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Eigenvalue of a semi-infinite elastic strip

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A semi-infinite elastic strip, subjected to traction free boundary conditions, is studied in the context of in-plane stationary vibrations. By using normal (Rayleigh-Lamb) mode expansion the problem of existence of the strip eigenmode is reformulated in terms of the linear dependence within infinite system of normal modes. The concept of Gram's determinant is used to introduce a generalized criterion of linear dependence, which is valid for infinite systems of modes and complex frequencies. Using this criterion, it is demonstrated numerically that in addition to the edge resonance for the Poisson ratio $\nu = 0$, there exists another value of $\nu \approx 0.22475$ associated with an undamped resonance. This resonance is best explained physically by the orthogonality between the edge mode and the first Lamé mode. A semi-analytical proof for the existence of the edge resonance is then presented for both described cases with the help of the augmented scattering matrix formalism.

Keywords: edge resonance, vibration, semi-infinite strip.

1. Introduction

The phenomenon of so-called *edge resonance* was first observed by Shaw (1956) in his experiments on vibration of circular disks. When excited at a particular frequency, seemingly independent of the disc radius, vibration tended to localize near the disc edge. However, the observed vibration frequency lied below the first cut-off frequency of the corresponding infinite layer and could not therefore be related to thickness vibration modes. The phenomenon defied an explanation, for it seemed that the discovered vibration mode was not a disc natural mode.

First explanation was provided by Mindlin & Onoe (1957). They noted that at every particular frequency, in addition to a finite number of propagating Rayleigh-Lamb modes, an unbounded plate possesses an infinite family of non-propagating modes associated with complex wave numbers. These exponentially decaying modes form an infinite system of standing waves, which can be used to satisfy the boundary conditions at the edge of a semi-infinite plate. To illustrate this point, Gazis & Mindlin (1960) used a refined plate theory, approximating first couple of complex modes, to reproduce an edge resonance type response in the problem of the fundamental mode reflecting at the plate edge. The resonance phenomena was linked to excitation of the higher complex modes.

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The resonance frequency predicted by the somewhat crude approximation of Mindlin and Gazis substantially differed from the experimental observations of Shaw. Thus, Torvik (1967) implemented a more elaborate numerical procedure that used the fundamental and 10 pairs of complex modes to reproduce the experimentally observed edge vibration frequency with 1% accuracy. The method of Torvik was based on expanding the solution into an infinite series of normal (Rayleigh-Lamb) modes, which was subsequently truncated and used to approximately satisfy certain variational condition at the strip end.

Most of the following work followed this pattern. In particular, a version of the microwave network technique has been implemented for the solution of the same reflection problem by Auld & Tsao (1977). These authors have provided detailed numerical analysis of the reflection in a vicinity of the edge resonance, as well as derived a simple variational formula, which enabled an accurate estimation of the edge resonance frequency using only one pair of complex modes. Another numerical procedure based on the "method of projection" was presented by Gregory & Gladwell (1983). Although their main interest lied in the behavior of higher reflected normal modes, Gregory and Gladwell demonstrated very strong resonance-like behavior in a strip with traction free faces that is composed of an isotropic elastic material with Poisson's ratio $\nu = 1/4$. Thorough discussion of the edge vibration has also been given in the book by Grinchenko & Meleshko (1981).

While the numerical work has been mainly concerned with normal strip modes (with a notable exception of Grinchenko & Meleshko (1981), who used regular Fourier series), some progress has been recently made for stationery vibrations of a strip with mixed boundary conditions at the faces. For this problem Kaplunov et al. (2000) found an infinite discrete spectrum of edge-localized solutions, which are naturally related to the surface wave traveling along the strip edge. In her later work Wilde (2004) has also been able to use this spectrum in order to provide an empirical formulae for the edge vibration frequencies in the case of strip with free faces. However, in the latter case any direct results that would link edge vibration with edge surface waves are still evading the researchers.

Although various numerical techniques enabled quite accurate prediction of the frequency and modal shapes for the edge vibration phenomena, some important questions still remained unanswered. On the one hand, the complex mode excitation amplitudes in the problem of fundamental mode reflecting at the strip end were repeatedly demonstrated to be *finite*. This suggested that the observed resonance must in fact be damped. On the other hand, significant variations between observed relative amplitudes (from about 35 by Torvik (1967) to over 3000 by Gregory & Gladwell (1983)) did not allow such straightforward interpretations, as truncating an infinite system of equations could, in principle, introduce errors sufficient to damp the resonance.

First, and so far the only, significant result in this direction was obtained by Roitberg et al. (1998), who were able to prove that when Poisson's ratio $\nu = 0$ the problem indeed possesses a positive real eigenvalue, therefore proving the existence of edge resonance in this case. The authors also conjectured that in the case of non-zero Poisson ratios the symmetry, which enabled their solution, would be destroyed. The associated eigenvalue would then move into the complex plane, thus indicating a damped resonance.

We aim to extend the existing understanding of the edge vibration phenomena for non-zero Poisson's ratios. Section 2 is devoted to precise description of the considered boundary value problem. By introducing normal mode expansion for the stress field at the end of the strip, the existence of the edge mode is described in terms of the linear dependence within the infinite system of normal stress modes. An appropriate criterion for the linear dependence, based on the notion of Gram's determinant, is introduced in Section 3. It is valid for infinite systems of stress modes and complex frequencies. Particular kinds of internal symmetries of the governing elasticity operator are also considered. In addition to the internal symmetry that enables edge resonance at $\nu = 0$, we demonstrate that similar symmetry is available at the frequencies associated with Lamé (purely distortional) modes, see Lamb (1917). The developed methodology is applied in Section 4 to determine complex eigenvalue of the semi-infinite elastic strip that lies below first cut-off frequency. The dependence of the eigenvalue on the Poisson ratio is investigated in great detail. While generally the eigenvalue is demonstrated to be complex and, therefore, correspond to a damped resonance, an undamped resonance is observed for two values of the Poisson ratio, $\nu = 0$ and $\nu \approx 0.22475$. The resonance at the non-zero Poisson ratio is linked to interaction between the edge and Lamé modes. Finally, in Section 5 the analysis of the augmented scattering matrix in the spirit of Kamotskii & Nazarov (2002) is used to prove semi-analytically the existence of the edge resonance in both observed cases. This technique has previously been used to study eigenvalues of the acoustic half-plane with periodic boundary, see Kamotskii & Nazarov (1999a,b).

