $k = 1 \dots S$ (free variable); thus, $T_1, \dots, T_S$ are the outcomes of a random variable equal in distribution to the funding ratio;

Z_k= The mean of the worst k outcomes of the funding ratio, or other said, $\text{ScaledTail}_{k/S}(F)$; $Z_k = (T_1 + \dots + T_k)/k$, $k = 1 \dots S$ (free variable);

$\delta = \min_{k=1 \dots S}(Z_k - asp_k)$= the worst partial achievement (free variable);

$d_{k,s}$= Non-negative variables, $d_{k,s} = [T_k - F_s]^+$ that is

$$d_{k,s} = \begin{cases} 0, & \text{if } \mathrm{F}_s \geq T_k \\ T_k - F_s, & \text{otherwise} \end{cases} \tag{3.16}$$

In addition to the parameters discussed in Section 2.1 representing returns of the assets, liabilities, contributions and initial portfolio, there are parameters which are chosen by the decision maker:

asp_k = The target or aspiration level for $\text{ScaledTail}_{k/S}(F) = Z_k$, $k = 1 \dots S$;

$d > 0$ = Desired rate of return over the investment horizon;

$\epsilon > 0$ = The weighting coefficient of the regularisation term in the objective function.

Proposition 3.5. For any choice of aspiration levels and of $\epsilon > 0$, the optimal solution of the above model represents a first stage decision allocation $H_{i,0}$, $i = 1 \dots I$ such that the corresponding funding ratio F, represented by equally likely outcomes F_s, $s = 1 \dots S$, is non-dominated with respect to SSD.

Proof. Consider an optimal solution of (SSD-Scaled) with first stage decision $H^*_{i,0}$, $i = 1 \dots I$ and the corresponding funding ratio distribution F^*, with outcomes $F^*_s = \sum_{i=1}^{I} H^*_{i,0} R_{i,1,s}/L_{1,s}$, $s = 1 \dots S$. It is clear that the corresponding $(T^*_k)_{k=1..S}$ and $(Z^*_k)_{k=1..S}$ represent the k-th worst outcomes and the mean of the k worst outcomes of F^*, respectively. Otherwise by solving

$$\textbf{Max } [Z_k = T_k - \frac{1}{k}\sum_{i=1}^{S} d_{k,i}]$$

Subject to:

$$T_k - F^*_s \leq d_{k,s} \ , \qquad s = 1 \dots S$$

$$d_{k,s} \geq 0 \ , \qquad k, s = 1 \dots S$$

and using the optimums, we obtain a solution that strictly improves on the objective function in (SSD-Scaled), which is a contradiction.

Now, suppose that this optimal solution of (SSD-Scaled) is not SSD efficient; hence, there exists another feasible first stage decisions $H'_{i,0}$, $i = 1 \dots I$ with funding ratio F' having outcomes $F'_s = \sum_{i=1}^{I} H'_{i,0} R_{i,1,s}/L_{1,s}$, $s = 1 \dots S$, such that $F' \succ_2 F^*$.
For each $k \in \{1, .., S\}$ we solve:

$$\textbf{Max} \quad [T_k - \frac{1}{k}\sum_{s=1}^{S} d_{k,s}]$$

Subject to:

$$T_k - F'_s \leq d_{k,s} \ , \qquad s = 1 \dots S$$

$$d_{k,s} \geq 0 \ , \qquad s = 1 \dots S$$

and denote by T'_k and $d'_{k,s}$, $s = 1 \dots S$ the optimal solution and by $Z'_k = T'_k - \frac{1}{k}\sum_{s=1}^{S} d'_{k,s}$.
$F' \succ_2 F^* \Leftrightarrow Z' \geq Z^*$, $\forall k = \{1..S\}$ with al least one inequality strict.
$\Leftrightarrow Z' - asp_k \geq Z^* - asp_k$, $\forall k = \{1..S\}$ with al least one inequality strict.

$$\Rightarrow \min_{k=1 \dots S} (Z' - asp_k) \geq \min_{k=1 \dots S} (Z^* - asp_k)$$

$$\Rightarrow \min_{k=1 \dots S} (Z' - asp_k) + \sum_{s=1}^{S} (Z' - asp_k) > \min_{k=1 \dots S} (Z^* - asp_k) + \sum_{s=1}^{S} (Z^* - asp_k)$$

Hence, we have obtained a feasible solution of (SSD-Scaled) that results in a strictly better value of the objective function, which is a contradiction. $\square$

Furthermore, if the aspiration levels are not attainable, this model does not result in infeasibility; it produces a solution that comes close to these levels. In other words, it does as well as it can. This is due to the generalised goal programming approach used in this model: the constraint (3.11) does not require that all cumulated outcomes be greater than or equal to their aspiration levels but just expresses the worst partial achievement (i.e. δ can be negative, and it is, if the aspiration level is not attainable). That is, $\text{ScaledTail}_{k/S}(F^*) \geq \text{ScaledTail}_{k/S}(F)$, $\quad k = 1 \dots S$ with at least one inequality strict.

Remark 3.5. The sign of the optimal value of δ or of the optimal value of the objective function in (SSD-Scaled) is an indication of whether the aspiration levels have been achieved. A strictly positive δ indicates that the aspiration levels have been strictly improved upon. If the aspiration distribution is exactly matched, that is, if the optimal solution results in a funding ratio distribution whose scaled tails are exactly the reference points, the optimum value of the objective function is 0. Finally, a strictly negative optimal value of δ indicates that there is at least one scaled tail that did not achieve its target.

A regularisation term is added to the objective to deal with the case that optimisation of the worst partial achievement has multiple optimal solutions; in this case, the overall better solution should be chosen, that is, the solution for which the sum of the objective function values is higher. The complete algebraic formulation of the (SSD-Scaled) model could be found in Appendix **A** and the AMPL code used to solve it is in Appendix **B**.

3.4.2 The SSD Unscaled Model Formulation

By considering unscaled tails in the multi-objective optimisation problem (3.9), with a similar treatment as for the SSD scaled model, we obtain the (**SSD-Unscaled**) model:

$$\textbf{Max} \quad \delta' + \epsilon(\sum_{k=1}^{S} Z'_k - \sum_{k=1}^{S} asp'_k)$$

Subject to:

$$Z'_k = kT_k - \sum_{s=1}^{S} d_{k,s} \ , \qquad k = 1 \dots S$$

$$Z'_k - asp'_k \geq \delta' \ , \qquad k = 1 \dots S$$

also subject to (2.2) to (2.12) and (3.12) to (3.15).

Here, Z'_k is the sum of the worst k outcomes of the funding ratio, or in other words, $S \times \text{Tail}_{k/S}(F)$; $Z'_k = T_1 + \dots + T_k$, $k = 1 \dots S$.

Remark 3.6. Both (SSD-Scaled) and (SSD-Unscaled) models have (possibly different) optimal solutions such that their distribution of funding ratio is SSD efficient. The modelling difference resides in how the "closeness" to the target distribution is measured, more precisely, how the shortfalls below target points are penalised. With the unscaled model, the accumulation of outcomes below their targets is penalised, rather than the magnitude of the shortfalls, which is more severely penalised in the scaled model. This becomes more obvious when the target distribution is deterministic, having one single possible outcome. It is shown in the next section that, in this case, the scaled model maximises the worst outcome of the funding ratio, i.e the largest deviation from the (single) target point, while the unscaled model minimises the expected shortfall below the target, taking thus into account all situations when the target is not achieved.

Remark 3.7. Both (SSD-Scaled) and (SSD-Unscaled) models provide an SSD efficient solution, irrespective of the aspiration levels chosen by the decision maker; this choice cannot lead to infeasibility either. This follows from the "better than target" property of the Reference Point Method in multi-objective optimisation. It was shown in [69] that the maximisation of the achievement function results in a Pareto optimal solution of the multi-objective model irrespective of the reference points chosen by the decision maker. If the reference points do not form a Pareto optimal vector for the multi-objective model, the maximisation of the achievement function improves on the reference points until Pareto optimality is attained. If the reference points form a Pareto optimal vector, the optimal solution in the maximisation of the achievement function results in objective function values equal to the reference points. Finally, if at least one of the reference points is unattainable / too high, we obtain a Pareto optimal solution in which the worst difference between objective function values and reference points is optimised.
In the current setting, the multiple objective functions represent tails (or scaled tails) of the funding ratio distribution and Pareto optimal solutions represent SSD efficient distributions. The three cases above relate to whether the target distribution of funding ratio is (1) SSD dominated; (2) SSD efficient or

(3) unattainable, in the sense that there is no feasible solution that could attain or improve on all of its tails. In all three cases, the optimal solutions of both scaled and unscaled models are SSD efficient, corresponding to cases (1) better than target; (2) matching the target; (3) coming close to the target distribution.

3.5 Connection with Risk Minimisation

Consider the particular case in which the target level for every outcome of the funding ratio distribution is equal to a target funding ratio λ; $\lambda_1 = \lambda_2 = \ldots = \lambda_S = \lambda$. This makes the target distribution a deterministic distribution.

3.5.1 The SSD Scaled Model with Deterministic Target

In this case, the aspiration levels for the scaled tails of the funding ratio are also all equal to λ: $asp_k = \lambda$, $\forall k \in \{1, \ldots, S\}$.
It is clear that the worst partial achievement $\delta = \min_{k=1\ldots S}(Z_k - asp_k)$ corresponds to the worst outcome of the funding ratio distribution, irrespective to the value of λ. Thus, maximising the worst partial achievement is equivalent to maximising the worst outcome. A minimax mean-risk model, in which risk is defined as the maximum possible loss, was proposed by Young [73], who also showed that such a model can be formulated as an LP. The minimax model maximises the minimum return, subject to the restriction that the average return of the portfolio exceeds some pre-specified minimum level. That is, the minimax portfolio minimises the maximum loss, where loss is defined as the negative of return.

In our case, if we exclude the regularisation term, maximising the worst partial achievement can be formulated as a (**Maximin**) model, which optimises the worst outcome of funding ratio, subject to a constraint on the expected terminal wealth.

$$\textbf{Max} \quad \delta$$

Subject to:

$$F_s \geq \delta \ , \qquad s = 1 \ldots S$$

subject to (2.2) to (2.12), (3.13) and (3.15).

We have developed in Section 3.3, that maximisation of worst partial achievement is not guaranteed to be Pareto optimal; the addition of the regularisation term ensures Pareto optimality. Hence, in case the model above has a unique optimal solution, SSD efficiency is guaranteed. However, just as with the general SSD model, in case of non-unique optimal solutions, the SSD efficiency is not guaranteed; a regularisation term should be added in the objective function. The regularisation term in the SSD scaled model is the sum of tails/ cumulated outcomes; in order to formulate it, we need additional S^2 variables d_{ki} which adds substantially to the computational complexity. In order to avoid this, we can add in the objective function above a term such as $\epsilon \sum_{s=1}^{S} F_s$ which brings no extra computational complexity; we obtain the model (**Maximin 2**):

$$\textbf{Max} \quad (\delta + \epsilon \sum_{s=1}^{S} F_s)$$

Subject to:

$$F_s \geq \delta \ , \qquad s = 1 \ldots S$$

also subject to (2.2) to (2.12), (3.13) and (3.15).

Just as before, ϵ has to be chosen as a small enough number such that the optimisation is basically that of the worst outcome. The optimal solution of this model will result in a funding ratio that has the highest "worst" outcome and also the highest expected value amongst all optimal solutions of (Maximin). Notice that, although the chance of getting an SSD inefficient solution is substantially decreased at no extra computational complexity, SSD efficiency is still not guaranteed as there is theoretically the possibility that (Maximin 2) has multiple optimal solutions.

Thus, an SSD scaled model in which the reference distribution is deterministic could be in most cases written as a maximin model of maximising the worst case value of the funding ratio. The single outcome of the reference distribution is irrelevant.

3.5.2 The SSD Unscaled Model with Deterministic Target

The aspiration levels for the cumulated outcomes of the funding ratio are: $asp'_k = \frac{1}{S}k\lambda$, $k = 1 \dots S$. As in Section 4.1, denote by $T_1 \leq \dots \leq T_S$ the ordered outcomes of the funding ratio. The worst partial achievement is:

$$\frac{1}{S}\min_{k=1\dots S}(T_1 + \dots + T_k - k\lambda)$$

As each outcome below λ is penalised, the minimum is achieved for an index j in $\{1, \dots, S\}$ such that $T_j \leq \lambda \leq T_{j+1}$.
The worst partial achievement is thus

$$\frac{1}{S}[(T_1 - \lambda) + \dots + (T_j - \lambda)] = \frac{1}{S}\sum_{T_k<\lambda}(T_k - \lambda)$$

Thus, maximising the worst partial achievement is equivalent to minimising

$$\frac{1}{S}\sum_{T_k<\lambda}(\lambda - T_k)$$

which is the Lower Partial Moment of order 1 and target λ of the funding ratio, also called the expected shortfall below target λ.

The model that minimises the expected shortfall below target λ can be formulated as an LP by introducing S additional variables representing the shortage of the funding ratio with respect to target λ under each scenario:

$$\textbf{Min} \quad \frac{1}{S}\sum_{s=1}^{S} Sh_s$$

Subject to:

$$F_s - \lambda + Sh_s \geq 0 \ , \qquad s = 1 \dots S$$

$$Sh_{t,s} \geq 0 \ , \qquad s = 1 \dots S$$

subject to (2.2) to (2.12), (3.13) and (3.15).

We notice several things here.

First, without the addition of a regularisation term, minimisation of the expected shortfall is not guaranteed to result in an SSD efficient solution. One case in which SSD efficiency might not occur is the situation when there are multiple optimal solutions; at least one of them is SSD efficient but not necessarily all of them. The other case in which SSD efficiency might not occur is when the optimum in the minimisation of expected shortfall is zero, that is, $T_1 \geq \lambda$. In this case the optimal solution may be ANY solution such that the corresponding funding ratio has all outcomes above the target λ. Adding a regularisation term ensures that the optimal solution is improved until SSD efficiency is achieved - an example of "better than target" situation. However, a regularisation term as in the (SSD-Unscaled) model involves the introduction of additional S^2 variables. Similarly to the previous subsection, we may add a term in the objective function such that, out of all solutions that minimise the expected shortfall below λ, the one with the highest expected value is chosen:

$$\textbf{Min} \quad \frac{1}{S}\sum_{s=1}^{S} Sh_s - \epsilon \sum_{s=1}^{S} F_s$$

with ϵ a small enough number.

Secondly, the model that minimises expected shortfall below λ is closely connected to an ICCP model [34], in which the integrated chance constraint penalises shortfalls of the funding ratio distribution with respect to target λ. The connection is in the following sense. With the former, the expected shortfall is in the objective and a constraint on the terminal expected asset value is imposed. With the latter, the expected shortfall is the left hand side of a constraint, while maximising terminal expected asset value may be part of the objective. With appropriate choices of the right hand sides involved, the two models have the same optimal solution. We give an example of such a situation in the numerical experiment conducted in Chapter 5.

Remark 3.8. In the general case of the SSD models presented in this chapter, the reference distribution could be constructed in many different ways. A possible strategy is to start by implementing either an ICCP or Maximin model and analyse the resulting distribution of funding ratio. Should this be not acceptable, one can implement a generic SSD model, by setting a (non-deterministic) target distribution based on the outcomes of the funding ratio already obtained. For example, the targets for the worst case scenario and the left tails can be increased, should these values be too low in the ICCP solutions. Similarly, the targets for the tails at higher confidence levels (e.g. the expected value) may be increased, should the Maximin model provide a solution with poor performance apart from worst case scenarios. The fact that there is not one single way to choose the aspiration levels should be regarded as an advantage of this method. Different target distributions lead to different SSD efficient solutions. We can analyse the distribution of the portfolio that we obtain, and, if not satisfactory, we can modify the aspiration levels and obtain a further candidate solution portfolio. Improved distributions of funding ratios may be thus achieved, compared to the existing risk models for ALM. Numerical experiments in Chapter 5 support this claim.

3.6 Introducing Reservation Levels

The SSD models presented in Section 3.4 can be extended to include *reservation levels* in addition to aspiration levels. As the name implies, the difference between aspiration and reservation levels is that reservation levels are "minimum requirements" and should be achieved if at all possible, whilst it is desirable to achieve aspiration levels.

In this section, we extend the (SSD-Scaled) model by including both aspiration asp_k and reservation res_k levels for each objective function Z_k, $k = 1, ..., S$ representing the cumulative outcomes of the funding ratio distribution. Following [55], we denote the partial achievement functions by $\vartheta_{asp_k,res_k}(Z_k)$; Z_k

depends on the first stage decision variables $\{H_{i,0},\ B_{i,0},\ S_{i,0},\ i = 1 \ldots I\}$, while asp_k and res_k are chosen by the decision maker. Such partial achievement functions must satisfy certain requirements: being increasing functions with respect to Z_k, taking value 0 if $Z_k = res_k$ and value 1 if $Z_k = asp_k$ [71]. In this model, we use partial achievement functions defined as piecewise linear functions as follows:

$$\vartheta_{asp_k,res_k}(Z_k) = \begin{cases} \frac{\alpha[Z_k - res_k]}{asp_k - res_k} & for \quad Z_k \leq res_k \\ \frac{Z_k - res_k}{asp_k - res_k} & for \quad res_k < Z_k < asp_k \\ \frac{\beta[Z_k - asp_k]}{asp_k - res_k} + 1 & for \quad Z_k(x) \geq asp_k \end{cases} \tag{3.17}$$

where α, β are positive parameters and should be chosen in such a way that partial achievement functions are not only monotone, but also concave. As can be seen in Figure 3.1, under the condition $0 < \beta < 1 < \alpha$, the partial achievement function (3.17) is strictly increasing and concave. Thus, (3.17) is a typical utility function for a rational and risk-averse DM.

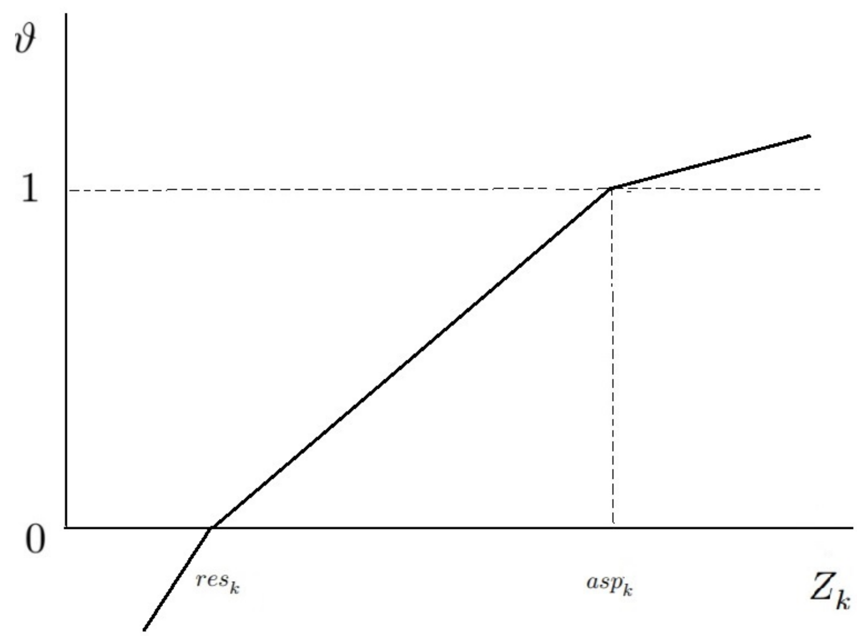

Figure 3.1: Example of a partial achievement function as in (3.17)

The parameter $\alpha > 1$ represents the decision maker's dissatisfaction for outcomes worse than their corresponding reservation level, while the parameter $\beta < 1$ represents satisfaction for outcomes better than their aspiration level. Thus, they map outcome values onto a normalized scale of the decision maker's satisfaction:

- If the outcome is below its reservation level ($Z_k \leq res_k$), the partial achievement function takes negative values;
- If ($Z_k = res_k$), the partial achievement function takes value 0;
- For outcomes between the reservation and the aspiration level $res_k \leq Z_k \leq asp_k$, it takes a value between 0 and 1;
- For ($Z_k = asp_k$), it takes value 1;
- For outcomes better than the aspiration level ($Z_k \geq asp_k$), the partial achievement function takes a value greater than 1.

The largest possible values of $\vartheta_{asp_k,res_k}(Z_k)$ are required for every Z_k, $k = 1 \ldots S$. Thus, we can express the partial achievement functions using a set of linear constraints and formulate the following (**SSD-res**) model as follows:

$$\textbf{Max} \quad \delta + \epsilon \sum_{k=1}^{S} \delta_k$$

Subject to:

$$\delta_k \geq \delta \,, \qquad k = 1 \ldots S \tag{3.18}$$

$$\delta_k \leq \frac{\alpha[Z_k - res_k]}{asp_k - res_k} \,, \qquad k = 1 \ldots S \tag{3.19}$$

$$\delta_k \leq \frac{Z_k - res_k}{asp_k - res_k} \,, \qquad k = 1 \ldots S \tag{3.20}$$

$$\delta_k \leq \frac{\beta[Z_k - asp_k]}{asp_k - res_k} + 1 \,, \qquad k = 1 \ldots S \tag{3.21}$$

$$Z_k = T_k - \frac{1}{k}\sum_{s=1}^{S} d_{k,s} \ , \qquad k = 1 \dots S \tag{3.22}$$

$$T_k - F_s \leq d_{k,s} \ , \qquad k, s = 1 \dots S \tag{3.23}$$

$$F_s = \sum_{i=1}^{I} H_{i,0} R_{i,1,s} / L_{1,s} \ , \qquad s = 1 \dots S \tag{3.24}$$

$$d_{k,s} \geq 0 \ , \qquad k, s = 1 \dots S \tag{3.25}$$

$$\frac{1}{S}\sum_{s=1}^{S} A_{T,s} \geq \sum_{i=1}^{I} OP_i (1 + d) \tag{3.26}$$

and also subject to equations (2.2) to (2.12).

In addition to the variables used in (SSD-Scaled) model, there are another S free variables: δ_k, for $k = 1...S$, representing the values of the partial achievement functions:

$$\delta_k = \min\{\frac{\alpha[Z_k - res_k]}{asp_k - res_k}, \ \frac{Z_k - res_k}{asp_k - res_k}, \ \frac{\beta[Z_k - asp_k]}{asp_k - res_k} + 1\}, \quad k = 1...S$$

The free variable δ represent $\min_{k=1...S} \delta_k$, the worst partial achievement functions.

With the same treatment,the SSD unscaled model can be extended to include reservation levels in addition to aspiration levels.

Remark 3.9. The advantage of adding reservation levels is explored through numerical experiments presented in Chapter 5. In the cases where the distribution obtained by (SSD-Scaled) or (SSD-Unscaled) models are considered unsatisfactory, we show that we can improve on the resulting distribution by appropriately selecting reservation levels.
Hence, the models could be used interactively by specifying, whenever needed,

new aspiration and/or reservation levels in order to obtain portfolios with progressively better funding ratio distributions. These reservation levels should be achieved - if possible - first, before attempting to come close to the aspiration levels.

3.7 Concluding Remarks

In this chapter, we have formulated ALM models in which the risk of underfunding is controlled using Second Order Stochastic Dominance (SSD). We obtain short-term funding ratio distributions that are SSD efficient, while a constraint is imposed on the expected terminal wealth. In addition to being SSD efficient, the funding ratio distribution comes close, in a well defined sense, to a benchmark distribution of funding ratio, whose outcomes are specified by the decision maker. The formulated models provide a meaningful solution, corresponding to the risk-averse attitude observed in investment.

The closeness is measured as follows, progressively larger left tails of the funding ratio distribution are considered, either scaled (equivalent to averages of a progressively higher number of worst case values), or unscaled (equivalent to sums of a progressively higher number of worst case values); these are compared with the tails of the benchmark distribution and the worst difference is optimised. Thus, the funding ratio distribution could be shaped and "crafted" to a desirable form, to the extent that is achievable. A regularisation term is added to ensure SSD efficiency in case of multiple optimal solutions.

Both the scaled and unscaled models result in (possibly different) SSD efficient distributions of funding ratio. The SSD scaled model gives however a greater importance to the magnitude of a shortfall below the target. A good way to grasp the difference between the models is by considering a particular case with interesting connections to risk minimisation. Special cases are obtained when the target distribution is deterministic, specified by a single outcome such as a required target funding ratio λ.

In most situations, the SSD scaled model is equivalent to a risk minimisation model, where risk is measured by the maximum loss. More precisely, the SSD scaled model can be reformulated as a (computationally much simpler) Maximin model which maximises the worst case value of the funding ratio.
The SSD unscaled model is equivalent in most cases to a risk minimisation model, where risk is measured by the lower partial moment of order 1 of the funding ratio around target λ, also called the expected shortfall below target λ. The well established ICCP model has the expected shortfall below target as a constraint, not in the objective function. By setting appropriate right hand side values , the SSD unscaled formulation and the ICCP formulation lead to the same optimal solutions.

There are situations in which the SSD models and risk minimisation models above may not be equivalent, most notably, when risk minimisation has multiple optimal solutions. A regularisation term should be added in this case to the objective function in the risk minimisation models in order to guarantee SSD efficiency. However, this increases the computational complexity to the level of the SSD formulations.

Thus, two established and computationally less expensive models, namely Maximin and ICCP, are under mild conditions, particular cases of the SSD models developed in this thesis. A natural question that arises is: can we obtain improved distributions of funding ratio by considering non-deterministic target distributions? The computational study in Chapter 5 offers insight into this problem.

We extend the SSD models by adding reservation levels, in addition to aspiration/target levels. This offers greater modelling power, as not achieving a reservation level is penalised more than non-achievement of an aspiration level. Reservation distributions could be set as minimal requirements for outcomes of the funding ratio distribution.

Chapter 4

Scenario Generation Models for Cash Inflows and Outflows

4.1 Introduction

An important issue in ALM problems is the representation of the underlying key stochastic parameters. In an ideal situation, we can represent the whole universe of possible outcomes in a set of scenarios with associated probabilities of occurrence. However, in most cases the underlying random variables (representing, for example, the asset returns at a specific time point) are continuous, or discrete with many possible realisations. In order to use an SP approach, one has to represent the distributions of interest (in a single-period case) via a limited number of possible outcomes "scenarios", or to represent the stochastic processes (in a multi-period case) via a scenario tree; a process known as *scenario generation (SG)*. A formal study of ALM problems usually involves modelling plausible scenarios for liabilities and returns of instruments in the portfolio to be used as an input to the optimisation model.

The scenarios needed for a two-stage stochastic program can be represented by a scenario tree in the form of a fan, as in Figure 4.1. The root node in this tree represents the information known today or the deterministic data of the present states of the world and the other nodes represent the uncertainties at later stages. Each path through the tree is a scenario which represents one possible sequence of outcomes of the stochastic elements throughout the time horizon under consideration.

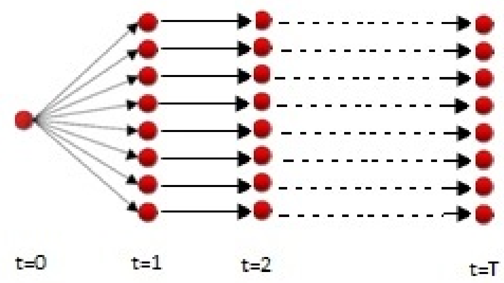

Figure 4.1: Two-stage scenario tree

The scenario tree captures the dynamics of the decision making process, since decisions are adjusted to the realisations of the uncertainties [25]. In this context, the root node in the scenario tree can be looked at as the decisions that need to be determined today (first-stage decision) and the other nodes represent recourse decisions at later stages. In two-stage SP, the stages are fixed to two; however, the number of time periods for recourse actions depends on the planning horizon to be considered. It is important to realise that the stages do not necessarily refer to time periods; they correspond to steps in the decision process.

Scenario generation for stochastic programming has been the subject of extensive research. There is a vast literature on different methods of scenario generation techniques, each with its strengths and weaknesses. For an overview of scenario tree generation methods applied in finance and economic decision making, see [66].

Different scenario generation techniques could be used for different purposes. A scenario tree could describe certain time-varying processes in nature or economics. For instance, it could be used to represent the evolution of a financial time series or a population dynamics. In an ALM research we need to generate scenarios for the returns of the instruments in the portfolio, the contributions as well as future obligatory payments.

Our derivation of the plan's future cash flow profile requires the specification of both demographic and economic variables as a first step. We model the uncertainty in both the pension fund's cash inflows and outflows. The fund's cash inflow is from the investment revenues and the funding, which in turns consists of contributions and money received from another pension funds when new participants join the GOSI's scheme from previous jobs. On the other hand, the cash outflow (liabilities) is modelled considering four different types of obligations. We consider: pension payments, lump sum payments, money transferred out of the fund due to the transferring of some participants who will join another pension fund's scheme and death grants to next of kin upon death.
For investment returns, we used bootstrapping to generate in-sample scenarios to be used as input to the SP models presented in Chapters 2 and 3. Using a copula method, we generate sets of larger number of scenarios by considering a larger number of historical observations; we use these sets for out of sample testing.
The scenarios for the funding and liabilities have the same underlying source of uncertainty; they are generated based on population models and a salary model, assuming that a fixed percentages of salaries are to be paid in, as contributions, or out, as liabilities. The dynamics of the pension fund's population is modeled by a "Birth, Immigration, Death, Emigration" (BIDE) population model (see for example, [45] and [58]).

To summarise, in order to generate scenarios for contributions and liabilities, we:

1. Construct population models to quantify the future population of the GOSI's active and retired participants.

2. Construct a salary model to define the future salary of the active workforce and the wage of the pensioners; this is simply done by considering a steady annual growth.

By combining the salary model of active participants together with the contributors population model, we compute funding. In the same way, by considering pensioners population and wage, we compute the liabilities.

The rest of the chapter is organised as follows. In Section 4.2, we describe the assets returns scenario generation methods. In Section 4.3, the population model is presented. Section 4.4 covers the salaries model. In Section 4.5, we explore how we assemble the funding scenario tree. Section 4.6 treats the liabilities scenarios. Concluding remarks are provided in Section 4.7.

4.2 Assets Returns Model

4.2.1 In-Sample Scenarios Using Bootstrapping

The simplest approach for generating scenarios is to use only the available observed data without any mathematical modelling. It bootstraps, that is, samples with replacement, a set of historical records. It is common method of obtaining parameters necessary as an input in optimisation models

4.2.2 Out-of-Sample Scenarios Using the Empirical Copula

For out-of-sample analysis we generate a larger set of scenarios for the rate of returns of the instruments in the portfolio using *the copula*. The name copula was first used by Sklar [62] to define a tool that describes the multi-variate structure of a distribution (i.e. dependence structure between the variables) irrespective of the marginal distributions [31]. Using copulas allow to separate the multivariate structure (the "shape" of the distribution) from the marginal distributions, thus allowing the marginals to be independently modelled.

Using copulas to control the dependence structure of a multi variate random variables overcome some limitations of other methods, such as using a correlation matrix [31]. For example, Pearson's correlation coefficient describes only the degree of linear dependence between each pair of random variables by one number and does not capture any non-linear dependencies [31].

An d-dimensional copula is the joint cumulative distribution function (cdf) of any d-dimensional random vector with standard uniform marginal distributions, i.e. a function $C : [0, 1]^d \to [0, 1]$ [31]. Sklar's theorem states that "any multivariate distribution can be written in terms of univariate marginal distribution functions and a copula".

Sklar's theorem

Let F be an d-dimensional joint cumulative distribution function of a random vector $(X_1, X_2, \ldots, X_d)$ with margins $F_1, ..., F_d$, $F(x_1, ..., x_d) = P(X_1 \leq x_1, ..., X_d \leq x_d)$. Then, there exist an d-dimensional copula C such that for all x in $\mathbb{R}^d$

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_n(x_d))$$

.

Moreover, if all the marginal cdfs F_i are continuous, then C is unique [46].

An immediate consequence is that, for every $(u_1, ..., u_d) \in [0, 1]^d$,

$$C(u_1, ..., u_d) = F(F_1^{-1}(u_1), ..., F_d^{-1}(u_d)),$$

where F_i^{-1} is the generalised inverse of F_i:

$$F_i^{-1}(u) = inf\{t : F_i(t) \geq u\}, \qquad u \in [0, 1] \quad [10] \tag{4.1}$$

In this research, we use a special kind of copula, the so-called *empirical copula*, see for example [46]. We adapt a similar method to [31]; we create the empirical copula and generate samples for each univariate margin. Using the copula, the univariate samples are combined to form a sample from the multivariate distribution.

The basic motivation of using this method is to generate, starting from a large (historical) sample of multivariate data, other samples of the same size. Generating samples of a specified size from univariate data is a standard problem. One way of doing this is to find the distribution that best fit the univariate data using statistical software packages, then, sample from the fitted distribution.
The challenge is to assemble the samples from univariate distributions and preserve the dependence structure of the multi-variate data. To this end, we use the historical copula, created based on the historical sample.

Suppose we have available N samples (observations) from an d-variate distribution of a random vector $(X_1, X_2, ..., X_d)$; denote these samples by $\mathcal{S} = \{(x_1^i, x_2^i, ..., x_d^i)\}_{i=1}^N$; then, our goal is to generate a matrix X of the size $N \times d$ of other possible outcomes using the empirical copula.
The main idea is to create a matrix $N \times d$ of "ranks"; element c_j^i in this matrix is $\frac{k}{N}$, where k is the "rank" of observation x_j^i among the observed values of variable X_j ; that is element c_j^i corresponds to the k-th worst value out of the values of the observations x_j^i, $i = 1 \dots N$ of the random variable X_j, $j = 1 \dots d$. Thus, we can interpret a row of this matrix as a "scenario" of dependence between the d random variables, for example, one scenario may be "the maximum of margin 1 occurs at the same time with the second worst value of margin 2 together with the minimum of margin 3, etc.".

In this approach, where the true marginal distribution functions F_j, $j = 1...d$ are usually unknown, we use instead the empirical distribution with marginal cdfs given by:

$$F_j^e(x_j^i) = \frac{rank(x_j^i, x_j)}{N}, \quad j = 1..d \tag{4.2}$$

where $rank(x^i, x)$ is the rank (order) of value x^i in a vector x, with values between 1 and N.

Once we have the copula (dependence structure) on one hand and the samples of the margins on the other, the samples from the margins are then combined according to the copula matrix. Denote by u_{ij} ($i = 1 \dots N$ and $j = 1 \dots d$) the elements of this matrix; they are all numbers of the form $\frac{k}{N}$, $k = 1 \dots N$. Denote by r_{ij} ($i = 1 \dots N$ and $j = 1 \dots d$) the samples generated from margin j and suppose we have ordered them in ascending order $r_{1j} < r_{2j} < \cdots < r_{Nj}$.

The element r'_{ij} of the matrix of new samples from the multivariate distribution is obtained according to the generalised inverse of the cumulative distribution function F_j of the distribution with outcomes $\{r_{1j}, \dots, r_{Nj}\}$. More precisely,

$$r'_{ij} = \inf_{k=1 \dots N} \{r_{kj}; F_j^{-1}(u_{ij}) \geq r_{kj})\}$$

In this way, we obtain a new sample from the multi-variate distribution r_{ij} ($i = 1 \dots N$ and $j = 1 \dots d$).

By repeating the process in generating other samples from the univariate distributions and combining them in the same manner, we obtain as many samples from the multi-variate distribution as desired.

4.3 Population Model

The inflow and outflow of the fund members are simulated using the BIDE population model [58]. Population models are used in population ecology to model the dynamics of wildlife or human populations. All populations can be modeled by one simple equation:

$$N_{t+1} = N_t + B_t - D_t + I_t - E_t$$

where:

- N_t represents the population size at time t;
- B_t is the number of births within the population between N_t and N_{t+1};
- D_t is the number of deaths within the population between N_t and N_{t+1};
- I_t is the number of individuals immigrating into the population between N_t and N_{t+1};

- E_t is the number of individuals emigrating from the population between N_t and N_{t+1}.

This equation is called a BIDE model (Birth, Immigration, Death, Emigration model).