2. Statement of the problem

We consider a semi-infinite strip $\{(x, y) : x \geqslant 0\,,\ |y| \leqslant 1\}$, composed of a homogeneous, isotropic, linearly elastic material, see Figure 1. The strip boundary is subjected to traction free boundary conditions. In this paper we deal with stationary vibrations, i.e. the time dependence is assumed to be in the form $e^{-\mathrm{i}\omega t}$, with the (possibly complex) angular frequency ω. The resulting displacement field may be described by the vector equation of motion

$$k^2 \Delta \boldsymbol{u} + (K^2 - k^2)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + k^2 K^2 \boldsymbol{u} = 0\,, \tag{2.1}$$

and the following boundary conditions

$$\tau_{xy}(x, \pm 1) = \sigma_{yy}(x, \pm 1) = 0\,, \qquad x \geqslant 0\,, \tag{2.2}$$

$$\sigma_{xx}(0, y) = \tau_{xy}(0, y) = 0\,, \qquad |y| \leqslant 1\,. \tag{2.3}$$

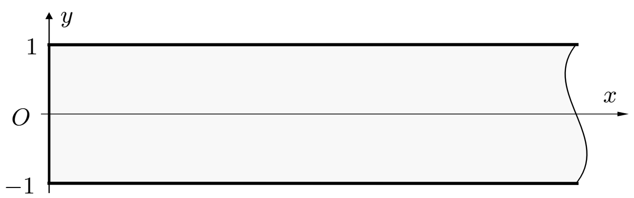

Figure 1. Semi-infinite strip configuration.

Scalars k and K are non-dimensional frequency parameters associated with the angular frequency ω through $k = \omega/c_1$ and $K = \omega/c_2$ and normalised to the unit half-thickness. k and K are related as

$$\frac{K}{k} = \sqrt{\frac{2-2\nu}{1-2\nu}}\,, \tag{2.4}$$

where ν is the Poisson ratio.

If for some value of angular frequency $\omega_e \in \mathbb{R}$ there exists a function $\boldsymbol{u}(x,y)$ that satisfies equation (2.1), boundary conditions (2.2), (2.3) and is summable over the region occupied by the strip, we shall call it a *pure eigenmode.* The associated angular frequency ω_e would be the corresponding eigenvalue. It is worth reiterating that the existence of such mode has been proven by Roitberg et al. (1998) for the case of zero Poisson's ratio.

For other values of Poisson's ratio, this definition ought to be generalized. The natural expectation is that for non-zero Poisson ratios the eigenvalue (and therefore the associated vibration frequency) moves off the real axis into the complex plane. A negative imaginary part of the angular frequency implies decaying in time, so the corresponding eigenmode will have a component that does not decay along axis Ox, thus leaking the energy to infinity.

It is well known that an infinite strip with the traction free faces $y = \pm 1$ may sustain two-dimensional waves of the form

$$\boldsymbol{u}(x,y,t) = \boldsymbol{U}(y)e^{\mathrm{i}(\alpha x - \omega t)}, \tag{2.5}$$

where scalar α is a wave number and ω is the associated angular frequency. In this paper we restrict our attention to solutions, which are symmetric with respect to the strip thickness. Thus, wave numbers α must satisfy the symmetric Rayleigh-Lamb secular equation, given by

$$(2\alpha^2 - K^2)^2 \cosh\gamma \sinh\delta - 4\alpha^2\gamma\delta \sinh\gamma \cosh\delta = 0\,, \tag{2.6}$$

$$\gamma = \sqrt{\alpha^2 - k^2}\,, \qquad \delta = \sqrt{\alpha^2 - K^2}\,, \tag{2.7}$$

see e.g. Graff (1991). Displacement solutions to satisfy (2.6) have the following form

$$\boldsymbol{U}(y) = \begin{pmatrix} u_1(y) \\ u_2(y) \end{pmatrix} \equiv \begin{pmatrix} i\alpha\cosh\gamma y + \delta B \cosh\delta y \\ \gamma \sinh\gamma y - i\alpha B \sinh\delta y \end{pmatrix}, \tag{2.8}$$

$$B = \frac{(2\alpha^2 - K^2)^2 \cosh\gamma}{2\mathrm{i}\alpha\delta\cosh\delta}\,. \tag{2.9}$$

Further developments would ask for stresses at the end of the strip. The appropriate stress components of the waves (2.5) may be written as

$$\begin{pmatrix} \sigma_{xx} \\ \tau_{xy} \end{pmatrix} = \mu \boldsymbol{S}(y) e^{\mathrm{i}(\alpha x - \omega t)}\,, \tag{2.10}$$

in which

$$\boldsymbol{S}(y) = \begin{pmatrix} \sigma(y) \\ \tau(y) \end{pmatrix} \equiv \begin{pmatrix} -(2\gamma^2 + K^2)\cosh\gamma y + 2\mathrm{i}\alpha\delta B \cosh\delta y \\ 2\mathrm{i}\alpha\gamma\sinh\gamma y + (2\alpha^2 - K^2) B \sinh\delta y \end{pmatrix}, \tag{2.11}$$

and μ is the shear modulus.