In closed population models, we focus on estimating the population size, N , but in open population models we are interested in the dynamics that arise between years or seasons and thus we focus not only on N_t but on the processes that drive population changes (i.e. the parameters governing these processes) [45].
In this research, we consider open population models for both the contributors' and retirees' populations. In the next sections, we explain how we construct the scenario trees for these populations and how to generate scenarios for the funding levels and liability payments accordingly.

4.3.1 The Contributors' Population Scenario Tree

The BIDE equation is adopted as follows:

$$N_{t+1,s} = N_{t,s}+New_{t,s}-R_{t,s}+TI_{t,s}-TO_{t,s}-D_{t,s}, \quad t=0,1\dots T-1,\ s=1\dots S \tag{4.3}$$

Where:

- $N_{t,s}$ the total number of contributors in employment by the end of time period t under scenario s;

- $New_{t,s}$ the numbers of new employees at time t (i.e. the total number of new employees who have been hired between the states $N_{t,s}$ and $N_{t+1,s}$) under scenario s;

- $R_{t,s}$ the total number of contributors who leave the scheme (due to retirement or death) between the states $N_{t,s}$ and $N_{t+1,s}$ under scenario s;

- $TI_{t,s}$ the total number of employees who enter the system from another pension fund (i.e. transferred) between the states $N_{t,s}$ and $N_{t+1,s}$ under scenario s;

- $TO_{t,s}$ the total number of employees who leave the system and transferred to another pension fund between the states $N_{t,s}$ and $N_{t+1,s}$ under scenario s;

- $D_{t,s}$ the total number of cases received lump sum payments and leave the scheme between the states $N_{t,s}$ and $N_{t+1,s}$ under scenario s.

Using the historical data, we computed the ratios of employment, retirement, transfers in, transfers out and lump sum payment cases in the last ten years. Let γ_t denote the employment ratio at time t, defined as:

$$\gamma_t = New_t/N_t$$

Similarly, μ_t is the ratio of the retirement (including deaths) at time t, defined as:

$$\mu_t = R_t/N_t$$

Similarly, let us denote by η_t the ratio of the transfers out of the fund at time t , we have:

$$\eta_t = TI_t/N_t$$

ϕ_t is the ratio of the transfers out of the fund at time t:

$$\phi_t = TO_t/N_t$$

Finally, Δ_t represent the ratio of leaving the scheme at time t, defined as:

$$\Delta_t = D_t/N_t$$

Each of these ratios is a random variable that affects the total number of contributors in each year. By sampling from these observed (historical) ratios we obtain a vector (γ, μ, η, ϕ, Δ) that represents a possible scenario for these ratios for the next year (time period). By using this sample of ratios and the number of contributors in the last year, we simulate the total number of new employees, retires, the number of contributors who will transfer to the fund, the number of contributors who will transfer out, and the number of lump sum payment payable cases for the next year as follows:

$$\begin{aligned} New_t &= \gamma_t \; N_t \quad t = 0, ..., T-1 \\ R_t &= \mu_t \; N_t \quad t = 0, ..., T-1 \\ TI_t &= \eta_t \; N_t \quad t = 0, ..., T-1 \\ TO_t &= \phi_t \; N_t \quad t = 0, ..., T-1 \\ D_t &= \Delta_t \; N_t \quad t = 0, ..., T-1 \end{aligned} \tag{4.4}$$

In order to simulate a sample path (i.e. a scenario or one sequence of possible values) for the number of contributors at times $t = 1 \ldots T$, we use the following procedure.
By considering N_0, the number of contributes at $t = 0$ (observed), in equations (4.4), then substituting the set of resulting values together with N_0 in equation (4.3), we compute the total number of contributors at $t = 1$. By re-sampling for (γ, μ, η, ϕ, Δ) and using the simulated value for the number of contributors at time $t = 1$, we can obtain a simulation for the number of contributors at $t = 2$ under the same scenario. Repeating the process, we obtain one scenario for the number of contributors at times $t = 1 \ldots T$.
By generating S scenarios, we construct a scenario tree for the contributors population in the form of a fan.

4.3.2 The Retirees' Population Scenario Tree

We adopt the BIDE model to represent the dynamics of the retirees population:

$$NR_{t+1,s} = NR_{t,s} + R_{t,s} - G_{t,s} \qquad t = 0, 1, ..T-1, \; s = 1 \ldots S \tag{4.5}$$

Where :

- $NR_{t,s}$ the total number of retirees who receive pension at time t under scenario s (i.e. retires' population size at time t under scenario s);

- $G_{t,s}$ the number of cases that stop receiving pension (leaving the retirees population) between the states $NR_{t,s}$ and $NR_{t+1,s}$ under scenario s;

- $R_{t,s}$ the total number of contributors who leave the scheme = enter the pensioner's population (due to retirement or death) between the states $NR_{t,s}$ and $NR_{t+1,s}$ under scenario s;

A similar approach, based on observed "rates", that is, percentages of cases leaving/entering the population out of the initial population, is employed in order to construct sample paths for the retirees population numbers. .

4.4 Salary Model

Salary is an important factor for determining future contributions and pension payments. Based on the historical data, we compute the average annual salary of a plan member at t=0 and estimate an annual growth of 1.008% each year (1.008% is the average of the growth in the Saudi workers salaries in the last ten years in Saudi Arabia). Hence, the annual salary of a plan member at time t, denoted by $S(t)$, can be simply derived as:

$$S_t = S_{t-1} * 1.008 \quad , \quad t = 1 \dots T \tag{4.6}$$

4.5 Funding Scenario Tree

The funding scenario tree is constructed by combing two benefit types. Firstly, the future realisations of the contributions, which is computed by combining the salary model of active participants together with the contributors' population model and the contribution rate using the formula:

$$CN_{t,s} = 0.18 * S_t * N_{t,s}, \qquad t = 1 \dots T,\ s = 1 \dots S \tag{4.7}$$

Where:

- $CN_{t,s}$ is the total amount of contributions (in Saudi Riyals) at time t under scenario s;

- 18% is the fixed contribution percentage;

- S_t the average annual salary for each member in the plan at time t (from (4.6));

- $N_{t,s}$ the total number of contributors at time t under scenario s (from the population model, equation (4.3)).

Secondly, due to the transferring of some members from one pension fund to the pension fund that we consider, some money will be transferred into the fund's account too. To model the amount of money transferred into the fund from another pension fund at time t; we multiply the simulated number of contributors who transferred to the fund during year t (from the population model) by the expected amount of money to be transferred to the fund "per each member joining the GOSI's from previous job". We calculate the average amount of money to be transfered using historical data and assume a growth on this average through the planning horizon. These calculations are illustrated by the following formula:

$$I_{t,s} = TI_{t,s} * MI_t \quad , \qquad t = 1 \dots T,\ s = 1 \dots S \tag{4.8}$$

Where:

- $I_{t,s}$ is the total amount of money transferred to the fund's account from another pension fund at time t under scenario s;

- $TI_{t,s}$ is the total number of individuals that transfer to the fund from another pension fund at time t under scenario s (simulated as in the population model (4.3));

- MI_t the average amount of money transferred to the fund per member at time t (computed from historical data).

Finally, we calculate the funding at each time point t and under each scenario s by considering the values of equations (4.7) and (4.8) for every t and s. Denote the funding at time t and scenario s by $C_{t,s}$ we have:

$$C_{t,s} = CN_{t,s} + I_{t,s} \quad , \qquad t = 1 \dots T,\ s = 1 \dots S \tag{4.9}$$

4.6 Liabilities Scenario Tree

The scenario generation model for the future liabilities takes into account that there are four types of payments to be made: (a) pension payments, (b) money to be transferred out of the fund's account due to participants transferring, (c) lump sum payments for active participants who leave the scheme before being eligible to receive pension, (d) death grants to next of kin upon death. This follows the structure of payments in the GOSI, described in Section 1.4 and used in our numerical experiments.

We model pension payments by combining the pensioners' population model with a salary model, with which we deduce an average annual wage per each member for each year t in the planning horizon. According to the GOSI's regulations, the retirement pension is obtained by multiplying one-fortieth of the average monthly wage for the last two years by the number of contribution years and months. According to the GOSI's system, the average monthly wage is defined as "an average of the total contributory wages received throughout the last two years of contribution period" [1].
Using the salary model, we compute the average annual salary of each two successive years, denoted here by Av_t, as follows:

$$Av_t = \frac{S_{t-1} + S_{t-2}}{2}$$

We compute the average number of years of contribution (historically); for the GOSI's data, this is 35.2 years. Thus, we work out the average annual pension using the formula:

$$PN_t = Av_t \frac{35.2}{40} \quad , \quad t = 1 \dots T$$

where:

- PN_t the annual pension average during year t;
- Av_t the average annual salary during year t.

The total annual pension payments for all pensioners for each realisation of the retirement population will then equal:

$$P_{t,s} = PN_t * NR_{t,s} \quad , \quad t = 1 \dots T, \quad s = 1 \dots S \tag{4.10}$$

here $P_{t,s}$ denote the total annual pension payments at time t under scenario s and $NR_{t,s}$ is the total number of retirees at time t under scenario s simulated as in (4.5).

The second type of payments occurs in the case when an active member transfers to another job which is covered by another insurance scheme. In this case, the fund has to transfer the accumulated contributions of that member to the new pension fund.
We employ the population model described in Section 4.3.1 in order to generate scenarios for the number of individuals who move to another fund. We also compute, based on historical data, the average amount of money transfered to another fund per individual and assume a growth by a fixed inflation rate:

$$O_{t,s} = TO_{t,s} * MO_t \quad , \quad t = 1 \dots T,\ s = 1 \dots S \tag{4.11}$$

Where:

- $O_{t,s}$ is the total amount of money transferred out of the fund to another pension fund at time t under scenario s;
- $TO_{t,s}$ is the total number of individuals who transfer out of the fund to another pension fund at time t under scenario s;
- MO_t the average amount of money transferred out of the fund per member at time t.

In a similar fashion, i.e. (a) using population models and generating scenarios for the number of individuals and (b) using observed average lump sum payments per individual and assuming a growth rate imposed by inflation, we generate scenarios for the third and fourth categories: lump sum payments ($LS_{t,s}$) and death grants ($DG_{t,s}$) respectively for every t and s.

The scenario tree for liabilities is then obtained by adding all of these payment types together:

$$L_{t,s} = P_{t,s} + O_{t,s} + LS_{t,s} + DG_{t,s}, \quad t = 1 \dots T,\ s = 1 \dots S \tag{4.12}$$

4.7 Concluding Remarks

This chapter gives an insight into how do we generate scenarios for the pension fund cash inflow and outflow. Equations (4.9) and (4.12) represent respectively the generated scenarios for the amount of money received from, and paid to, the fund participants.

For investment returns, we used bootstrapping and a copula method. We use the generated scenarios to evaluate the models proposed in Chapter 3 both in-sample and out-of-sample.

Chapter 5

Numerical Experiments

5.1 Introduction and Motivation

In this chapter, we implement the SSD based models proposed in Sections 3.4 and 3.5 using data sets drawn from the GOSI. Precisely, we implement an ICCP type of model (equivalent to an SSD unscaled model with deterministic target distribution), a Maximin type of model (equivalent to an SSD scaled model with deterministic target distribution) and SSD models, both scaled and unscaled, in which the target distribution is non-deterministic.

Our main question is whether by using different appropriate target distributions one can obtain better solutions, in the sense of more desirable distributions of funding ratio and asset value (wealth)?
To this end, we compare statistics from these distributions, as obtained from the four different models; we are particularly concerned with statistics describing the left tail for funding ratios. We perform both in-sample and out-of-sample analysis.

We also investigate the effect of introducing reservation levels, in addition to aspiration levels. To this end, we implement the SSD based model introduced in Section 3.6 by starting from a model with aspiration levels only and introducing reservation levels for the tails.

In the next section, we describe the data set we have used as input in our optimisation models. In Section 5.3, we discuss the computational set up and the results of *Experiment 1*, in which we compare the four models (SSD-Scaled), (SSD-Unscaled), (Maximin) and (ICCP). We illustrate the benefit of using generic SSD models over the special cases (ICCP) and (Maximin) models. In Section 5.4, we examine by *Experiment 2* whether the funding ratio distribution can be further improved by including reservation levels in addition to aspiration levels. We discussed how do we evaluate the models out-of-sample in Section 5.5. Finally, conclusions are given in Section 5.6.

5.2 Dataset

We consider a large defined benefit pension fund in Saudi Arabia, the General Organisation for Social Insurance GOSI [1]. As in [12] and [44], in this research, we consider a planning horizon of 10 years; $t = 0$ refers to year 2016. The GOSI can typically invest in four major investment fields; shares, bonds, loans, and real estate investments and it is keen to focus on domestic investments. Because of lack of data in the Saudi bonds' and real estate's indices they will not be included among the considered investment instruments in this research.

We consider 16 asset classes: the Saudi equities represented by 15 sectors indices beside cash. Investment decisions have to be taken "now" ($t = 0$) and then rebalanced every year, $t = 1 \dots 9$. We generate a set of $S = 300$ sample paths for the asset returns, contributions and liability values at times $t = 1 \dots 10$. These scenarios results from combining 30 scenarios of the asset returns and 10 scenarios correspond to contributions and liabilities based on the population model.

The scenarios for the asset returns are obtained by bootstrapping from historical data drawn from the Saudi Arabian stock market index (TASI) [2] for the period from Jun 2007 to Nov 2015. For the risk-free rate of return (interest rate) we consider the current Saudi Arabian interest rate of 2% following [3],

and assumed that it will stay at this level for the remaining time of the investment period.
The scenarios for the liability and contribution values have the same underlying source of uncertainty; they are generated based on populations and salary models. We followed the GOSI regulations in setting the percentage of salary to be paid in, as contributions, or out, as liabilities. The dynamics of the pension's fund population is modeled by a BIDE (Birth, Immigration, Death, Emigration) model [58], as described in Chapter 4. We use historical data from the GOSI's population as an input to this model [1]; that includes for example the number of participants, number of retirees, employment and retirement rates for the last 10 years, salary average and average salary growth.

5.3 Experiment 1

5.3.1 Computational Set Up

We implement the models (SSD-Scaled) and (SSD-Unscaled) developed in Chapter 3 with deterministic and non-deterministic target distributions of funding ratio. As a non-deterministic target distribution, we use a synthetic one with 300 equally likely scenarios, the lowest outcome is 0.9 and there is an increase by 0.0016 under each scenario. As a deterministic target distribution, we use one defined by the single outcome $\lambda = 1.1$.
We refer to the SSD scaled model with deterministic target distribution as (Maximin); as exposed in Section 3.5.1, it is equivalent to a maximisation of the worst outcome of the funding ratio.
We refer to the SSD unscaled model with deterministic target distribution as (ICCP), the reason for this is as follows. We have shown in Section 3.5.2 that the SSD unscaled model is equivalent, under mild conditions, to a model in which the expected shortfall (below the single target) is minimised. Further on, if the expected shortfall is used in a constraint rather than the objective (an ICCP type of model) and optimise the asset value, we obtain the same optimal solution, provided that the right hand side in these models are chosen

appropriately.
In our numerical experiments, we implement an ICCP model in which we maximise the expected terminal asset value and we impose a constraint on the expected shortfall of the funding ratio below the target 1.1 for the first time period. In order to decide on the right hand side for the integrated chance constraint, we ran an optimisation model that minimises the expected shortfall of the funding ratio distribution with respect to the target 1.1; this is in order to obtain the lowest right hand side for which the model is feasible. We found that the minimum expected shortfall is 0.042254. We implemented an ICCP model in which we maximise the expected terminal asset value and we constrain the expected shortfall of the funding ratio below 1.1 to be no more than 0.043. We obtained the portfolio displayed in Figure 5.1, the third pie chart. The integrated chance constraint is active, as the expected shortfall of the returns is exactly 0.043. We recorded the optimal value of the objective function of the ICCP model; let us denote it by A_T. We implemented the SSD unscaled model with a deterministic target of 1.1 with a constraint on the expected terminal asset value: to be no less than A_T. This is the reason why we refer to the SSD unscaled model with deterministic target distribution as "the ICCP model", they result in the same optimal solution. For all the models, we set A_T as the right hand side of the constraint on terminal asset value.

We have thus four SSD based models that we refer to as (SSD-Scaled), (SSD-Unscaled), (Maximin) and (ICCP). In all models , the right hand side of the constraint on the expected terminal asset value is the same (equal to A_T which corresponds to a cumulated terminal wealth of 581.5548 billions of Saudi Riyals (SAR)). The value of ϵ is fixed to 0.0001. We implement the models in AMPL and solve them using CPLEX 12.5.1.0.

The models (SSD-Scaled) and (SSD-Unscaled) have $(s^2 + 3(nt + t + 1)s + 3n + 3)$ decision variables and constraints, where n is the number of the available assets, t is the number of time periods and s is the number of scenarios. The solving time for these models, with the datasets described in Chapter 4

(S=300, n=15, t=10) are 4 minutes and 16 minutes respectively (this is the "_total_solve_elapsed_time" reported by AMPL). The algebraic formulation of the SSD Scaled model is in Appendix A and AMPL code is in Appendix B.

5.3.2 Computational Results

Figure 5.1 shows the optimum first stage decisions / portfolio allocations obtained by the four models within each sector. The (SSD-Scaled) and (Maximin) model are the most similar to each other; they mostly invest in cash investments, while the (SSD-Unscaled) and (ICCP) model mostly invests in the Retail sector. In view of the discussion in Sections 3.5.1 and 3.5.2 about the special case of the SSD models with deterministic target, it does not come as a surprise that the portfolio's compositions, and the performance, of (SSD-Scaled) and (Maximin) model on one hand and (SSD-Unscaled) and (ICCP) model on the other are somewhat comparable.

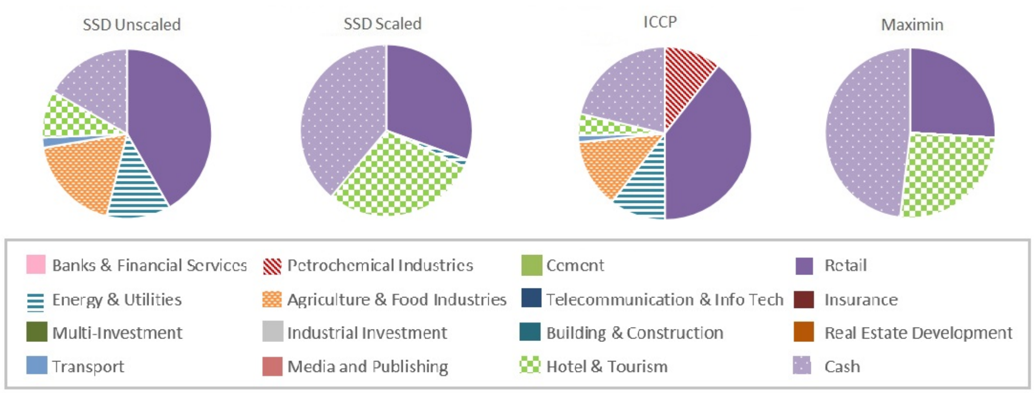

Figure 5.1: **The optimum first stage investment decisions for (SSD-Unscaled), (SSD-Scaled), (ICCP), and (Maximin) models: the proportion of wealth devoted to each asset class**

Tables 5.1 and 5.2 present performance measures for the first stage decisions of each of the four considered models by considering both the resulting return and funding ratio distributions. Table 5.1 lists statistics and risk - adjusted performance measures of the rate of return of the portfolio such as:

1. **The Sharpe ratio**: it measures the excess return per unit of risk in the investment; thus, the higher the Sharpe ratio value, the better the risk adjusted performance. For a random return R, the Sharpe ratio is defined as :

$$Sharpe\ ratio = \frac{E[R] - \tau}{\sigma(R)}$$

   where τ is a target return.

   The numerator represents the expected value of the return in excess of the target τ and σ is the standard deviation of the random return R. All models have the same target rate of return of $\tau = 2\%$ for the first time period.

2. **Sortino ratio**: Unlike the Sharpe ratio that penalises both upside as well as downside return deviations, the Sortino ratio penalises only those returns falling below a user-specified target rate of return τ; we fix it for all models at 2%.

   The Sortino ratio of a portfolio return R is defined as follows [63]:

$$Sortino\ ratio = \frac{E[R] - \tau}{\sqrt[2]{LPM_2(\tau, R)}}$$

   where

$$LPM_2(\tau, R) = E[\max\{\tau - R, 0\}^2]$$

3. **Value at Risk (VaR) and Conditional Value at Risk (CVaR)**: For the formal definitions of VaR_α and CVaR_α please refer to Section 2.4.2. We compute VaR and CVaR at confidence level α=95%. We consider as loss distribution the negative of the rate of return.

Table 5.2 lists statistics of the funding ratio. The α%-Scaled tail is the average of the worst α% outcomes of the funding ratio. All measures in these tables are calculated for the first time period.

Table 5.1: **The performance measures related to the rate of return: (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) models.**

Comparison criteria	SSD-Unscaled	SSD-Scaled	ICCP	Maximin
Expected rate of return	16.33 %	15.72 %	14.41 %	13.89 %
Sharpe ratio	0.7757	0.7158	0.7656	0.7108
Sortino ratio	2.9161	3.8335	2.3188	3.8356
$\mathbf{VaR}_{0.95}$	0.103	0.085	0.1273	0.07
$\mathbf{CVaR}_{0.95}$	0.139	0.0922	0.1637	0.0754

Table 5.2: **The performance measures related to the funding ratio: (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) models.**

Comparison criteria	SSD-Unscaled	SSD-Scaled	ICCP	Maximin
Expected funding ratio	1.160	1.154	1.141	1.135
Minimum funding ratio	0.8168	0.876	0.7926	0.8932
Expected shortfall of FR with respect to 1.1	0.0455	0.0526	0.0430	0.0517
1%-Scaled tail	0.8200	0.8794	0.7957	0.8967
5%-Scaled tail	0.8526	0.8969	0.8283	0.9124
10%-Scaled tail	0.8725	0.9165	0.8556	0.9290
15%-Scaled tail	0.8946	0.9284	0.8915	0.9392
20%-Scaled tail	0.9207	0.9383	0.9227	0.9471
25%-Scaled tail	0.9420	0.9477	0.9464	0.9551

The values in Tables 5.1 and 5.2 illustrate well the main differences between the models - and also supports the motivation of this work.
The results in Table 5.1 reinforce the similarity between (SSD-Unscaled) and

(ICCP), as well as between (SSD-Scaled) and (Maximin). The rate of returns of the (SSD-Unscaled) and (ICCP) solutions have higher expected values and Sharpe ratios but have poorer statistics regarding left tails / unfavorable outcomes: $\mathrm{VaR}_{0.95}$ and $\mathrm{CVaR}_{0.95}$ are (much) higher, indicating larger losses under unfavorable scenarios; also the Sortino ratio is considerably lower, indicating poorer downside risk return adjusted performance. We notice that the (SSD-Unscaled) solution performs better than the (ICCP) solution in all reported measures: expected value, risk-adjusted performance measures, left tail statistics.

The (SSD-Scaled) and (Maximin) solutions are similar in that the statistics on left tails and downside risk (as measured by $\mathrm{VaR}_{0.95}$and $\mathrm{CVaR}_{0.95}$) are considerably better at the expense of average performance. The (Maximin) solution has clearly the return distribution with the best left tail, but also with the lowest expected value. The (SSD-Scaled) solution provides a compromise between acceptable left tails and higher expected value.

From Table 5.2, as before, the similarity between (SSD-Unscaled) and (ICCP) models resides in a better overall performance at the expense of left tails / worst case scenarios. In contrast, (SSD-Scaled) and (Maximin) solutions result in funding ratios with the best statistics for left tails (measured up to 25% of left tails). Particularly in the worst case scenarios, these models perform much better - the differences start to decrease as we move along the left tails and consider more outcomes of the distributions. As before, the solution of the (SSD-Scaled) model provides a compromise between reasonable left tail statistics and better overall / average performance. While the (SSD-Unscaled) and (ICCP) solutions have rather similar characteristics, we note that the (SSD-Unscaled) solution results in better left tails up to 15%, including higher minimum and even better average performance, at the expense of a marginal increase in expected shortfall below the target.

Figure 5.2 plots, for each of the four models, the left tails of the funding ratio distributions; more precisely, the outcomes of the funding ratio distributions that are below the target 1.1. The main differences and similarities between the models are well illustrated in this figure.

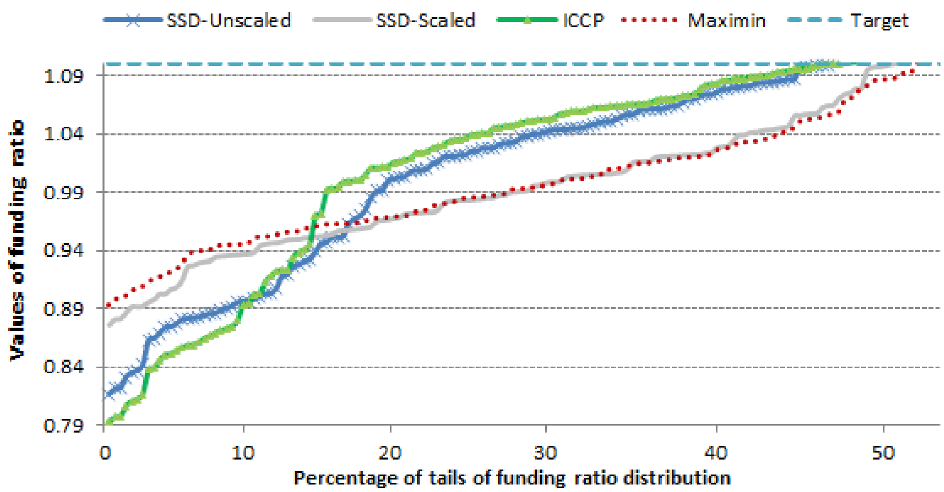

Figure 5.2: **Left tails of the funding ratio distributions resulting by (SSD-Unscaled), (SSD-Scaled), (ICCP), and (Maximin) models first stage decisions compared with the target 1.1**

From this figure, it can be easily seen that (ICCP) funding ratio "starts low" and it has the lowest outcomes up to 15% of the distribution. After this, it has the highest outcomes; hence overall it results in the lowest average shortfall below the target. The (SSD-Unscaled) distribution is closer in shape to the (ICCP) one; it has however higher outcomes under the worst 15% of scenarios.

In contrast, the (Maximin) model has the "best" worst outcome of the funding ratio. However, the performance in the rest of the distribution, although not a bad one, does not keep the best attributes; the model results in the lowest average funding ratio. From Figure 5.2, we can see that the lower part of the funding ratio distribution obtained by the (SSD-Scaled) model is very similar to the one obtained by the (Maximin) model; however, given that the (SSD-Scaled) had a higher average funding ratio, we can conclude that the gap between the curves correspond to the (Maximin) and the (SSD-Scaled) model increase after exceeding the target 1.1.

Different funding ratio distributions could be obtained by setting different aspiration levels. The fact that there is not a unique way of choosing the aspiration levels should be regarded as an advantage of this method. The process is interactive; we can analyse the obtained distribution of interest, and, if not satisfactory, we can modify the aspiration levels and obtain a further candidate solution. Thus, this funding ratio distribution can be further shaped, if necessary.

In the next section, we present another experiment in which we investigate whether we can further improve on the resulting funding ratio distribution by considering reservation levels, in addition to aspiration levels.

5.4 Experiment 2

5.4.1 Computational Set Up and Motivation

The SSD models were extended, as explained in Chapter 3, in order to gain greater modelling power. In this experiment, we provide a numerical example to illustrate the benefit of including the reservation levels to (SSD-Unscaled) model. A main objective of this experiment is to clarify that the models could be used interactively by specifying, whenever needed, new aspiration and/or reservation levels in order to obtain portfolios with progressively better funding ratio distributions.

We run the (SSD-Unscaled) model, first with aspiration levels only. These aspiration levels are set to be the optimums of each tail, forming thus an "unattainable" distribution. More precisely, for every $k = 1...S$, the new aspiration level asp_k is the optimum value that the tail of the funding ratio distribution can attain at confidence level $\frac{k}{S}$, $k = 1...S$. The vector $asp = (asp_1 \dots asp_S)$ results from solving the following optimisation model for every $k = 1..S$:

$$\textbf{Max} \quad [kT_k - \sum_{i=1}^{S} d_{k,i}]$$

Subject to:

$$T_k - F_s \leq d_{k,s} \; , \qquad s = 1 \dots S$$

$$d_{k,s} \geq 0 \; , \qquad s = 1 \dots S$$

also subject to (2.2) to (2.12).

We solved (SSD-Unscaled) and (SSD-Scaled) models using the new aspiration levels and examine the resulting funding ratio distributions to identify any undesirable aspects, the performance measures of the obtained solutions are presented in Table 5.3 and Table 5.4. We refer to these models by (**SSD-Unscaled2**) and (**SSD-Scaled2**) respectively.

Firstly, we found that (SSD-Unscaled2) provides the highest average rate of returns and the highest average funding ratio among all the models that we presented so far but at the expense of worsening the left tails. We set reservation levels for the 1% and 5% scaled tails of the funding ratio distribution to be 0.85 and 0.9 respectively, for the rest of the tails we use the solution without any modifications; we refer to this models as (**SSD-res1**).

Secondly, we consider the optimum solution of the (SSD-Scaled2). We use the reservation levels in order to increase the resulting average funding ratio; thus, we include a reservation level for 100% scaled tail of the funding ratio distribution, equivalent to the average, to be 1.21. We refer to the SSD model with a reservation level for the mean as (**SSD-res2**). The numerical results of this experiment are included in the next section.

5.4.2 Computational Results

Table 5.3 illustrates some performance measures of the return and the funding ratio distributions obtained by (SSD-Unscaled2) and (SSD-res1). All the measures are calculated for the first time period.
The values in Table 5.3 support the motivation of this experiment and support the claim that including reservation levels increase the modelling power of our SSD based optimisation models; we reshaped the left tail of the funding ratio distribution obtained by (SSD-Unscaled2) model. It could be seen from this table that by including reservation levels for 1% and 5% scaled tails we obtain a better 25% lower part of the funding ratio distribution, with a decrease of 0.023 in the average funding ratio. Moreover, although the solution obtained by (SSD-res1) model provides a lower expected rate of return; it results in a better/higher Sortino ratio and a better/lower $\mathrm{VaR}_{0.95}$ and $\mathrm{CVaR}_{0.95}$.

Table 5.3: **The performance measures related to the funding ratio distribution obtained by (SSD-Unscaled2) and (SSD-res1).**

Comparison criteria	**SSD-Unscaled2**	**SSD-res1**
Expected rate of return	23.42%	21.17%
Sortino ratio	3.7584	3.8354
$\mathbf{VaR}_{0.95}$	0.1465	0.12457
$\mathbf{CVaR}_{0.95}$	0.1643	0.1332
Expected funding ratio	1.230	1.207
1%-Scaled tail	0.8043	0.8390
5%-Scaled tail	0.8276	0.8567
10%-Scaled tail	0.8573	0.8822
15%-Scaled tail	0.8800	0.8980
20%-Scaled tail	0.8972	0.9109
25%-Scaled tail	0.9139	0.9221

Table 5.4 illustrates some performance measures of the return and the funding ratio distributions obtained by the models (SSD-Scaled2) and (SSD-res2). All the measures are calculated for the first time period.

Table 5.4: **The performance measures related to the funding ratio distribution obtained by (SSD-Scaled2) and (SSD-res2).**

Comparison criteria	**SSD-Scaled2**	**SSD-res2**
Expected rate of return	20.02%	20.90%
Sortino ratio	3.8352	3.8355
$\mathbf{VaR}_{0.95}$	0.1159	0.1225
$\mathbf{CVaR}_{0.95}$	0.1240	0.1310
Expected funding ratio	1.196	1.205
1%-Scaled tail	0.8481	0.8411
5%-Scaled tail	0.8657	0.8588
10%-Scaled tail	0.8899	0.8840
15%-Scaled tail	0.9046	0.8996
20%-Scaled tail	0.9167	0.9123
25%-Scaled tail	0.9274	0.9234

The values in Table 5.4 illustrate that in this example we slightly improved on the funding ratio distribution on average at the expense of a slight reduction in the lower tails. Interestingly, we can notice that the slight increase in the expected returns is associated with an increase in the VaR and CVaR values, as expected, but the Sortino ratio was not affected in a negative manner.

Remark 5.1. The reservation levels in these examples are not attainable, as some represent the tails of an SSD efficient solution and for some confidence levels we increase these numbers. The difference lyes in the fact that the new distribution will try to come close to these reservation levels before attempting to come close to the aspiration levels, hence it improved on tails.

5.5 Out-of-Sample Testing: Decision Evaluation

In this section, the first-stage decisions obtained by the different models analysed in the previous sections are evaluated out-of-sample. We follow the same order as in the in-sample analysis. Firstly, we compare the SSD based models with deterministic and non-deterministic targets: we analyse the first stage investment decisions of (SSD-Unscaled), (SSD-Scaled), (ICCP), and (Maximin) models out-of-sample. A similar out-of-sample analysis is done for (SSD-Unscaled2) versus (SSD-res1) on one hand and (SSD-Scaled2) versus (SSD-res2) on the other.

5.5.1 Datasets

Out-of-sample analysis is conducted over 11 different data sets. One of them is obtained by considering all observed historical returns of the component assets; these are, annual returns observed on a daily basis from the Saudi stock market between Jun 2007 and Nov 2015. We have computed 1937 scenarios for the annual rates of returns of the assets (fifteen stock indices); e.g. one scenario is the rate of return between first of January 2010 and first of January 2011.
Using the BIDE population model, in the same manner as explained in Chapter 4, we generate a larger number of 500 scenarios for the liabilities, to be used for out-of-sample tests. Hence, the set of out-of-sample scenarios has a large cardinality (968500 scenarios).

The remaining 10 data sets for asset returns are obtained using the Empirical copula and sampling from the margins, as explained in Chapter 4. For each marginal, we use the historical samples and fit into a univariate distribution, using the **R** package *gamlss*; Appendix **D** includes plots showing the histogram of the historical data of each stock index and overlaying the fitted distribution chosen for that variable. We also generate other 1937 samples from the fitted distribution for each margin, then combine them via the empirical copula. Each of these sets is then combined with 500 scenarios for liabilities; hence, we end up with 10 data sets each of them containing 968500 scenario.

To summaries, the out-of-sample analysis is conducted over 11 sets of scenarios each of size 968500.

5.5.2 Design of Computational Experiment

Our approach to out-of-sample analysis is described below:

1. Generate the in-sample scenarios for the optimisation problems.
2. Solve the models (SSD-Unscaled), (SSD-Scaled), (ICCP), (Maximin), (SSD-Unscaled2), (SSD-Scaled2), (SSD-res1) and (SSD-res2) using the in-sample scenarios.
3. Generate 11 larger sets of out-of-sample scenarios.
4. Use the first stage investment decisions obtained at 2 and compute the realisations of the rate of returns distribution and the funding ratio distribution, considering an out-of-sample scenario set generated in 3.
5. Compute performance and risk-adjusted performance measures.
6. Repeat the last two steps for each of the 11 out-of-sample scenario sets.

5.5.3 Computational Results

We evaluate the first-stage investment decisions obtained by the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) out-of-sample using 11 different data sets. Table 5.5 illustrates the results obtained considering the first out-of-sample data set; it is obtained using all available "historical" data, we refer to this data set by (Data set 1).