For every real value of angular frequency ω equation (2.6) possesses at least one positive real solution α (generally, a finite number of such solutions is available). These solutions are associated with propagating waves (2.5) and are typically referred to as normal or Rayleigh-Lamb (symmetric) modes. It is worth noting that with the exception of some special cases that will be further discussed in this paper, the superposition of propagating Rayleigh-Lamb modes cannot usually satisfy the boundary conditions at the strip edge, $x = 0$. In addition to these real solutions, for every value of the angular frequency there exists an *infinite* number of the complex solutions of (2.6). In the context of harmonic wave propagation in an infinite strip the associated wave solutions transport no energy and, if $\Im(\alpha) < 0$, exponentially grow as $x \to \infty$ and therefore are non-physical. However, modes with complex wave numbers α such that $\Im(\alpha) > 0$ form an infinite system of standing waves that are bounded for $x \geqslant 0$ and may therefore be used to satisfy the free-end boundary conditions at the strip edge.

Taking this into account we shall say that function $\boldsymbol{u}(x, y)$ is an eigenmode of the problem (2.1)–(2.3) provided it satisfies boundary conditions (2.3) and can be represented as a superposition of Rayleigh-Lamb modes, associated with both outgoing propagating waves (carrying energy to infinity) and exponentially decaying standing modes. Thus, we assume that the associated stress field is given by

$$\boldsymbol{S}(x, y) \equiv \sum_{j=1}^{N} A_j \boldsymbol{S}_j(y) e^{\mathrm{i}\alpha_j x} + \sum_{j=N+1}^{\infty} A_j \boldsymbol{S}_j(y) e^{\mathrm{i}\alpha_j x}\,, \tag{2.12}$$

with the first (finite) sum taken over Rayleigh-Lamb modes propagating energy in the positive x-direction ($\alpha_j \in \mathbb{R}$, $j = 1, 2, 3, .., N$, please note that α_j may be negative for modes whose phase and group velocities are of opposite signs). The second (infinite) sum in (2.12) is taken over damped modes ($\alpha_j \in \mathbb{C}$, $\Im(\alpha_j) > 0$, $j = N+1, N+2, \dots$). When the frequency is real the concepts of phase and group velocity are well-defined and the described choice of modes constitutes the assumed radiation conditions. We will also use expansion (2.12) for negative imaginary perturbations of the frequency. This is common technique used e.g. in scattering theory, see Veksler (1993) and references therein. Boundary conditions (2.3) take then the form

$$\boldsymbol{S}(0, y) \equiv \sum_{j=1}^{\infty} A_j \boldsymbol{S}_j(y) = \boldsymbol{0}\,. \tag{2.13}$$

The use of the normal mode expansions (2.12), (2.13) assumes completeness of the infinite family of stress modes. To the best of our knowledge the completeness of the dynamical problem for a semi-infinite strip with the stress boundary conditions at the end has never been proved and is usually assumed *a priory.* The proofs of completeness for the elastic strip modes are currently available in elastostatics, see Ustinov & Iudovich (1973), Kovalenko (1992), and in elastodynamics for displacement, see Kostyuchenko & Orazov (1977), and bonded interface boundary conditions at the strip end, see Kirrmann (1995).

3. Properties of normal modes

(a) *Linear dependence*

Since we are interested in non-trivial solutions for $\boldsymbol{S}(x,y)$, equation (2.13) may be interpreted as the requirement of linear dependence within the infinite system of modes $\boldsymbol{S}_j(y)$. With this in mind we introduce linear space $L_2^2(-1,1)$ of the two-dimensional vector functions $\boldsymbol{S}(y)$ with the scalar product

$$(\boldsymbol{S}_1,\boldsymbol{S}_2)=(\sigma_1,\sigma_2)+(\tau_1,\tau_2)=\int_{-1}^{1}\sigma_1(y)\overline{\sigma_2(y)}dy+\int_{-1}^{1}\tau_1(y)\overline{\tau_2(y)}dy\,, \tag{3.1}$$

where $\sigma(y),\tau(y)\in L_2(-1,1)$, see (2.11). A characteristic function of linear dependence can now be constructed using the so-called Gram determinant. For a given finite set of elements $\boldsymbol{S}_1,\boldsymbol{S}_2,\ldots,\boldsymbol{S}_N\subset L_2^2(-1,1)$, the Gram determinant

$$\mathrm{Gr}(\boldsymbol{S}_1,\boldsymbol{S}_2,\ldots,\boldsymbol{S}_N)=\begin{vmatrix}(\boldsymbol{S}_1,\boldsymbol{S}_1) & (\boldsymbol{S}_1,\boldsymbol{S}_2) & \ldots & (\boldsymbol{S}_1,\boldsymbol{S}_N)\\ (\boldsymbol{S}_2,\boldsymbol{S}_1) & (\boldsymbol{S}_2,\boldsymbol{S}_2) & \ldots & (\boldsymbol{S}_2,\boldsymbol{S}_N)\\ \ldots & \ldots & \ldots & \ldots\\ (\boldsymbol{S}_N,\boldsymbol{S}_1) & (\boldsymbol{S}_N,\boldsymbol{S}_2) & \ldots & (\boldsymbol{S}_N,\boldsymbol{S}_N)\end{vmatrix} \tag{3.2}$$

is equal to the squared volume of N-dimensional parallelepiped formed by the vectors $(\boldsymbol{S}_1,\boldsymbol{S}_2,\ldots,\boldsymbol{S}_N)$. Gram's determinant for a finite set of vectors is equal to zero if and only if these vectors are linearly dependent. However, we cannot use this fact directly because for the normalized sequence of normal mode vectors $(\boldsymbol{S}_1,\boldsymbol{S}_2,\ldots,\boldsymbol{S}_N)$ determinant (3.2) tends to zero everywhere as $N\to\infty$. It is therefore necessary to introduce new function

$$\phi_N(\boldsymbol{S}_j)=\sqrt{\frac{\mathrm{Gr}\left(\boldsymbol{S}_1,\boldsymbol{S}_2,\ldots,\boldsymbol{S}_{j-1},\dfrac{\boldsymbol{S}_j}{\|\boldsymbol{S}_j\|},\boldsymbol{S}_{j+1},\ldots,\boldsymbol{S}_N\right)}{\mathrm{Gr}(\boldsymbol{S}_1,\boldsymbol{S}_2,\ldots,\boldsymbol{S}_{j-1},\boldsymbol{S}_{j+1},\ldots,\boldsymbol{S}_N)}} \tag{3.3}$$

that is equal to the distance between normalized vector $\boldsymbol{S}_j$ and linear space Λ_j^N formed by the vectors $(\boldsymbol{S}_1,\boldsymbol{S}_2,\ldots,\boldsymbol{S}_{j-1},\boldsymbol{S}_{j+1},\ldots,\boldsymbol{S}_N)$. It may alternatively be defined as the solution for the following optimization problem

$$\phi_N(\boldsymbol{S}_j)=\min_{\substack{C_n\in\mathbb{C}\\ n\neq j}}\left\|\frac{\boldsymbol{S}_j}{\|\boldsymbol{S}_j\|}-\sum_{\substack{n=1,\ldots,N\\ n\neq j}}C_n\boldsymbol{S}_n\right\|. \tag{3.4}$$

Definitions (3.3) and (3.4) imply that

$$0\leqslant\phi_N(\boldsymbol{S}_j)\leqslant 1\,,\qquad \phi_{N+1}(\boldsymbol{S}_j)\leqslant\phi_N(\boldsymbol{S}_j)\,, \tag{3.5}$$

which enables us to formulate characteristic function for infinite vector systems in the form

$$\phi(\boldsymbol{S}_j)=\lim_{N\to\infty}\phi_N(\boldsymbol{S}_j)\,. \tag{3.6}$$

Let $\overline{\Lambda_j}$ be the closure of infinite-dimensional space spanned by $(\boldsymbol{S}_1,\boldsymbol{S}_2,\ldots,\boldsymbol{S}_{j-1},$ $\boldsymbol{S}_{j+1},\ldots)$. It is easy to see that $\phi(\boldsymbol{S}_j)$ exists for every system of vectors and has the following properties

$$1.\quad \phi(\boldsymbol{S}_j)=0\ \Leftrightarrow\ \boldsymbol{S}_j\in\overline{\Lambda_j}\,; \tag{3.7}$$

$$2. \quad \phi(\boldsymbol{S}_j) = 1 \;\Leftrightarrow\; \boldsymbol{S}_j \perp \overline{\Lambda_j}\,. \tag{3.8}$$

Thus we constructed a tool for investigating linear dependence within the system of stress modes (2.11). In the following sections it will be applied to find a complex eigenvalue and the associated eigenmode of the semi-infinite strip.

(*b*) *Orthogonality*

The proof for the existence of pure eigenmode by Roitberg et al. (1998) has been enabled by the special symmetry of the governing elasticity operator for zero Poisson's ratio. The crucial simplification cames from the fact that when $\nu = 0$ the fundamental mode of an elastic strip becomes non-dispersive, with $\alpha_1 = \pm K/\sqrt{2}$. Associated stresses

$$\boldsymbol{S}_1 = C\begin{pmatrix}1\\0\end{pmatrix}, \qquad C = \text{const}\,, \tag{3.9}$$

may then be combined into a standing wave, which would satisfy zero traction boundary conditions at the end of the strip. In other words, the fundamental mode impinging at the end of the strip would reflect into itself without generating decaying modes. This circumstance allowed Roitberg et al. (1998) to separate the essential spectrum of the elasticity operator for frequencies below the first cut-off† and, therefore, to isolate the strip eigenvalue.

The situation is best formulated in terms of mode orthogonality. Indeed, by considering formulae (2.11), (3.1) and (3.9), as well as taking into account dispersion relation (2.6), it may be shown that

$$(\boldsymbol{S}_1, \boldsymbol{S}_j) = C\int_{-1}^{1}\left(\frac{(2\alpha_j^2 - K^2)\cosh\gamma_j}{\cosh\delta_j}\cosh\delta_j y - (2\gamma_j^2 + K^2)\cosh\gamma_j y\right)dy = 0\,, \quad j \geqslant 2\,, \tag{3.10}$$

so we are justified to use the following notation

$$\boldsymbol{S}_1 \perp \boldsymbol{S}_j\,, \qquad j \geqslant 2\,. \tag{3.11}$$

Thus, any linear combination of modes $\boldsymbol{S}_j$, $j \geqslant 2$, cannot excite fundamental mode $\boldsymbol{S}_1$ and vice versa; this is true for all frequencies provided $\nu = 0$. Since fundamental mode is the only one propagating mode for frequencies below the first cut-off, the conclusion can be made that if there is an eigenmode below the first cut-off then it must be a pure eigenmode.

A similar kind of internal symmetry is available in the case of the so-called Lamé modes. These belong to the special kind of Rayleigh-Lamb modes, which produce purely distortional motions. There exists an infinite number of such modes associated with the frequencies

$$K = \sqrt{2}\left(\frac{\pi}{2} + \pi n\right), \qquad n = 0, 1, \ldots \tag{3.12}$$

† Here the *physical* cut-off frequency ω_c is implied, i.e. the frequency at which second Rayleigh-Lamb mode becomes propagating. It must not be mistaken for a long-wave high-frequency limit of the second mode ω_e, since $\omega_c < \omega_e$ for most physically significant values of the Poisson ratio.

We are mostly interested in the first Lamé mode ($n = 0$) with the corresponding stress field given by

$$\boldsymbol{S}_1(y) = C\begin{pmatrix}\cos\frac{\pi}{2}y\\ 0\end{pmatrix}, \qquad C = \mathrm{const}\,. \tag{3.13}$$

By considering scalar products $(\boldsymbol{S}_1, \boldsymbol{S}_j)$, $j \geqslant 2$, it is possible to demonstrate that in this case again

$$\boldsymbol{S}_1 \perp \boldsymbol{S}_j\,, \qquad j \geqslant 2\,. \tag{3.14}$$

It is worth reiterating that this solution only exists at a particular frequency, which for most positive values of the Poisson ratio lies below the first cut-off. Nevertheless, this kind of internal symmetry is very similar to the symmetry present in the case of zero Poisson's ratio. It is therefore natural to expect that if the frequency associated with Lamé mode coincides with the strip eigenvalue then a pure eigenmode exists. In the next section we will demonstrate that this is indeed possible and happens at a particular non-zero value of the Poisson ratio.