Table 5.5: **Out-of-sample analysis for the first-stage decisions of the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) using (Data set 1).**

Comparison criteria	**SSD-Unscaled**	**SSD-Scaled**	**ICCP**	**Maximin**
Expected rate of return	14.02%	13.74%	12.34%	12.21%
Sortino ratio	1.5947	2.1166	1.3460	2.1986
Expected FR	1.1198	1.1169	1.1036	1.1020
Minimum Funding ratio	0.5803	0.6516	0.5813	0.6903
Expected shortfall of FR with respect to 1.1	0.0718	0.0720	0.0708	0.0699
1%-Scaled tail	0.6527	0.7256	0.6450	0.7601
5%-Scaled tail	0.7089	0.7810	0.6964	0.8103
10%-Scaled tail	0.7534	0.8180	0.7396	0.8424
15%-Scaled tail	0.7901	0.8448	0.7801	0.8650
20%-Scaled tail	0.8243	0.8663	0.8211	0.8834
25%-Scaled tail	0.8545	0.8847	0.8557	0.8991

From Table 5.5, it can be seen that the out-of-sample results are mostly in line with the in-sample results, although (as expected) the worst case realisations are considerably lower, for all models considered. The solution of (Maximin) model has the (out-of-sample) funding ratio distribution with the highest/best worst case values and the highest left tails up to 25% of the distribution; on the other hand, the expected rate of return of the corresponding portfolio is the lowest, compared with the rest of the models. Interestingly, the solution of (ICCP) model does not result in the distribution with the lowest expected shortfall - it is the (Maximin) model that does, although the difference is marginal. Similarly to the in-sample results, the (SSD-Unscaled) and (SSD-Scaled) with non-deterministic target distributions have similar performances to (ICCP) and (Maximin) models, respectively, but do bring something new to the table. The solution of the (SSD-Unscaled) model improves on the very left tails of the funding ratio distribution, as compared to the (ICCP) model, while the solution of the (SSD-Scaled) model improves on the right tail / overall performance, compared to (Maximin) model.

We analyse the model's first-stage decisions out-of-sample over another ten different data sets in which the asset returns are generated using the empirical copula and univariate sampling described in Section 4.2.2. Over these data sets, we observe a similar pattern as in Table 5.5. The differences in the performance of each model's first-stage investment decisions are small; thus, the main features of the results in Table 5.5 are preserved. Table 5.6 illustrate the results obtained using one of these sets, we refer to this data set by (Data set 2).

Table 5.6: **Out-of-sample analysis for the first-stage decisions of the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) using (Data set 2).**

Comparison criteria	**SSD-Unscaled**	**SSD-Scaled**	**ICCP**	**Maximin**
Expected rate of return	14.02%	13.53%	12.50%	12.02%
Sortino ratio	1.7341	2.2270	1.5323	2.3081
Expected FR	1.1198	1.1149	1.1052	1.1001
Minimum FR	0.5928	0.6703	0.5934	0.7063
Expected shortfall of FR with respect to 1.1	0.0704	0.0700	0.0679	0.0681
1%-Scaled tail	0.6639	0.7348	0.6621	0.7675
5%-Scaled tail	0.7273	0.7904	0.7221	0.8182
10%-Scaled tail	0.7738	0.8280	0.7680	0.8502
15%-Scaled tail	0.8083	0.8546	0.8057	0.8727
20%-Scaled tail	0.8382	0.8751	0.8406	0.8903
25%-Scaled tail	0.8644	0.8921	0.8702	0.9052

The rest of the results, corresponding to the other 9 data sets, can be found in Tables E.1 to E.9 in Appendix **E**, we denote these data sets by (Data set 3) up to (Data set 11).
To give a general idea about the results of the out-of-sample analysis over the other data sets, we provide in Tables 5.7 and 5.8 a summary for some of the evaluated performance measures: average rate of returns and the 5% scaled tail of the funding ratio distribution computed based on the first stage decisions obtained by each model over all the data sets.

Table 5.7: **The average rate of return of the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) computed out-of-sample using 11 different out-of-sample data sets.**

Sample number	**SSD-Unscaled**	**SSD-Scaled**	**ICCP**	**Maximin**
Data set 1	14.02%	13.74%	12.34%	12.21%
Data set 2	14.02%	13.53%	12.50%	12.02%
Data set 3	14.32%	13.82%	12.86%	12.27%
Data set 4	14.30%	13.11%	12.82%	11.63%
Data set 5	13.75%	13.37%	12.24%	11.89%
Data set 6	13.97%	13.61%	12.52%	12.10%
Data set 7	13.97%	13.34%	12.55%	11.86%
Data set 8	13.39%	13.41%	12.01%	11.93%
Data set 9	13.76%	13.21%	12.24%	11.74%
Data set 10	13.53%	12.73%	12.20%	11.31%
Data set 11	13.79%	12.97%	12.29%	11.53%
Average	13.89%	13.35%	12.42%	11.86%
SD	0.0029	0.0033	0.0026	0.0029

Table 5.8: **The 5% scaled tail of the funding ratio distribution of the models: (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) computed out-of-sample using 11 different out-of-sample data sets.**

Sample number	SSD-Unscaled	SSD-Scaled	ICCP	Maximin
Data set 1	0.7089	0.7810	0.6964	0.8103
Data set 2	0.7273	0.7904	0.7221	0.8182
Data set 3	0.735	0.7927	0.7298	0.82
Data set 4	0.7344	0.7914	0.7288	0.8186
Data set 5	0.7217	0.7865	0.7193	0.815
Data set 6	0.7253	0.7872	0.721	0.8156
Data set 7	0.7263	0.7879	0.7202	0.8159
Data set 8	0.7251	0.7889	0.7205	0.8169
Data set 9	0.7273	0.7893	0.7226	0.8171
Data set 10	0.7215	0.7834	0.7153	0.8121
Data set 11	0.7301	0.7905	0.725	0.8179
Average	0.7257	0.7881	0.7201	0.8161
SD	0.0071	0.0035	0.0089	0.0029

From Table 5.7 and Table 5.8, it could be seen that the first stage decisions provided by the SSD based model with non-deterministic targets have consistently higher average rates of returns, and therefor higher average funding ratio, evaluated over all data sets used for out-of-sample analyses. The analysis shows that the first stage decisions obtained by (Maximin) model always results in the best/higher minimum funding ratio but lowest average rate of return.

In the second part of the experiment, we evaluate the first-stage decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2) and (SSD-res2) out-of-sample using the same data sets. The results illustrated in Table 5.9 corresponds to analysis made by using (Data set 1), corresponding to all historical observations.

Table 5.9: **Out-of-sample analysis of the first stage decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2) and (SSD-res2) using (Data set 1).**

Comparison criteria	**SSD-Unscaled2**	**SSD-res1**	**SSD-Scaled2**	**SSD-res2**
Expected rate of return	20.18%	18.45%	17.45%	18.21%
Sortino ratio	1.9154	2.0621	2.0763	2.0652
Expected FR	1.1798	1.1628	1.1531	1.1605
Minimum Funding ratio	0.4867	0.5512	0.5732	0.5564
Expected shortfall of FR with respect to 1.1	0.0837	0.0809	0.0789	0.0805
1%-Scaled tail	0.5715	0.6336	0.6542	0.6384
5%-Scaled tail	0.6511	0.7053	0.7223	0.7094
10%-Scaled tail	0.7083	0.7543	0.7685	0.7576
15%-Scaled tail	0.7529	0.7909	0.8029	0.7937
20%-Scaled tail	0.79	0.8198	0.8301	0.8223
25%-Scaled tail	0.8212	0.8440	0.8529	0.8461

From Table 5.9, we can see that the out-of sample results for these models are in line with the in-sample results. The advantage of adding reservation levels for 1% and 5% of the scaled tails of the funding ratio distribution in the model (SSD-res1) is preserved; it improves on the left tails at the expense of a decrease on the average funding ratio as compared with the optimum solution of (SSD-Unscaled2). On the other hand, with respect to (SSD-Scaled2) and (SSD-res2), the expected rate of return increased, and therefore the average funding ratio, with a decrease in the left tails.

We analyse the model's first stage decisions out-of-sample over the rest of data sets in which the asset returns are generated using the empirical copula and univariate sampling. The analysis using (Data set 2) is summarised in Table 5.10. The out-of-sample analysis using the rest of data sets could be found in Tables E.10 to E.18 in Appendix **E**.

Table 5.10: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2) and (SSD-res2) using (Data set 2).**

Comparison criteria	SSD-Unscaled2	SSD-res1	SSD-Scaled2	SSD-res2
Expected rate of return	19.87%	18.15%	17.18%	17.92%
Sortino ratio	2.029	2.1626	2.1777	2.1659
Expected FR	1.1768	1.1599	1.1504	1.1576
Minimum Funding ratio	0.5165	0.5770	0.5975	0.5819
Expected shortfall of FR with respect to 1.1	0.0809	0.0783	0.0765	0.0779
1%-Scaled tail	0.5878	0.6465	0.6663	0.6512
5%-Scaled tail	0.6677	0.7170	0.7333	0.7208
10%-Scaled tail	0.7283	0.7682	0.7815	0.7714
15%-Scaled tail	0.7727	0.8048	0.8158	0.8074
20%-Scaled tail	0.8066	0.8320	0.8414	0.8342
25%-Scaled tail	0.8342	0.8542	0.8625	0.8562

In Tables 5.11 and 5.12, we summarise the out -of-sample results regarding the average rate of return and the 5% scaled tail of the funding ratio distribution computed based on the first-stage investment decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2) and (SSD-res2); we consider the values obtained over all out-of-sample data sets.

Table 5.11: **The average rate of return in (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2) and (SSD-res2); out-of-sample evaluation using 11 different out-of-sample data sets.**

Comparison criteria	**SSD-Unscaled2**	**SSD-res1**	**SSD-Scaled2**	**SSD-res2**
Data set 1	20.18%	18.45%	17.45%	18.21%
Data set 2	19.87%	18.15%	17.18%	17.92%
Data set 3	20.34%	18.55%	17.55%	18.31%
Data set 4	19.37%	17.52%	16.59%	17.30%
Data set 5	19.59%	17.94%	16.98%	17.71%
Data set 6	19.97%	18.27%	17.29%	18.04%
Data set 7	19.62%	17.88%	16.92%	17.65%
Data set 8	19.67%	17.99%	17.03%	17.76%
Data set 9	19.42%	17.69%	16.75%	17.47%
Data set 10	18.70%	17.00%	16.10%	16.79%
Data set 11	19.08%	17.35%	16.42%	17.13%
Average	19.62%	17.89%	16.93%	17.66%
SD	0.0047	0.0047	0.0044	0.0047

Table 5.12: **The 5% scaled tail of the funding ratio distribution resulting by the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2) and (SSD-res2); out-of-sample evaluation using 11 different out-of-sample data sets.**

Sample number	**SSD-Unscaled2**	**SSD-res1**	**SSD-Scaled2**	**SSD-res2**
Data set 1	0.6511	0.7053	0.7223	0.7094
Data set 2	0.6677	0.7170	0.7333	0.7208
Data set 3	0.6722	0.7197	0.7358	0.7235
Data set 4	0.6709	0.7183	0.7345	0.7221
Data set 5	0.6610	0.7117	0.7284	0.7157
Data set 6	0.6621	0.7123	0.7289	0.7162
Data set 7	0.6641	0.7138	0.7303	0.7177
Data set 8	0.6654	0.7151	0.7315	0.7189
Data set 9	0.6666	0.7153	0.7317	0.7192
Data set 10	0.6568	0.7071	0.7240	0.7111
Data set 11	0.6688	0.7172	0.7335	0.7211
Average	0.6642	0.7139	0.7304	0.7178
SD	0.0062	0.0045	0.0042	0.0044

The out-of-sample results are in line with the in-sample ones. Although the spread of the distributions involved is larger when evaluated out-of-sample, including considerably lower worst case scenario than as suggested in-sample, the differences between shape of distributions obtained via different models remain the same.

5.6 Conclusions

In this chapter, we present computational experiments that illustrate how the proposed SSD models make a choice, as compared to some established approaches of imposing risk constraints in ALM models.

In Chapter 3, we showed that two established and computationally less expensive models, namely (Maximin) and (ICCP), are particular cases of the SSD models developed in this thesis; they are equivalent, under mild conditions, to SSD based models in which the target distribution is deterministic with one possible outcome. The numerical experiments presented in this chapter are mainly conducted to answer a natural question: can we obtain improved distributions of funding ratio by considering non-deterministic target distributions - and having thus the considerable extra computational difficulty of the generic SSD models? The computational study offers insight into this problem, by analysing solutions obtained from the (Maximin), the (ICCP) and the two SSD models (scaled and unscaled) with non-deterministic target distributions. The (ICCP) solution, although with lowest expected shortfall below target, has the lowest left tails, that is; lowest outcomes under worst case scenario, out of all solutions considered. The (Maximin) solution provides indeed the best outcome under the worst case scenario, however this advantage is not kept in the rest of the distribution. By using generic SSD formulations, we may obtain different solutions, in which the resulting distributions represent compromises between these two cases; that is, distributions with acceptable left tail and acceptable average, thus possibly more appealing to a range of decision makers.

Setting different aspiration levels offers a wide range of obtaining different/desirable funding ratio distributions. The fact that there is not a unique way of choosing the aspiration levels should be regarded as an advantage of this method. The process is interactive; we can analyse the obtained distribution of interest, and, if not satisfactory, we can modify the aspiration levels and obtain a further candidate solution.
A possible strategy is to start by implementing either an (ICCP) or (Maximin) model and analyse the resulting distribution of funding ratio. Should this be not acceptable, one can implement a generic SSD model, by setting a (non-deterministic) target distribution based on the outcomes of the funding ratio already obtained. For example, the targets for the worst case scenario and the left tails can be increased, should these values be too low in the ICCP

solutions. Similarly, the targets for the tails in the upper part of the distribution may be increased, should the (Maximin) model provide a solution with poor performance apart from worst case scenarios.

The numerical results of Experiment 2 illustrates that including reservation levels into the SSD model, beside aspiration levels, increase the modelling power of our SSD based optimisation models. By adding reservation levels, we improved in the left tails of the funding ratio distribution in one example and increase the average funding ratio in another. Thus, the funding ratio distribution can be shaped and "crafted" to a desirable form, to the extent that is achievable.

We analyse the model's first stage decisions out-of-sample over different and larger data sets, each containing 968500 scenarios. The asset returns are generated using the empirical copula and liabilities are generated by combining BIDE population model together with a salary model. The out-of-sample results are in line with the in-sample results, although, as expected, the worst case realisations are considerably lower, as there is a larger spread of the out-of-sample distributions. The shape of these distributions and the differences between them preserves the same pattern as indicated in-sample.

Chapter 6

Conclusions and Further Directions

6.1 Summary and Conclusions

In this research, we have formulated ALM models in which the risk of underfunding is controlled using Second Order Stochastic Dominance. This is a criterion of ranking random variables that takes the entire distribution of outcomes into account; in this case, the random variables of interest are funding ratios, that is, the ratio of asset value to liabilities. Random variables are compared with respect to SSD by point-wise comparisions of "α-tails" (unconditional expectation of A% worst case outcomes) where α=A% or "α-scaled tails" (conditional expectation of A% worst case outcomes) at different values of $\alpha \in (0, 1)$. A random variable is non-dominated with respect to SSD if there is no other random variable that results in better tails for all α. SSD eliminates the need to elect a utility function but works under the general and widely accepted assumptions of decision makers being rational (utility function is non-decreasing) and risk averse (utility function is concave).
Different approaches for modelling risk in the context of ALM can be found in the literature. They mainly stem from the single period asset allocation modelling framework and are related to the mean-risk paradigm, where the most common approach is to find investment decisions which result in a return

distribution with a high expected value and low value of risk. This modelling approach has been extended to the case of multi-period setting, liability driven investment by maximising the expected terminal fund wealth while imposing risk constraints on the funding ratio in the intermediate time periods. A risk constraint models one single aspect of the distribution. For example, one may impose an upper limit on the expected shortfall below a target; this does not guarantee a good enough left tail, corresponds to worst case scenarios, or an acceptable average performance.

The models proposed in this thesis find investment decisions such that the corresponding short-term funding ratio distribution is non-dominated with respect to SSD, while a constraint is imposed on the expected terminal wealth. In addition to being SSD efficient, the funding ratio distribution comes close, in a well defined sense, to a benchmark (target) distribution of funding ratio, whose outcomes are specified by the decision maker. Different target distributions lead to different SSD efficient solutions; the outcomes of the target distribution can be modified to satisfy specific requirements. Improved distributions of funding ratios may be thus achieved, compared to the existing risk models for ALM. As an application, we consider the planning problem of the General Organisation for Social Insurance (GOSI), which is a large defined benefit pension fund in Saudi Arabia.

There are two main SSD models presented in this thesis, a "scaled" model and an "unscaled" model. In these models, progressively larger left tails of the funding ratio distribution are considered, either scaled (equivalent to averages of a progressively higher number of worst case values), or unscaled (equivalent to sums of a progressively higher number of worst case values). Target values are considered for scaled and unscaled tails; the worst difference between a tail and its corresponding target value is optimised. A regularisation term is added to ensure SSD efficiency in case of multiple optimal solutions.

Target values for the tails correspond to target values for outcomes of the distributions, they thus determine a "target" or benchmark distribution. This

benchmark distribution is not required to be itself SSD efficient (or even the distribution corresponding to a feasible solution); the SSD efficiency of the resulting funding ratio distribution is guaranteed, irrespective of this choice.
Firstly, if the benchmark distribution is dominated with respect to SSD, the optimal solution results in a funding ratio distribution which is "better than target": it improves on the benchmark until SSD efficiency is attained. Secondly, if the benchmark is SSD efficient, the optimal solution of the model has a funding ratio distribution that exactly matches the benchmark. Finally, if the target is not attainable (in the sense that no feasible solution could match or improve on it), the optimal solution has a funding ratio distribution which is SSD efficient and comes as close as possible, in a well defined sense, to the target.

Both models result in (possibly different) SSD efficient distributions of funding ratio. With the SSD scaled model, the magnitude of deviations of outcomes below their corresponding targets weighs more. A good way to grasp the difference between the models is by considering that in the special case, the SSD scaled model is equivalent to maximising the lowest funding ratio, while the SSD unscaled model is equivalent to minimising the average of shortfalls below the target. Both models are formulated as LPs of large size (more than S^2 variables and constraints, where S is the number of scenarios).

The second contribution is of a theoretical nature; interesting results, connecting the proposed models to well established risk models and well established classes of SP models are derived for the particular case when the target distribution is deterministic, specified by one single outcome λ.
In most situations, the SSD scaled model is equivalent to a risk minimisation model, where risk is measured by the maximum loss. More precisely, the SSD scaled model can be reformulated as a (computationally much simpler) Maximin model which maximises the worst case value of the funding ratio.
The SSD unscaled model is equivalent, under mild conditions, to a risk minimisation model, where risk is measured by the lower partial moment of the funding ratio with order 1 and target λ, also called the expected shortfall below

target λ. The well established ICCP model has the expected shortfall below target as a constraint. By setting appropriate right hand side values, the SSD unscaled formulation and the ICCP formulation lead to the same optimal solutions.

There are situations in which the SSD models and risk minimisation models above may not be equivalent. This may happen when (a) the risk minimisation model has multiple optimal solutions; (b) in the case of minimisation of expected shortfall, the minimum is zero. In these cases, the optimal solution of the risk minimisation model is not guaranteed to be SSD efficient - unlike with the SSD formulations. A regularisation term should be added to the objective function in the risk minimisation models in order to guarantee SSD efficiency. However, this means increasing computational complexity to the level of the SSD formulations.
Thus, two established and computationally less expensive models, namely Maximin and ICCP, are particular cases of the SSD models developed in this thesis. Numerical experiments are conducted in order to answer a natural question that arises: can we obtain improved distributions of funding ratio by considering non-deterministic target distributions? The computational study offers insight into this problem, by analysing solutions obtained from Maximin, ICCP and the two SSD models (scaled and unscaled) with non-deterministic target distributions.
The ICCP solution, although with lowest expected shortfall below target, has the lowest left tails out of all solutions considered. The Maximin solution provides indeed the best outcome under the worst case scenario, however this advantage is not kept in the rest of the distribution. These can be regarded as extreme cases. By using generic SSD formulations, we may obtain better left tails, compared with the ICCP solution, and better overall performance, compared with Maximin solution. The resulting distributions of funding ratio may appeal to a large class of decision makers.

Setting different aspiration levels offers a wide range of obtaining different/desirable funding ratio distributions. Actually, the fact that there is not a unique way of choosing the aspiration levels should be regarded as an advantage of this method. The process is interactive; we can analyse the obtained distribution of interest, and, if not satisfactory, we can modify the aspiration levels and obtain a further candidate solution. Thus, the funding ratio distribution can be shaped and "crafted" to a desirable form, to the extent that is achievable.

A possible strategy is to start by implementing either an ICCP or Maximin model and analyse the resulting distribution of funding ratio. Should this be not acceptable, one can implement a generic SSD model, by setting a (non-deterministic) target distribution based on the outcomes of the funding ratio already obtained. For example, the targets for the worst case scenario and the left tails can be increased, should these values be too low in the ICCP solutions. Similarly, the aspiration levels for the tails including the upper part of the distribution may be increased, should the Maximin model provide a solution with poor performance apart from worst case scenarios.

The modelling power of SSD based models can be further increased by introducing reservation levels for the tails, in addition to the aspiration levels. The difference between reservations and aspirations is that reservation levels should "pre-empt" aspiration levels and thus should be achieved if at all possible. That is, optimisation should ensure that all reservation levels are achieved before attempting to come close to aspiration levels.

Modelling the uncertain parameters in ALM is much more challenging than in pure investment problems, due to the presence of liabilities and contributions (which require population models and salary models) and to the multi-stage nature of the model. The last contribution of this thesis is a framework for scenario generation based on the "Birth, Immigration, Death, Emigration" (BIDE) population model and the empirical copula. The empirical copula is used to generate new sets of scenarios that preserve the dependence structure among the asset classes under consideration. The scenarios for the liability and contribution values have the same underlying source of uncertainty; they are generated based on a BIDE population model, adapted to our ALM setting, and a salary model (assuming that a fixed percentage of salaries are to be paid in, as contributions, or out, as liabilities). The generated scenarios are then combined and used to evaluate all the proposed models and their special cases both in-sample and out-of-sample.

6.2 Further Directions

The SSD models proposed here are LPs of large size. This is due to the formulation of tails/scaled-tails; for each of the tails, a number of additional S variables and constraints are required. Since we consider tails at parameters $\frac{k}{S}$, $k = 1 \dots S$, we formulate an LP with S^2 additional variables and constraints, due to the tail representations. In real-life applications, when the number of scenarios is very large, such a model may not be tractable. Thus, it would be valuable to look into other representations for the tails, such as the cutting-plane representation [21].

In our approach, we have used the reference point method in order to find a Pareto optimal solution of a multi-objective optimisation problem (in which the objective functions represent tails of the funding ratio distribution). Another way is to optimise a weighted sum of the objective functions. This approach does not need the input of values for reference points but does need the input of weighting coefficients, which is also subjective. It would be interesting to investigate the connection between assigning specific reference points and assigning specific weighting coefficients.

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Appendices

Appendix A

Algebraic Formulation of SSD Model

In what follows, we present the formulation of the Second order Stochastic Dominance (SSD-Scaled) model. We use the following notations:

I = The number of financial assets available for investment

T = The number of time periods

S = The number of scenarios

The parameters of the model are defined as:

OP_i = The amount of money held in asset i at the initial time period $t = 0$; $i = 1 \ldots I$

L_0 = Aggregated liability payments to be made "now" $(t = 0)$

C_0 = The funding contributions received "now" $(t = 0)$

$L_{t,s}$ = Liability value for time period t under scenario s; $t = 1 \ldots T$, $s = 1 \ldots S$

$C_{t,s}$ = The contributions paid into the fund at time period t under scenario s; $t = 1 \ldots T$, $s = 1 \ldots S$

$R_{i,t,s}$ = The rate of return of asset i at time period t under scenario s; $i = 1 \dots I$, $t = 1 \dots T$, $\quad s = 1 \dots S$

u_i = The upper bound imposed on the investment in asset i; $i = 1 \dots I$

ψ = The transaction cost expressed as a percentage of the value of each trade

π_s = The probability of scenario s occurring; $s = 1 \dots S$

asp_k = The target or aspiration level for $\text{ScaledTail}_{k/S}(F) = Z_k$, $k = 1 \dots S$ (i.e. the aspiration levels for the mean of the worst k values of the funding ratio)

$d > 0$ = Desired rate of return over the investment horizon

$\epsilon > 0$ = The weighting coefficient of the regularisation term in the objective function.

Now, we define **the decision variables**.

Let us denote the **first stage decision variables** by:

$B_{i,0}$ = The monetary value of asset i to buy at the beginning of the planning horizon ($t = 0$); $i = 1 \dots I$

$S_{i,0}$ = The monetary value of asset i to sell at $t = 0$; $i = 1 \dots I$

$H_{i,0}$ = The monetary value of asset i to hold at $t = 0$; $i = 1 \dots I$

with $H_{i,0} = OP_i + B_{i,0} - S_{i,0}$, $\quad i = 1 \dots I$.

The additional variables for the SSD models are:

F_s = The funding ratio under scenario s at time t=1; (F_s=$A_{1,s}/L_{1,s}$); $s = 1 \dots S$

T_k= The k-th worst outcome of the funding ratio at time 1, k = 1 ... S (free variable); thus, $T_1, \dots, T_S$ are the outcomes of a random variable equal in distribution to the funding ratio

Z_k= The mean of the worst k outcomes of the funding ratio, or other said, $\text{ScaledTail}_{k/S}(F)$; $Z_k = (T_1 + \ldots + T_k)/k$, $k = 1 \ldots S$ (free variable)

$\delta = \min_{k=1\ldots S}(Z_k - asp_k)$= the worst partial achievement (free variable);

$d_{k,s}$= Non-negative variables, $d_{k,s} = [T_k - F_s]^+$ that is

$$d_{k,s} = \begin{cases} 0, & if \quad \mathrm{F}_s \geq T_k \\ T_k - F_s, & \quad \text{otherwise} \end{cases} \tag{A.1}$$

Recourse decision variables:

$B_{i,t,s}$ = The monetary value of asset i to buy at time t under scenario s; $i = 1 \ldots I, \quad t = 1 \ldots T-1, \quad s = 1 \ldots S$

$S_{i,t,s}$ = The monetary value of asset i to sell at time t under scenario s; $i = 1 \ldots I, \quad t = 1 \ldots T-1, \quad s = 1 \ldots S$

$H_{i,t,s}$ = The monetary value of asset i to hold at time t under scenario s; $i = 1 \ldots I, \quad t = 1 \ldots T, \quad s = 1 \ldots S$

$A_{t,s}$ = The assets value at time t under scenario s, **before** portfolio rebalancing.

THE Objective Function

The objective is to **maximise the minimum deviation between the mean of the worst k funding ratios at time t=1 and the k-th aspiration level**, a regularisation term is added to tackle the case of multiple optimal solutions:

$$\textbf{Max} \quad \delta + \epsilon\left(\sum_{k=1}^{S} Z_k - \sum_{k=1}^{S} asp_k\right)$$

Subject to

- **Asset Value Constraints**

$$A_{1,s} = \sum_{i=1}^{I} H_{i,0} R_{i,1,s} \ , \qquad s = 1 \dots S$$

$$A_{t,s} = \sum_{i=1}^{I} H_{i,t-1,s} R_{i,t,s} \ , \qquad t = 2 \dots T, \ s = 1 \dots S$$

- **Asset Holding Constraints**

$$H_{i,0} = OP_i + B_{i,0} - S_{i,0} \ , \qquad i = 1 \dots I$$

$$H_{i,1,s} = H_{i,0} R_{i,1,s} + B_{i,1,s} - S_{i,1,s} \ , \qquad i = 1 \dots I, \ s = 1 \dots S$$

$$H_{i,t,s} = H_{i,t-1,s} R_{i,t,s} + B_{i,t,s} - S_{i,t,s} \ , \qquad i = 1 \dots I, \ t = 2 \dots T-1, \ s = 1 \dots S$$

$$H_{i,T,s} = H_{i,T-1,s} R_{i,T,s} \ , \qquad i = 1 \dots I, \ s = 1 \dots S$$

- **Fund Balance Constraints**

$$\sum_{i=1}^{I} B_{i,0}(1+\psi) + L_0 = \sum_{i=1}^{I} S_{i,0}(1-\psi) + C_0$$

$$\sum_{i=1}^{I} B_{i,t,s}(1+\psi) + L_{t,s} = \sum_{i=1}^{I} S_{i,t,s}(1-\psi) + C_{t,s} \ , \qquad t = 1 \dots T-1, \ s = 1 \dots S$$

- **Short-Selling Constraints**

$$S_{i,0} \leq OP_i \ , \qquad i = 1 \dots I$$

$$S_{i,t,s} \leq H_{i,t-1,s} \ , \qquad i = 1 \dots I, \ t = 1 \dots T-1, \ s = 1 \dots S$$

- **Bound Constraints**

$$H_{i,t,s} \leq u_i \sum_{i=1}^{I} H_{i,t,s} \ , \qquad i = 1 \dots I, \ t = 1 \dots T, \ S = 1 \dots S$$

- **Funding Ratio Definition**

$$F_s = \sum_{i=1}^{I} H_{i,0} R_{i,1,s} / L_{1,s} \quad (F_s = A_{1,s}/L_{1,s}) \ , \qquad s = 1 \dots S$$

- **Additional Constraints to Formulate the SSD Model**

$$Z_k = T_k - \frac{1}{k}\sum_{s=1}^{S} d_{k,s} \;, \qquad k = 1 \dots S$$

$$Z_k - asp_k \geq \delta \;, \qquad k = 1 \dots S$$

$$T_k - F_s \leq d_{k,s} \;, \qquad k, s = 1 \dots S$$

$$d_{k,s} \geq 0 \;, \qquad k, s = 1 \dots S$$

- **Terminal Wealth Constraint**

$$\frac{1}{S}\sum_{s=1}^{S} A_{T,s} \geq \sum_{i=1}^{I} OP_i(1+d)$$

Appendix B

AMPL Code for (SSD-Scaled) Model

This Appendix includes the AMPL code for the (**SSD-Scaled**) model.

```
#Sets
set assets:=1..15 ;
set time;
set scenarios;

#Params
param Contributions {scenarios , time};
param Liabilities {scenarios , time};
param Original_Portfolio {assets} ;
param Return_Assets {assets , scenarios , time};
param Return_Banks {scenarios , time};
param Return_Petrochemecals {scenarios , time};
param Return_Cement {scenarios , time};
param Return_Retail {scenarios , time};
param Return_Energy {scenarios , time};
param Return_Agriculture {scenarios , time};
param Return_Telecom {scenarios , time};
param Return_Insurance {scenarios , time};
```

```
param Return_MultiInvestment {scenarios , time};
param Return_Industry {scenarios , time};
param Return_Building {scenarios , time};
param Return_RealEstate {scenarios , time};
param Return_Transportation {scenarios , time};
param Return_Media {scenarios , time};
param Return_Hotels {scenarios , time};
param upperbound {assets};
param transaction;
param lendprft;
param probability:=1/300;
# SSD parameters
param asp {scenarios};
param epsilon:= 0.0001;

    #Variables
var Market_Value_now >= 0;
var Assets_Value {time , scenarios} >=0;
var buy_now {assets} >=0;
var sell_now {assets} >=0;
var hold_now {assets} >=0;
var allamountHold_now >=0;
var lend_now >= 0;
var amounthold {a in assets, t in time , s in scenarios} >= 0;
var amountbuy {a in assets, t in 1..9 , s in scenarios} >= 0;
var amountsell {a in assets, t in 1..9 , s in scenarios} >= 0;
var allamountHold {t in time , scenarios} >= 0;
var lend {t in 1..9 , s in scenarios} >= 0;
# SSD variables
var fund_ratio {s in scenarios};
var ord_fund_ratio {s in scenarios};
var cumul_fund_ratio { s in scenarios};
var dev { k in scenarios , s in scenarios} >= 0;
```

var delta;

The objective function
maximize

minimum_deviation: delta + epsilon*sum{s in scenarios}(cumul_fund_ratio[s]/s-asp[s,1]/s);

#Constraints

subject to

#Market_Value
Market_Valuedeff0: Market_Value_now = allamountHold_now+lend_now;

#Assets_Value
Assets_Valuedeff1{s in scenarios}: Assets_Value[1,s] = sum{a in assets} (hold_now[a]* ReturnAssets[a,s,1])+lendprft*lend_now;
Assets_Valuedeffs{t in 2..10 , s in scenarios}: Assets_Value[t,s] = sum{a in assets} (amounthold[a,t-1,s]*ReturnAssets[a,s,t]) + lendprft*lend[t-1,s];

#Asset Holding Constraints:
assetholdingconstraints0{a in assets}: hold_now[a] = Original_Portfolio[a]+buy_now[a]-sell_now[a];
initialtotalHolding: allamountHold_now = sum{ a in assets }hold_now[a];
assetholdingconstraintassets1{a in assets , s in scenarios}: amounthold[a,1,s]= hold_now[a]*ReturnAssets[a,s,1]+ amountbuy[a,1,s]-amountsell[a,1,s];
assetholdingconstraintassets{a in assets, t in 2..9 , s in scenarios}: amounthold[a,t,s] = amounthold[a,t-1,s]*ReturnAssets[a,s,t]+ amountbuy[a,t,s]-amountsell[a,t,s];
assetholdingconstraintassets10{a in assets ,s in scenarios }: amounthold[a,10,s]= amounthold[a,9,s]*ReturnAssets[a,s,10];
amountHoldd{t in time , s in scenarios}: allamountHold[t,s] = sum{ a in assets }amounthold[a,t,s];

#Fund Balance Constraints:
fundbalanceconstraint0: sum{a in assets}(1+transaction)* buy_now [a]+ lend_now= sum {a in assets}(1-transaction)* sell_now[a];
fundbalanceconstraint1{s in scenarios}: sum{a in assets}(1+transaction)* amount-buy [a,1,s]+ Liabilities[s,1]+lend[1,s] = sum {a in assets}(1-transaction)* amountsell[a,1,s]+Contributions[s,1]+lendprft*lend_now;
fundbalanceconstraints{t in 2..9 , s in scenarios }: sum{a in assets} (1+trans-action)* amountbuy [a,t,s]+ Liabilities[s,t]+lend[t,s] = sum {a in assets}(1-transaction)* amountsell[a,t,s]+Contributions[s,t]+lendprft*lend[t-1,s];

#Short Sale Constraint:
shortsaleconstraint0{a in assets}: sell_now[a] <= Original_Portfolio[a];
shortsaleconstraint1{a in assets , s in scenarios}: amountsell[a,1,s] <= hold_now[a];
shortsaleconstraints{a in assets, t in 2..9 , s in scenarios}: amountsell[a,t,s] <= amounthold[a,t-1,s];

#Bound Constraints:
boundsconstraintup1{a in assets }: hold_now[a] <= upperbound[a]*allamountHold_now;
boundsconstraintup{a in assets , t in time , s in scenarios}: amounthold[a,t,s]<= upperbound[a]*allamountHold[t,s];

#Funding Ratio Definition
fundingratiodeff1{s in scenarios}: fund_ratio[s] = Assets_Value[1,s]/Liabilities[s,1];

#SSD Constraints:
minimumdeviation1{k in scenarios}: (cumul_fund_ratio[k]/k)-asp[k]/k >= delta;
cumulativeassetvaluedeff1{k in scenarios}: cumul_fund_ratio[k] = k*ord_fund_ratio[k]-sum{s in scenarios}dev[k,s];
deviationdeff1{k in scenarios, s in scenarios}: ord_fund_ratio[k]-fund_ratio[s] <= dev[k,s];

Appendix C

AMPL Code for Two-Stage SP and ICCP Models

This Appendix includes the complete AMPL code for the generic SP model and the additional integrated chance constraints.

```
    #Sets
set assets:=1..15 ;
set time;
set scenarios;

    #Params
param Contributions {scenarios , time};
param Liabilities {scenarios , time};
param Original_Portfolio {assets} ;
param Return_Assets {assets , scenarios , time};
param Return_Banks {scenarios , time};
param Return_Petrochemecals {scenarios , time};
param Return_Cement {scenarios , time};
param Return_Retail {scenarios , time};
param Return_Energy {scenarios , time};
param Return_Agriculture {scenarios , time};
param Return_Telecom {scenarios , time};
```

```
param Return_Insurance {scenarios , time};
param Return_MultiInvestment {scenarios , time};
param Return_Industry {scenarios , time};
param Return_Building {scenarios , time};
param Return_RealEstate {scenarios , time};
param Return_Transportation {scenarios , time};
param Return_Media {scenarios , time};
param Return_Hotels {scenarios , time};
param upperbound {assets};
param transaction;
param lendprft;
param probability:=1/300;

    #Variables
var Market_Value_now >= 0;
var Assets_Value {time , scenarios} >=0;
var buy_now {assets} >=0;
var sell_now {assets} >=0;
var hold_now {assets} >=0;
var allamountHold_now >=0;
var lend_now >= 0;
var amounthold {a in assets, t in time , s in scenarios} >= 0;
var amountbuy {a in assets, t in 1..9 , s in scenarios} >= 0;
var amountsell {a in assets, t in 1..9 , s in scenarios} >= 0;
var allamountHold {t in time , scenarios} >= 0;
var lend {t in 1..9 , s in scenarios} >= 0;

    #Objective Function

    maximize

    Terminal_wealth: sum {s in scenarios} (probability*Assets_Value[10,s]);
```