4. Numerical examples

Boundary conditions (2.13) for the end of the strip were interpreted in the previous section as the requirement of linear dependence within the infinite system of normal stress modes. If K^* is the eigenvalue associated with the edge mode, then the stress modes with non-zero amplitudes in the normal expansion (2.13) are linearly dependent. Therefore, we can use zeros of the characteristic functional $\phi(\boldsymbol{S}_j)$ to locate edge mode, see (3.7), but we need to ensure that the amplitude A_j of the associated mode $\boldsymbol{S}_j$ in (2.13) is not equal to zero. In particular, this may happen when the chosen stress mode is orthogonal to the rest of modes. For example, for zero Poisson's ratio $\boldsymbol{S}_1 \perp \boldsymbol{S}_n$, $n = 2, 3, \ldots$, see (3.11), and the characteristic functional $\phi(\boldsymbol{S}_1)$ cannot be used to locate the edge mode, because $\phi(\boldsymbol{S}_1) = 1$ for all K, see (3.8). Our tests indicated that $\boldsymbol{S}_2$ is not orthogonal to the rest of stress modes for Poisson ratios $\nu \in [0, 0.5]$ and frequencies below the first cut-off. Thus, $\phi(\boldsymbol{S}_2)$ will be used as a characteristic functional for linear dependence. $\phi(\boldsymbol{S}_1)$ will only be used to demonstrate cases when $\boldsymbol{S}_1$ is orthogonal to the rest of the modes.

In practical numerical computations functionals $\phi_N(\boldsymbol{S}_2)$ have to be used as estimates for $\phi(\boldsymbol{S}_2)$, see (3.3). Examples given in this section were computed using 60 complex Rayleigh-Lamb modes, so N=61. Instead of looking for zeros of the non-negative functional $\phi_{60}(\boldsymbol{S}_2)$, we seek local minima of the functional, which indicate proximity of eigenvalues.

A typical behaviour of the characteristic functional is presented in Figure 2 for $\nu = 0.31$. The associated eigenvalue of the semi-infinite strip is computed in this case as $K^* \approx 2.32816 - 0.00151\mathrm{i}$. The left-hand plot illustrates behavior of $\phi_{60}(\boldsymbol{S}_2)$ for $\Im(K) = 0$. The right-hand plot demonstrates the effect of moving into the complex domain with fixed $\Re(K) = 2.32816$. These plots show that the local minimum of $\phi_{60}(\boldsymbol{S}_2)$ is sufficiently well-defined to allow for accurate computation of both real and imaginary parts of the eigenvalue K^*.

It is well-known that edge vibration strongly depends on the Poisson ratio, see e.g. Grinchenko & Meleshko (1981). This is further illustrated in Figure 3, in which right-hand and left-hand plots correspond to $\nu = 0.2$ and $\nu = 0.31$, respectively.

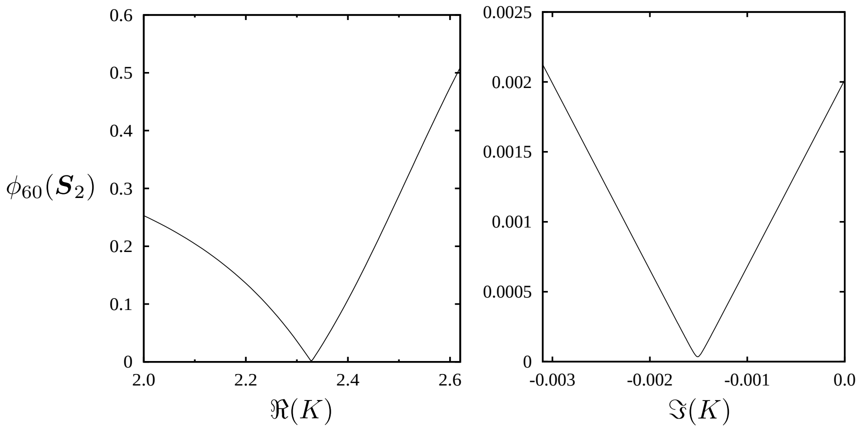

Figure 2. Behaviour of the characteristic functional $\phi_{60}(\boldsymbol{S}_2)$ in the vicinity of eigenvalue $K^* \approx 2.32816 - 0.00151\mathrm{i}$, where $\nu = 0.31$.

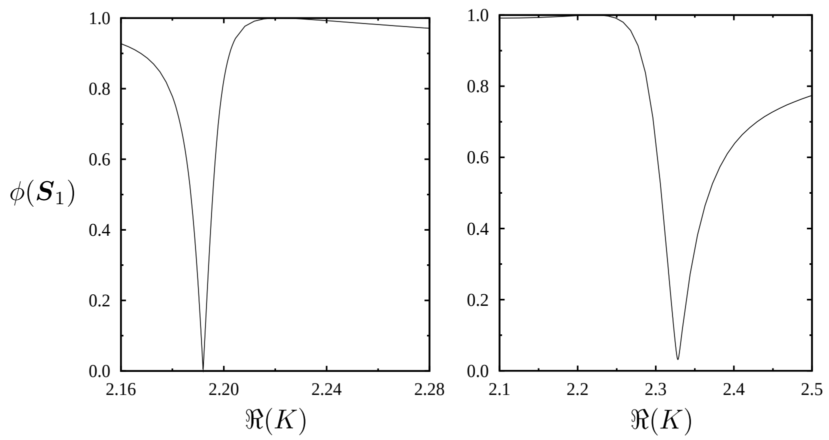

Figure 3. Behaviour of the characteristic functional $\phi(\boldsymbol{S}_1)$ in the vicinity of the Lamé reflection frequency. The Poisson ratio is $\nu = 0.2$ and $\nu = 0.31$ for the left-hand and right-hand plots, respectively.