#Constraints
subject to

#Market_Value
Market_Valuedeff0:Market_Value_now=allamountHold_now+lend_now;

#Assets_Value
Assets_Valuedeff1{s in scenarios}:Assets_Value[1,s]=sum{a in assets} (hold_now[a]* ReturnAssets[a,s,1])+lendprft*lend_now;
Assets_Valuedeffs{t in 2..10 , s in scenarios}:Assets_Value[t,s]=sum{a in assets} (amounthold[a,t-1,s]*ReturnAssets[a,s,t]) + lendprft*lend[t-1,s];

#Asset Holding Constraints:
assetholdingconstraints0{a in assets}:hold_now[a]=Original_Portfolio[a]+buy_now[a]-sell_now[a];
initial_total_Holding: allamountHold_now=sum{ a in assets }hold_now[a];
assetholdingconstraintassets1{a in assets , s in scenarios}: amounthold[a,1,s]= hold_now[a]*ReturnAssets[a,s,1]+ amountbuy[a,1,s]-amountsell[a,1,s];
assetholdingconstraintassets{a in assets, t in 2..9 , s in scenarios}: amounthold[a,t,s]= amounthold[a,t-1,s]*ReturnAssets[a,s,t]+ amountbuy[a,t,s]- amountsell[a,t,s];
assetholdingconstraintassets10{a in assets ,s in scenarios }: amounthold[a,10,s]= amounthold[a,9,s]*ReturnAssets[a,s,10];
amountHoldd{t in time , s in scenarios}: allamountHold[t,s]=sum{ a in assets}amounthold[a,t,s];

#Fund Balance Constraints:
fundbalanceconstraint0:sum{a in assets}(1+transaction)* buy_now [a]+ lend_now= sum {a in assets}(1-transaction)* sell_now[a];
fundbalanceconstraint1{s in scenarios}:sum{a in assets}(1+transaction)* amountbuy [a,1,s]+ Liabilities[s,1]+lend[1,s]=sum {a in assets}(1-transaction)* amountsell[a,1,s]+Contributions[s,1]+lendprft*lend_now;
fundbalanceconstraints{t in 2..9 , s in scenarios }:sum{a in assets} (1+transaction)* amountbuy [a,t,s]+ Liabilities[s,t]+lend[t,s]=sum {a in assets}(1-transaction)*

amountsell[a,t,s]+Contributions[s,t]+lendprft*lend[t-1,s];

#Short Sale Constraint:
shortsaleconstraint0{a in assets}:sell_now[a]<=Original_Portfolio[a];
shortsaleconstraint1{a in assets , s in scenarios}:amountsell[a,1,s]<=hold_now[a];
shortsaleconstraints{a in assets, t in 2..9 , s in scenarios}:amountsell[a,t,s]<=
amounthold[a,t-1,s];

#Bound Constraints:
boundsconstraintup1{a in assets}:hold_now[a] <=upperbound[a]*allamountHold_now;
boundsconstraintup{a in assets , t in time , s in scenarios}:amounthold[a,t,s]<=
upperbound[a]*allamountHold[t,s];

- -

To formulate **ICCP** we can add the following integrated chance constraints to the SP model above:

we need the additional parameters
param targetFR:=1.1;
param RHS:=0.043;

and the additional variables
var fund_ratio{s in scenarios};
var shortage{s in scenarios }>=0;

Hence, the following integrated chance constraint could be formulated
integratedchanceconstraints1{s in scenarios}: fund_ratio[s]-targetFR+shortage[s]>=0;
integratedchanceconstraints2: probability *sum {s in scenarios} shortage[s]<=
RHS;
where the funding ratio is defined as follows:
fundingratiodeff1{t in time , s in scenarios}: fund_ratio[s]=Assets_Value[1,s]/Liabilities[s,1];

Appendix D

Histograms of the Historical Data of the Stocks Indices

The following plots show the histograms of the historical data of 15 Saudi stock sector indices and the fitted distribution chosen for each of them. The distributions that best fit the historical data are determined using a function in the **R** package *gamlss* that is designed to fit all available (from a list) *gamlss.family* distributions and select the one with the smallest *Akaike Information Criterion (AIC)*: a measure of the relative quality of statistical model for a given set of data. These fitted distributions are then used to generate samples for the univariate random variables representing each stock index when we use the copula as explained in Chapter 4.

These samples were kindly provided via a cooperation with my colleague Alina Peluso.

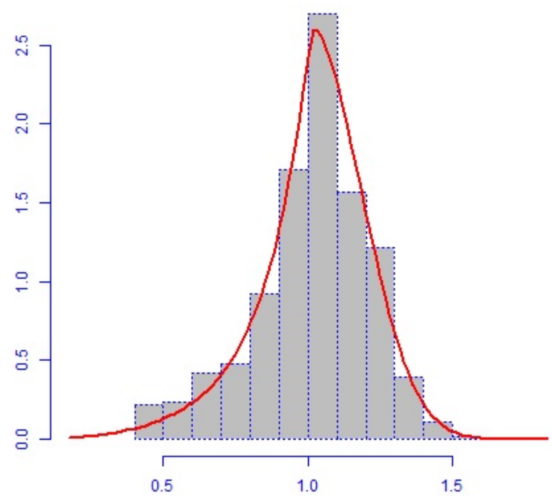

Figure D.1: The histogram of Banks sector and the fitted Box-Cox power exponential (BCPE) distribution

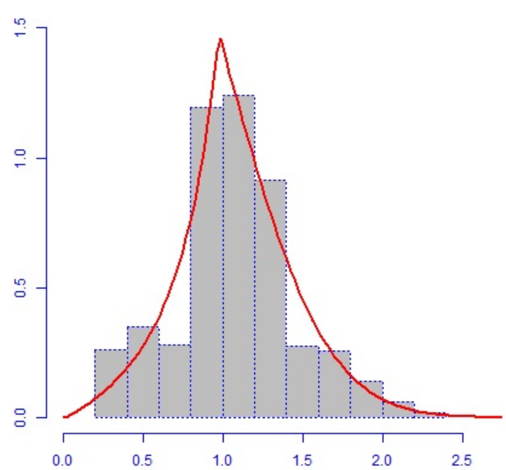

Figure D.2: The histogram of Petrochemical Industries sector and the fitted Box-Cox power exponential (BCPE) distribution

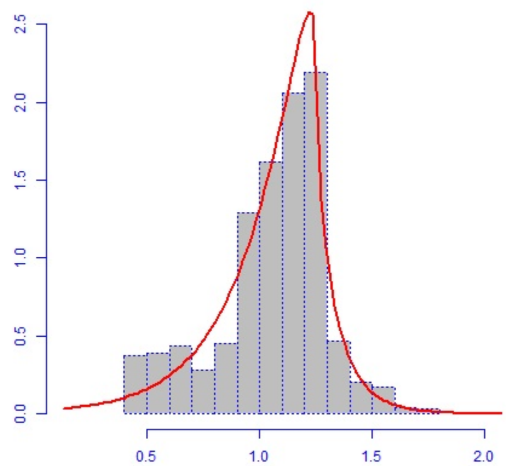

Figure D.3: The histogram of Cement sector and the fitted Skew Exponential Power type 2 (SEP2) distribution

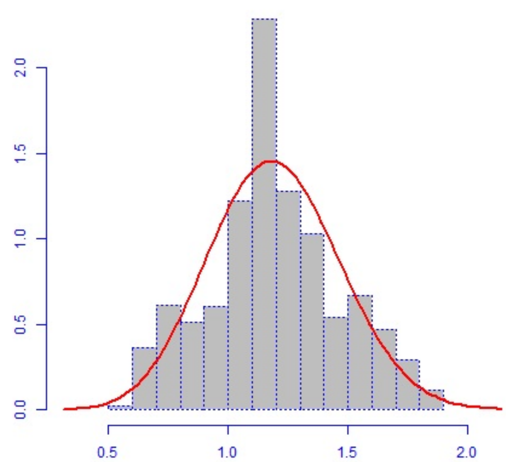

Figure D.4: The histogram of Retail sector and the fitted Generalised gamma (GG) distribution

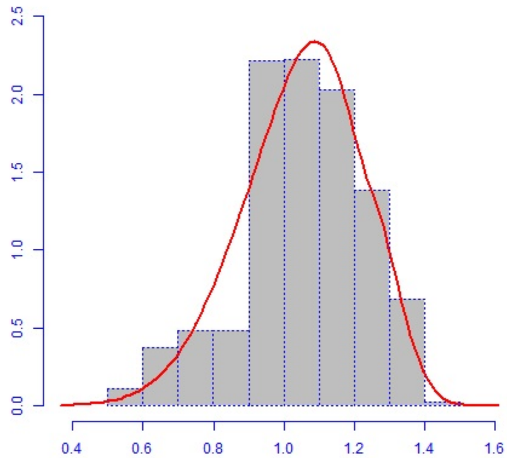

Figure D.5: The histogram of Energy and Utility sector and the fitted Skew Exponential Power type 2 (SEP2) distribution

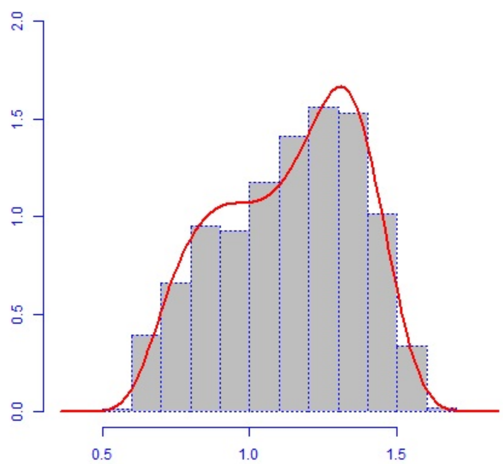

Figure D.6: The histogram of Agriculture and Food industries sector and the fitted Skew Exponential Power type 2 (SEP2) distribution

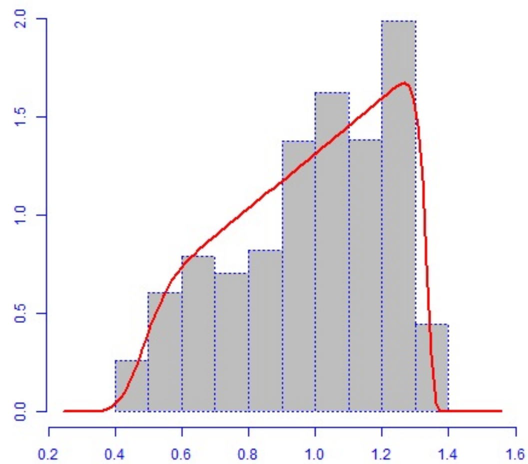

Figure D.7: The histogram of Telecommunication services sector and the fitted Box-Cox power exponential (BCPE) distribution

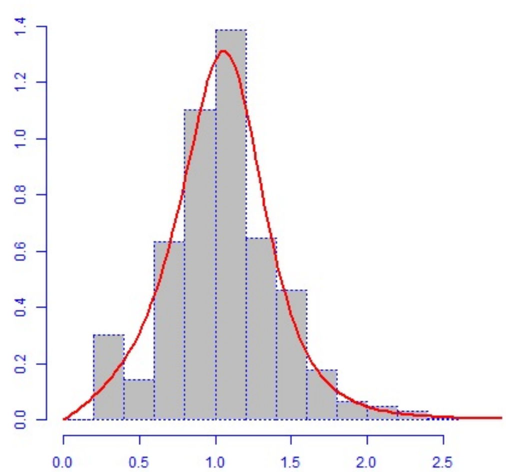

Figure D.8: The histogram of Insurance sector and the fitted Box-Cox t (BCT) distribution

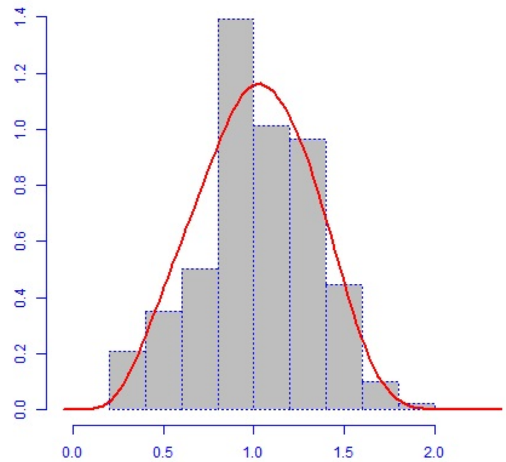

Figure D.9: The histogram of Multi-Investment sector and the fitted Skew Exponential Power type 1 (SEP1) distribution

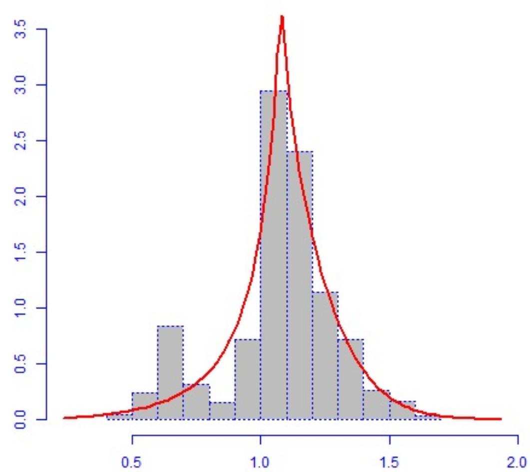

Figure D.10: The histogram of Industrial Investments sector and the fitted Box-Cox power exponential (BCPE) distribution

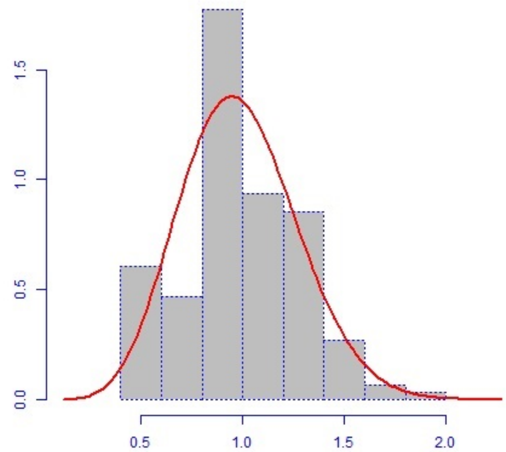

Figure D.11: The histogram of Building and Constructions sector and the fitted Generalised gamma (GG) distribution

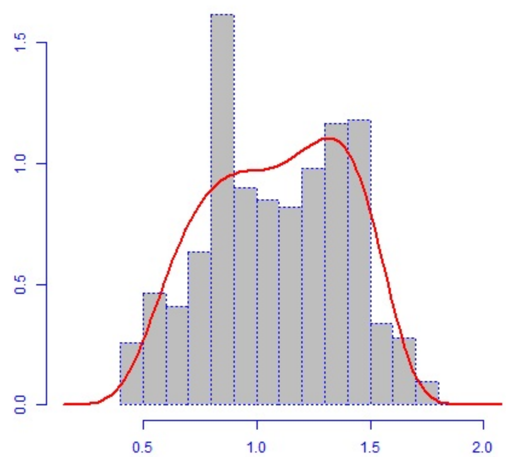

Figure D.12: The histogram of Real Estate Development sector and the fitted Skew Exponential Power type 2 (SEP2) distribution

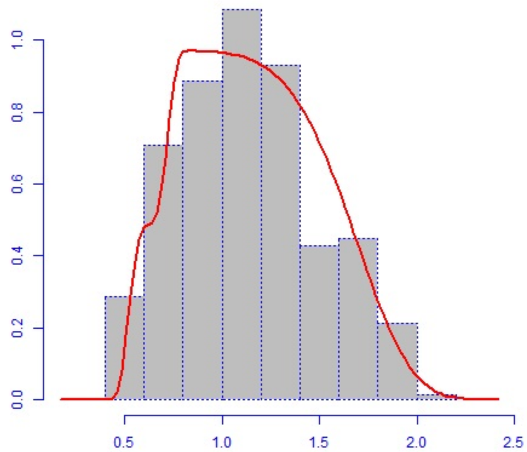

Figure D.13: The histogram of Transportation sector and the fitted Skew Exponential Power type 2 (SEP2) distribution

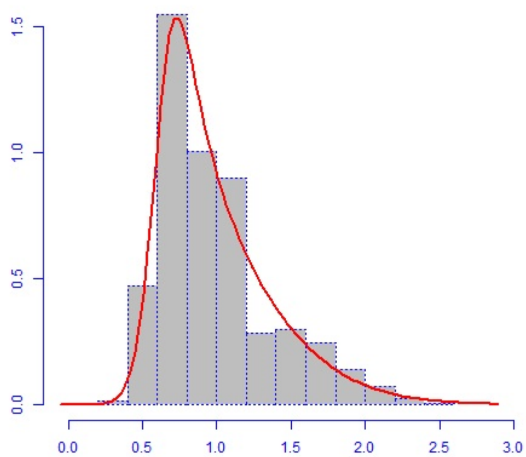

Figure D.14: The histogram of Media sector and the fitted Sinh-Arcsinh (SHASH) distribution

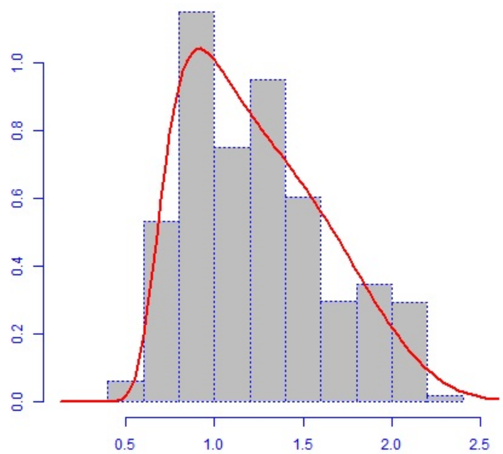

Figure D.15: The histogram of Hotels and Tourism sector and the fitted Sinh-Arcsinh (SHASH) distribution

Appendix E

Out-of-Sample Analysis

The following tables show the out-of-sample analysis for the first stage decisions of the models: SD-unscaled, SSD-Scaled, ICCP, Maximin, SD-unscaled2, SSD-Scaled2 and SSD with reservation levels (SSD-res1 and SSD-res2). The data sets for the asset returns are generated using copula.