Both of these plots illustrate the dependence of $\phi(\boldsymbol{S}_1)$ on $\Re(K)$, assuming that $\Im(K) = 0$. It follows from (3.8) that $\phi(\boldsymbol{S}_1) = 1$ when $\boldsymbol{S}_1 \perp \boldsymbol{S}_j$, $j = 2, 3, \ldots$, which is happening at the frequency of first Lamé mode $K = \pi/\sqrt{2} \approx 2.22$. Another interesting features are the local extrema, which correspond to the strip eigenvalue. Because of the continuity considerations, we may conclude that at the certain value of the Poisson ratio $\nu^* \in (0.2, 0.31)$ frequencies of the strip eigenmode and Lamé mode must coincide.

In order to fully characterize the dependence of complex eigenvalue on the Poisson ratio, Figures 4 and 5 were prepared. First of these figures provides real part of the eigenvalue for the Poisson ratios between 0 and 0.45. The value of ν where the frequency of the Lamé mode coincides with the real part of the strip eigenvalue may be accurately determined to be $\nu \approx 0.22475$. Similar figure was prepared by Le Clézio et al. (2003) for the range of Poisson's ratios between 0.2 and 0.4. Rela-

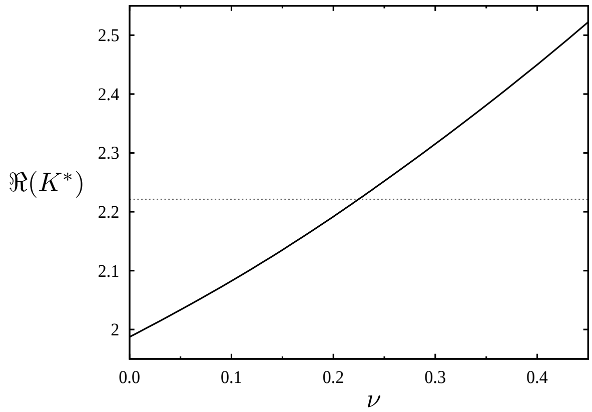

Figure 4. Solid line is a real part of complex eigenvalue of the semi-infinite strip, shown as a function of the Poisson ratio. Dotted line indicates a corresponding frequency of the Lamé mode.

tive flatness of the curve prompted the authors to suggest that the dependence of the strip eigenvalue on Poisson's ratio may be linear. Figure 4 clearly shows that $\Re(K^*(\nu))$ is not a linear function. It is, however, simple enough to be accurately approximated by a quadratic fit of the following form

$$\Re(K^*) \approx \frac{151 + 68\nu + 50\nu^2}{76}\,. \tag{4.1}$$

This formula predicts the frequency of the strip edge mode with the relative error below 0.05% for all positive Poisson's ratios.

The imaginary parts of the eigenvalues, shown in Figure 5, characterize the rate of energy leakage from the edge. Real (undamped) edge resonance is associated with

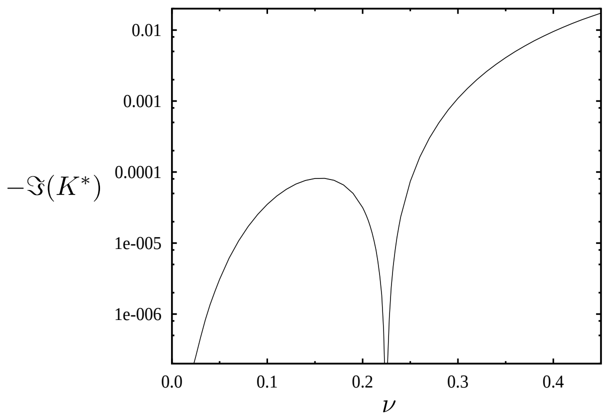

Figure 5. Imaginery part of complex eigenvalue of the semi-infinite strip, shown as a function of the Poisson ratio.

pure eigenmodes and corresponds to local extrema on Figure 5. As predicted in the previous section, two pure eigenmodes can be observed, first of them corresponds to $\nu = 0$, second — to $\nu \approx 0.22475$.

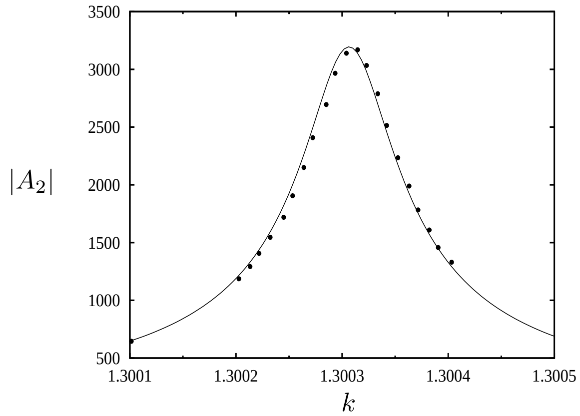

Figure 6. Dependence of relative amplitude of the first decaying mode on the frequency of the reflected mode ($\nu = 0.25$). Dots are taken from Figure 10 in the paper by Gregory & Gladwell (1983), solid line corresponds to the approximation (4.2).

The described numerical results clearly confirm our conjecture and indicate that there may exist second pure eigenmode associated with the Lamé mode. For a broad band of the Poisson ratios ($0 \leqslant \nu \leqslant 1/4$) the imaginary part of the eigenvalue is found to be under 10^{-4}, which indicates extremelly low amounts of energy leakage. The possible proximity of the second pure eigenmode serves as a good explanation for the unusually high edge excitation amplitudes, observed by Gregory & Gladwell (1983).

(*a*) *Application to free end reflection*

The analogy with the problem considered by Gregory & Gladwell (1983) may be elaborated yet further. It is worth reminding that these authors observed damped edge vibration in the problem of the Rayleigh-Lamb fundamental mode impinging at the end of the strip from the infinity for the Poisson ratio $\nu = 1/4$. The strip eigenmode manifested itself as a sharp increase in the amplitude of the reflection coefficient for the first complex mode.

Our technique does not allow accounting for the external excitation, therefore it is not possible to compute reflection coefficient directly. Instead we use a numerical estimate of the eigenvalue k^* to approximate observed edge vibration with the formula for a simple harmonic oscillator

$$|A_2| = \frac{C}{|k - k^*|}\,, \tag{4.2}$$

where $C = \text{const}$ has to be chosen to match heights of the amplitude peaks.

Numerical computations indicated that $k^* = 1.30031 - 0.00004\mathrm{i}$ when $\nu = 1/4$. The reflection coefficient for the first complex mode is shown in Figure 6. Approximation (4.2) (solid line) is shown along with individual dots taken from the paper by Gregory & Gladwell (1983). The real part of the eigenvalue matches the result of Gregory and Gladwell within fractions of a percent. Very good correlation between the widths of the damped resonance peaks indicates that we obtained correct result for the imaginary part of the eigenvalue as well.

5. Augmented scattering matrix

In order to prove the existence of the real eigenvalue associated with the Lamé mode, we use the method based on the theory of general elliptic problems with cylindrical outlets to infinity, see Nazarov & Plamenevsky (1994). The object that will be constructed in this section allows extracting discrete eigenvalues from the continuous spectrum. The main advantage of the described method is that it does not depend on internal operator symmetries and, consequently, does not require separation of the continuous spectrum, necessary for variational techniques.

Let us consider several pairs of the Rayleigh-Lamb modes, both growing and decaying exponentially along axis Ox. The augmented scattering matrix arises when one simulates physical radiation conditions† by selecting linear combinations of these modes. This requires some rearrangements, and first of all we enumerate the Rayleigh-Lamb modes and the corresponding roots of (2.6). It is worth reiterating that for every fixed real frequency secular equation (2.6) has finite number T of positive real roots, as well as T negative real roots, which we denote as

$$\alpha_1^+ < \cdots < \alpha_T^+ < 0 < \alpha_T^- < \cdots < \alpha_1^- \,. \tag{5.1}$$

Our arrangement assumes that the root multiplicity never exceeds one. In particular, this is true below the first cut-off frequency of the strip.

The secular equation (2.6) also has an infinite number of complex roots, but we only need to consider a finite subset of complex roots, situated close to the real axis. Let us specify $\gamma > 0$ such that the lines $\Im(\alpha) = \pm\gamma$ do not cross any roots of the equation (2.6). Since (2.6) is essentially a function of α^2, the number of roots between these two lines is even and we denote it as $2N$. $2T$ of these roots are real, the rest $2(N-T) = 4F$ are complex (for every complex root α equation (2.6) has three more roots $-\alpha$, $\overline{\alpha}$ and $-\overline{\alpha}$). The complex roots are denoted as follows

$$\Im(\alpha_j^+) > 0\,, \qquad j = T+1\dots N\,, \tag{5.2}$$

$$\Im(\alpha_{T+1}^+) = \Im(\alpha_{T+2}^+) < \cdots < \Im(\alpha_{N-1}^+) = \Im(\alpha_N^+)\,, \tag{5.3}$$

$$\Re(\alpha_{T+2j-1}^+) > 0\,, \qquad \Re(\alpha_{T+2j}^+) < 0\,, \qquad j = 1\dots F\,, \tag{5.4}$$

$$\alpha_j^- = \overline{\alpha_j^+}\,, \qquad j = T+1\dots N\,. \tag{5.5}$$

We denote by $w_j^\pm$ the Rayleigh-Lamb waves (2.5) corresponding to the roots $\alpha_j^\pm$. The quadratic form that describes generalized bi-orthogonality conditions may then be introduced as

$$q(w_i, w_j) = (U_i, S_j) - (S_i, U_j)\,, \tag{5.6}$$

where $(\cdot,\cdot)$ signifies the inner product on the edge $\{x = 0, y \in (-1,1)\}$. The reciprocity theorem states that $q(w_i, w_j) \neq 0$ if and only if $\alpha_i = \overline{\alpha_j}$, thus we introduce $c_i = q(w_i^+, \overline{w_i^+}) \neq 0$. For the propagating waves it describes the energy flux through the strip cross-section and, consequently, allows to formulate physical radiation conditions. Moreover, the form (5.6) allows generalizing radiation conditions for arbitrary Rayleigh-Lamb wave. In order to satisfy physical radiation conditions

† Hereafter, we are using the nomenclature of Kamotskii & Nazarov (1999a,b).

we bring quadratic form (5.6) to the principal axes. This is achieved by considering superpositions of the Rayleigh-Lamb waves, which define the new basis

$$u_i^{\pm} = |c_i|^{-1/2} w_i^{\pm}\,, \qquad i = 1\dots T\,, \tag{5.7}$$

$$u_i^{\pm} = \frac{1}{\sqrt{2}}\left((-c_i)^{-1/2} w_i^{+} \mp \mathrm{i}\,\overline{(-c_i)^{-1/2}} w_i^{-}\right)\,, \qquad i = T+1\dots N\,. \tag{5.8}$$

The quadratic form is brought to the principal axes in this basis and the following equalities hold

$$q(u_i^-, u_j^-) = i\delta_{ij}\,, \quad q(u_i^+, u_j^+) = -i\delta_{ij}\,, \quad q(u_i^+, u_j^-) = 0\,, \quad i,j = 1\dots N\,. \tag{5.9}$$

We shall call the waves $u_1^+,\dots,u_N^+$ ($u_1^-,\dots,u_N^-$) incoming (outgoing) waves. For propagating waves, $i = 1,\dots,T$, this definition obviously coincides with the physical radiation conditions, but is also extends the notion for the non-propagating waves, $i = T+1,\dots,N$. It turns out that it is more convenient to deal with the basis (5.7)–(5.8), rather than with the original basis, see Nazarov & Plamenevsky (1994).