Table E.1: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) using (Data set 3).**

Comparison criteria	**SSD-Unscaled**	**SSD-Scaled**	**ICCP**	**Maximin**
Expected rate of return	14.32%	13.82%	12.86%	12.27%
Sortino ratio	1.836	2.3309	1.6284	2.4107
Expected FR	1.1227	1.1177	1.1087	1.1026
Minimum Funding ratio	0.5913	0.6564	0.5993	0.6939
Expected shortfall of FR with respect to 1.1	0.0682	0.0678	0.0656	0.0661
1%-Scaled tail	0.6702	0.7353	0.6692	0.7679
5%-Scaled tail	0.735	0.7927	0.7298	0.82
10%-Scaled tail	0.7806	0.8309	0.7746	0.8525
15%-Scaled tail	0.815	0.8579	0.812	0.8753
20%-Scaled tail	0.8445	0.8787	0.8465	0.8932
25%-Scaled tail	0.8701	0.8959	0.8758	0.9083

Table E.2: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) using (Data set 4).**

Comparison criteria	SSD-Unscaled	SSD-Scaled	ICCP	Maximin
Expected rate of return	14.3%	13.11%	12.82%	11.63%
Sortino ratio	1.8068	2.1228	1.6063	2.1883
Expected FR	1.1225	1.1108	1.1083	1.0963
Minimum Funding ratio	0.5959	0.6606	0.6003	0.6975
Expected shortfall of FR with respect to 1.1	0.0695	0.0719	0.0667	0.0699
1%-Scaled tail	0.6742	0.7385	0.6714	0.77
5%-Scaled tail	0.7344	0.7914	0.7288	0.8186
10%-Scaled tail	0.7784	0.8266	0.7728	0.8486
15%-Scaled tail	0.8119	0.8521	0.8096	0.8703
20%-Scaled tail	0.841	0.872	0.8438	0.8874
25%-Scaled tail	0.8665	0.8888	0.873	0.902

Table E.3: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) using (Data set 5).**

Comparison criteria	**SSD-Unscaled**	**SSD-Scaled**	**ICCP**	**Maximin**
Expected rate of return	13.75%	13.37%	12.24%	11.89%
Sortino ratio	1.6588	2.1289	1.4814	2.2076
Expected FR	1.1171	1.1133	1.1026	1.0989
Minimum Funding ratio	0.5859	0.6545	0.5926	0.6925
Expected shortfall of FR with respect to 1.1	0.0718	0.0719	0.0689	0.0697
1%-Scaled tail	0.659	0.7316	0.6595	0.7646
5%-Scaled tail	0.7217	0.7865	0.7193	0.815
10%-Scaled tail	0.7686	0.8238	0.7656	0.8466
15%-Scaled tail	0.8036	0.8502	0.8034	0.8689
20%-Scaled tail	0.8339	0.8706	0.8381	0.8865
25%-Scaled tail	0.8603	0.8876	0.8676	0.9013

Table E.4: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) using (Data set 6).**

Comparison criteria	SSD-Unscaled	SSD-Scaled	ICCP	Maximin
Expected rate of return	13.97%	13.61%	12.52%	12.10%
Sortino ratio	1.7188	2.2084	1.5358	2.2885
Expected FR	1.1193	1.1156	1.1054	1.1009
Minimum Funding ratio	0.5852	0.6565	0.5908	0.6943
Expected shortfall of FR with respect to 1.1	0.0714	0.07	0.0684	0.068
1%-Scaled tail	0.6631	0.7332	0.6616	0.7657
5%-Scaled tail	0.7253	0.7872	0.721	0.8156
10%-Scaled tail	0.7724	0.8252	0.7679	0.8479
15%-Scaled tail	0.8076	0.8526	0.8063	0.8709
20%-Scaled tail	0.8374	0.8736	0.8409	0.889
25%-Scaled tail	0.8631	0.891	0.8701	0.9042

Table E.5: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) using (Data set 7).**

Comparison criteria	**SSD-Unscaled**	**SSD-Scaled**	**ICCP**	**Maximin**
Expected rate of return	13.97%	13.34%	12.55%	11.86%
Sortino ratio	1.6957	2.121	1.5128	2.1944
Expected FR	1.1195	1.1133	1.106	1.0988
Minimum Funding ratio	0.5843	0.6559	0.5882	0.6938
Expected shortfall of FR with respect to 1.1	0.0719	0.0714	0.0689	0.0693
1%-Scaled tail	0.6647	0.7349	0.6614	0.7671
5%-Scaled tail	0.7263	0.7879	0.7202	0.8159
10%-Scaled tail	0.7706	0.8235	0.7646	0.8463
15%-Scaled tail	0.8045	0.8496	0.8021	0.8684
20%-Scaled tail	0.8341	0.87	0.8371	0.886
25%-Scaled tail	0.8601	0.8872	0.8667	0.9009

Table E.6: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) using (Data set 8).**

Comparison criteria	**SSD-Unscaled**	**SSD-Scaled**	**ICCP**	**Maximin**
Expected rate of return	13.39%	13.41%	12.01%	11.93%
Sortino ratio	1.6151	2.1771	1.4442	2.2579
Expected FR	1.1139	1.114	1.1006	1.0995
Minimum Funding ratio	0.5913	0.6646	0.5931	0.7012
Expected shortfall of FR with respect to 1.1	0.0727	0.0696	0.0698	0.0677
1%-Scaled tail	0.6627	0.735	0.6615	0.7673
5%-Scaled tail	0.7251	0.7889	0.7205	0.8169
10%-Scaled tail	0.7695	0.8256	0.7646	0.8483
15%-Scaled tail	0.8034	0.8524	0.8016	0.8709
20%-Scaled tail	0.8328	0.8733	0.8361	0.8889
25%-Scaled tail	0.8587	0.8907	0.8657	0.9041

Table E.7: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) using (Data set 9).**

Comparison criteria	SSD-Unscaled	SSD-Scaled	ICCP	Maximin
Expected rate of return	13.76%	13.21%	12.24%	11.74%
Sortino ratio	1.6857	2.1398	1.493	2.2122
Expected FR	1.1175	1.1121	1.1029	1.0977
Minimum Funding ratio	0.5848	0.6531	0.5913	0.6911
Expected shortfall of FR with respect to 1.1	0.072	0.0715	0.0693	0.0695
1%-Scaled tail	0.6643	0.734	0.6632	0.7665
5%-Scaled tail	0.7273	0.7893	0.7226	0.8171
10%-Scaled tail	0.7727	0.8262	0.7675	0.8484
15%-Scaled tail	0.8066	0.8524	0.8046	0.8705
20%-Scaled tail	0.8359	0.8726	0.8388	0.888
25%-Scaled tail	0.8614	0.8895	0.8679	0.9027

Table E.8: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) using (Data set 10).**

Comparison criteria	**SSD-Unscaled**	**SSD-Scaled**	**ICCP**	**Maximin**
Expected rate of return	13.53%	12.73%	12.20%	11.31%
Sortino ratio	1.6189	1.9846	1.4461	2.0529
Expected FR	1.1153	1.1073	1.1025	1.0934
Minimum Funding ratio	0.5885	0.6625	0.5914	0.6994
Expected shortfall of FR with respect to 1.1	0.0729	0.0738	0.0698	0.0717
1%-Scaled tail	0.6607	0.7304	0.6573	0.7632
5%-Scaled tail	0.7215	0.7834	0.7153	0.8121
10%-Scaled tail	0.7667	0.8204	0.7605	0.8436
15%-Scaled tail	0.8018	0.8471	0.7989	0.8662
20%-Scaled tail	0.8321	0.8678	0.8346	0.8839
25%-Scaled tail	0.8584	0.885	0.8648	0.8989

Table E.9: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled), (SSD-Scaled), (ICCP) and (Maximin) using (Data set 11).**

Comparison criteria	**SSD-Unscaled**	**SSD-Scaled**	**ICCP**	**Maximin**
Expected rate of return	13.79%	12.97%	12.29%	11.53%
Sortino ratio	1.7059	2.0833	1.5071	2.1534
Expected FR	1.1178	1.1097	1.1034	1.0956
Minimum Funding ratio	0.589	0.6621	0.5945	0.6991
Expected shortfall of FR with respect to 1.1	0.0717	0.0726	0.0692	0.0705
1%-Scaled tail	0.6689	0.7376	0.6671	0.7693
5%-Scaled tail	0.7301	0.7905	0.725	0.8179
10%-Scaled tail	0.7748	0.8258	0.7688	0.848
15%-Scaled tail	0.8083	0.8512	0.8055	0.8695
20%-Scaled tail	0.8373	0.8709	0.8394	0.8866
25%-Scaled tail	0.8627	0.8875	0.8684	0.901

Table E.10: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2), and (SSD-res2) using (Data set 3).**

Comparison criteria	**SSD-Unscaled2**	**SSD-res1**	**SSD-Scaled2**	**SSD-res2**
Expected rate of return	20.34%	18.55%	17.55%	18.31%
Sortino ratio	2.1346	2.2619	2.2773	2.2652
Expected FR	1.1814	1.1639	1.1541	1.1615
Minimum Funding ratio	0.4963	0.5571	0.5788	0.5622
Expected shortfall of FR with respect to 1.1	0.0778	0.0756	0.0738	0.0752
1%-Scaled tail	0.5888	0.6461	0.666	0.6508
5%-Scaled tail	0.6722	0.7197	0.7358	0.7235
10%-Scaled tail	0.7338	0.7717	0.7848	0.7748
15%-Scaled tail	0.7793	0.8091	0.8198	0.8116
20%-Scaled tail	0.8134	0.8367	0.8458	0.8388
25%-Scaled tail	0.8409	0.8591	0.8671	0.861

Table E.11: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2), and (SSD-res2) using (Data set 4).**

Comparison criteria	**SSD-Unscaled2**	**SSD-res1**	**SSD-Scaled2**	**SSD-res2**
Expected rate of return	19.37%	17.52%	16.59%	17.3%
Sortino ratio	1.9599	2.0457	2.0605	2.049
Expected FR	1.1719	1.1537	1.1446	1.1517
Minimum Funding ratio	0.503	0.5629	0.5842	0.5679
Expected shortfall of FR with respect to 1.1	0.0828	0.0808	0.0789	0.0804
1%-Scaled tail	0.5965	0.6526	0.6718	0.6572
5%-Scaled tail	0.6709	0.7183	0.7345	0.7221
10%-Scaled tail	0.7275	0.7658	0.7792	0.769
15%-Scaled tail	0.7703	0.8008	0.8120	0.8035
20%-Scaled tail	0.8031	0.8272	0.8369	0.8295
25%-Scaled tail	0.83	0.8489	0.8575	0.8509

Table E.12: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2), and (SSD-res2) using (Data set 5).**

Comparison criteria	SSD-Unscaled2	SSD-res1	SSD-Scaled2	SSD-res2
Expected rate of return	19.59%	17.94%	16.98%	17.72%
Sortino ratio	1.9374	2.0669	2.0815	2.0701
Expected FR	1.1741	1.1579	1.1485	1.1557
Minimum Funding ratio	0.4930	0.5547	0.5766	0.5599
Expected shortfall of FR with respect to 1.1	0.0838	0.0809	0.0789	0.0804
1%-Scaled tail	0.5835	0.6427	0.6626	0.6474
5%-Scaled tail	0.6610	0.7117	0.7284	0.7157
10%-Scaled tail	0.7213	0.7624	0.776	0.7656
15%-Scaled tail	0.7656	0.7986	0.81	0.8013
20%-Scaled tail	0.7993	0.8256	0.8355	0.828
25%-Scaled tail	0.8268	0.8478	0.8564	0.8499

Table E.13: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2), and (SSD-res2) using (Data set 6).**

Comparison criteria	**SSD-Unscaled2**	**SSD-res1**	**SSD-Scaled2**	**SSD-res2**
Expected rate of return	19.97%	18.27%	17.29%	18.04%
Sortino ratio	2.013	2.1451	2.16	2.1484
Expected FR	1.1778	1.1611	1.1515	1.1588
Minimum Funding ratio	0.4955	0.5576	0.5793	0.5628
Expected shortfall of FR with respect to 1.1	0.081	0.0783	0.0764	0.0778
1%-Scaled tail	0.5872	0.6457	0.6653	0.6503
5%-Scaled tail	0.6621	0.7123	0.7289	0.7162
10%-Scaled tail	0.7237	0.7640	0.7775	0.7672
15%-Scaled tail	0.7699	0.8018	0.813	0.8044
20%-Scaled tail	0.8046	0.8297	0.8393	0.832
25%-Scaled tail	0.8326	0.8525	0.8609	0.8545

Table E.14: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2), and (SSD-res2) using (Data set 7).**

Comparison criteria	**SSD-Unscaled2**	**SSD-res1**	**SSD-Scaled2**	**SSD-res2**
Expected rate of return	19.62%	17.88%	16.92%	17.65%
Sortino ratio	1.9409	2.0541	2.0687	2.0573
Expected FR	1.1746	1.1575	1.1482	1.1553
Minimum Funding ratio	0.4944	0.5569	0.5786	0.5621
Expected shortfall of FR with respect to 1.1	0.0833	0.0805	0.0785	0.08
1%-Scaled tail	0.5892	0.6478	0.6673	0.6524
5%-Scaled tail	0.6641	0.7138	0.7303	0.7177
10%-Scaled tail	0.7211	0.7616	0.7753	0.7649
15%-Scaled tail	0.7652	0.8246	0.809	0.8002
20%-Scaled tail	0.7989	0.8246	0.8345	0.827
25%-Scaled tail	0.8265	0.8469	0.8557	0.849

Table E.15: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2), and (SSD-res2) using (Data set 8).**

Comparison criteria	**SSD-Unscaled2**	**SSD-res1**	**SSD-Scaled2**	**SSD-res2**
Expected rate of return	19.67%	17.99%	17.03%	17.76%
Sortino ratio	1.9803	2.1145	2.1294	2.1177
Expected FR	1.1751	1.1587	1.1493	1.1564
Minimum Funding ratio	0.5084	0.5688	0.5898	0.5738
Expected shortfall of FR with respect to 1.1	0.0809	0.0781	0.0762	0.0776
1%-Scaled tail	0.589	0.6474	0.6671	0.6521
5%-Scaled tail	0.6654	0.7151	0.7315	0.7189
10%-Scaled tail	0.7239	0.7645	0.778	0.7677
15%-Scaled tail	0.7690	0.8016	0.8128	0.8042
20%-Scaled tail	0.8035	0.8294	0.839	0.8316
25%-Scaled tail	0.8316	0.8522	0.8606	0.8542

Table E.16: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2), and (SSD-res2) using (Data set 9).**

Comparison criteria	SSD-Unscaled2	SSD-res1	SSD-Scaled2	SSD-res2
Expected rate of return	19.42%	17.69%	16.75%	17.47%
Sortino ratio	1.9604	2.0694	2.0843	2.0727
Expected FR	1.1727	1.1557	1.1465	1.1535
Minimum Funding ratio	0.4911	0.5526	0.5746	0.5578
Expected shortfall of FR with respect to 1.1	0.0827	0.0803	0.0784	0.0799
1%-Scaled tail	0.5879	0.6458	0.6655	0.6505
5%-Scaled tail	0.6666	0.7153	0.7317	0.7192
10%-Scaled tail	0.7264	0.7654	0.7789	0.7686
15%-Scaled tail	0.7704	0.8015	0.8126	0.8041
20%-Scaled tail	0.8037	0.8283	0.8379	0.8306
25%-Scaled tail	0.8308	0.8503	0.8588	0.8523

Table E.17: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2), and (SSD-res2) using (Data set 10).**

Comparison criteria	**SSD-Unscaled2**	**SSD-res1**	**SSD-Scaled2**	**SSD-res2**
Expected rate of return	18.70%	17.00%	16.10%	16.79%
Sortino ratio	1.8171	1.9153	1.9296	1.9184
Expected FR	1.1656	1.1489	1.1401	1.1468
Minimum Funding ratio	0.5051	0.5659	0.5871	0.5710
Expected shortfall of FR with respect to 1.1	0.0855	0.0830	0.0810	0.0826
1%-Scaled tail	0.5830	0.6416	0.6615	0.6463
5%-Scaled tail	0.6568	0.7071	0.7240	0.7111
10%-Scaled tail	0.7160	0.7571	0.7711	0.7605
15%-Scaled tail	0.7612	0.7941	0.8057	0.7968
20%-Scaled tail	0.7957	0.8217	0.8317	0.8241
25%-Scaled tail	0.8236	0.8442	0.8530	0.8462

Table E.18: **Out-of-sample analysis for the first stage decisions of the models (SSD-Unscaled2), (SSD-res1), (SSD-Scaled2), and (SSD-res2) using (Data set 11).**

Comparison criteria	**SSD-Unscaled2**	**SSD-res1**	**SSD-Scaled2**	**SSD-res2**
Expected rate of return	19.08%	17.35%	16.42%	17.13%
Sortino ratio	1.9103	2.0118	2.0265	2.0150
Expected FR	1.1694	1.1523	1.1433	1.1502
Minimum Funding ratio	0.5043	0.5654	0.5867	0.5705
Expected shortfall of FR with respect to 1.1	0.0843	0.0818	0.0798	0.0814
1%-Scaled tail	0.5951	0.6517	0.6710	0.6563
5%-Scaled tail	0.6688	0.7172	0.7335	0.7211
10%-Scaled tail	0.7257	0.7650	0.7785	0.7682
15%-Scaled tail	0.7681	0.7998	0.8111	0.8025
20%-Scaled tail	0.8008	0.8260	0.8358	0.8283
25%-Scaled tail	0.8275	0.8476	0.8562	0.8496