Our argument is based on two principal facts, which are reformulated theorems from the paper by Kamotskii & Nazarov (1999a).

Theorem 1 (Kamotskii & Nazarov (1999a, p. 121, Theorem 2.11)). *There exist functions $Y_1,\dots,Y_N$ that satisfy equation of motion* (2.1) *and boundary conditions* (2.2)*,* (2.3) *and can be represented in the form*

$$Y_i = u_i^+ + \sum_{n=1}^{N} S_{in} u_n^- + o(e^{-\gamma x})\,, \qquad x \to \infty\,, \tag{5.10}$$

where matrix $\boldsymbol{S} = (S_{jn})_{j,n=1}^N$ is unitary, i.e. $\boldsymbol{S}^* = \boldsymbol{S}^{-1}$.

The matrix $\boldsymbol{S}$ is called the *augmented scattering matrix*. It is closely related to the standard scattering matrix and, in fact, may be represented in terms of it. Let us now introduce block $\boldsymbol{s}$ of the matrix $\boldsymbol{S}$, which corresponds to the non-propagating Rayleigh-Lamb waves, namely $\boldsymbol{s} = (S_{jn})_{j,n=T+1}^N$. The next result enables using $\boldsymbol{s}$ to gather information about edge eigenmodes.

Theorem 2 (Kamotskii & Nazarov (1999a, p. 124, Theorem 2.17)). *If* $\det(\boldsymbol{s} - \boldsymbol{I}) = 0$ *then the boundary value problem* (2.1)–(2.3) *has a pure eigenmode that can be represented in the following form*

$$Y = \sum_{n=T+1}^{N} A_n w_n^+ + o(e^{-\gamma x})\,, \qquad x \to \infty\,, \tag{5.11}$$

in which some of A_n are not equal to zero.

In the vicinity of a real eigenvalue only one propagating Rayleigh-Lamb mode exists, so $T = 1$. We also know from a numerical experiment that $A_2 \neq 0$ in the expansion (5.11). It is therefore sufficient to take $F = 1$ or $N = T + 2F = 3$. In both cases of the fundamental mode reflecting totally at the edge, as described in Section 3.2, the first function in (5.10) has the form

$$Y_1 = u_1^+ - u_1^-\,, \qquad S_{11} = -1\,, \qquad S_{1n} = 0\,, \qquad n = 2\dots N\,. \tag{5.12}$$

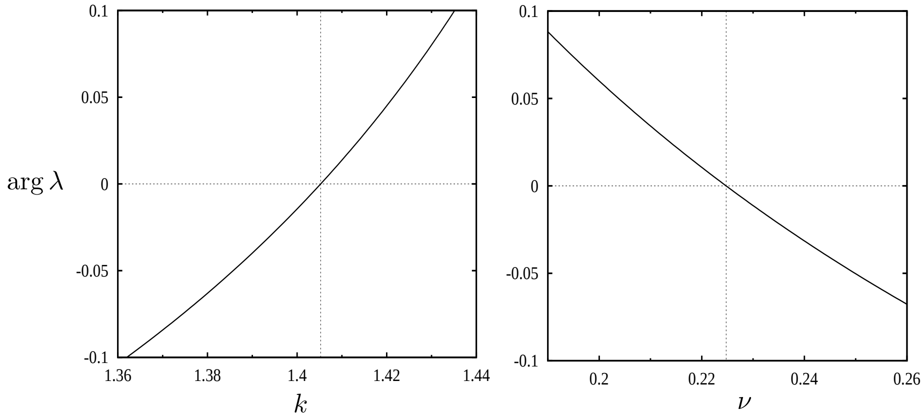

Figure 7. Behaviour of $\arg\lambda$ in the vicinity of real eigenvalues.

Since stresses associated with the damped waves are orthogonal at the edge to the stresses formed by the reflected fundamental mode, we have

$$Y_i = u_i^+ + \sum_{n=2}^{N} S_{in} u_n^- + o(e^{-\gamma x})\,, \qquad S_{i1} = 0\,, \qquad i = 2\ldots N\,. \tag{5.13}$$

Therefore, both when $\nu = 0$ and when $K = \pi/\sqrt{2}$, the augmented scattering matrix has the following form

$$\boldsymbol{S} = \begin{pmatrix} -1 & 0 \;\; 0 \\ 0 & \\ 0 & \boldsymbol{s} \end{pmatrix}, \tag{5.14}$$

and since $\boldsymbol{S}$ is a unitary matrix then the matrix $\boldsymbol{s}$ must be unitary as well. Thus, if λ is an eigenvalue of $\boldsymbol{s}$ then $|\lambda| = 1$.

Analytical expression for $\boldsymbol{S}$ is rather cumbersome, therefore we continued this investigation numerically. The computation demonstrated that in both cases of interest the function $\arg\lambda$ changes its sign, which is depicted in Figure 7. Left-hand plot demonstrates the dependence of $\arg\lambda$ on the frequency when $\nu = 0$, right-hand plot shows the dependence of $\arg\lambda$ on the Poisson ratio at the frequency of the Lamé wave, i.e. when $K = \pi/\sqrt{2}$. Because of the continuity of $\arg\lambda$, it means that in both cases we found the value of parameter, for which matrix $\boldsymbol{s}$ has the unit eigenvalue $\lambda = 1$. Therefore, the Theorem 2 guaranties the existence of the pure eigenmode in both of these cases.

The formulation and solution of the problem for the pure edge mode associated with the non-zero Poisson ratio may appear somewhat unconventional. The frequency of the edge mode is fixed, so the Poisson ratio assumes the role of spectral parameter. This situation is reminiscent of the concept of the Cosserat spectrum, which has been proved useful in elastostatics, see Markenscoff & Paukshto (1998) and references therein.

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