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Anti-symmetric motion of a pre-stressed incompressible elastic layer near shear resonance

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Abstract. A 2-dimensional model is derived for anti-symmetric motion in the vicinity of the shear resonance frequencies in a pre-stressed incompressible elastic plate. The method of asymptotic integration is used and a second order solution, for infinitesimal displacement components and incremental pressure, is obtained in terms of the long wave amplitude. The leading order hyperbolic governing equation for the long wave amplitude is observed to be not wave-like for certain pre-stressed states, with time and one of the in-plane spatial variables swapping roles. This phenomena is shown to be intimately related to the possible existence of negative group velocity at low wave number, i.e. in the vicinity of shear resonance frequencies.

Keywords: pre-stress, elastic plates, dispersion, shear resonance, asymptotics.

1. Introduction

The development and utilisation of lower dimensional (static) structural theories has been widespread for many years, resulting in such theories as Kirchhoff plate theory, Kirchhoff-Love shell theory and the refined Timoshenko-Reissner theories. In the case of static problems, only one type of asymptotic approximation, coupled with careful boundary layer analysis near the edge, is required, see for example [1]. In recent years the asymptotic approach has started to be extended to the dynamic case, for which high frequency motion is an additional feature of the problem. Moreover, high frequency motion will in general consist of both long and short wave contributions. A full detailed account of the asymptotic methodology required to determine the dynamic response of thin-walled elastic structures may be found, in the context of linear isotropic elasticity, in [2]. In this present paper we attempt to develop a model to help elucidate the effects of pre-stress on the dynamic response of an incompressible elastic plate. Specifically, this will involve deriving a 2-dimensional model to describe 3-dimensional anti-symmetric motion in the vicinity of the shear resonance (cut-off) frequencies.

Largely motivated by the widespread industrial application of rubber-like material, aspects of the effects of pre-stress on the dynamic properties of incompressible elastic media has been an area of considerable research activity in recent years, see for example [3], [4], [5], [6] and [7]. As a specific application we cite the use of rubber-like components in vibration control devices, especially as a method of protection against earthquake damage to bridges and tall buildings, see [8]. The main motivation for the present study is to further explicate the effects of pre-stress on dynamic material characteristics, by developing a greatly simplified 2-dimensional theory which is asymptotically consistent with the 3-dimensional theory, something clearly not required in the linear isotropic elastic context. We also remark that for any general dynamic loading problem waves of all wave

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lengths will contribute to the transient response, as well as those associated with all dispersion curve branches. For specific loads, or boundary conditions, motion in the vicinity of the cut-off frequencies may dominate. However, for all loads it will provide some contribution to transient response and the theory developed will therefore in part form the basis of possible future hybrid asymptotic-numerical methods to efficiently determine transient response. In the proposed context, and with pre-stress synthesising a spectrum of possible material response, some leading to possible loss of infinitesimal stability, the development of much faster methods to determine transient response is a highly desirable longer-term goal. There are also potential applications of this type of motion to fluid-structure interaction, particularly to jumps in radiation power and first order resonances of high frequency Lamb waves in scattering, see [9]. A final noteworthy motivation is the possible dominance of this type of motions in problems with fixed faces, such problems being characterised by the absence of a fundamental mode, see [10].

In Section 2 of this paper the basic equations of small amplitude time-dependent motions super-imposed upon a pre-stressed incompressible elastic solid are briefly reviewed. An appropriate dispersion relation is derived in Section 3 together with its appropriate approximations, which help to reveal the asymptotic structure of displacement components and incremental pressure. Asymptotically approximate equations are established in Section 4 and these are integrated exactly, in the vicinity of the first family of cut-off frequencies, to derive a leading order solution in terms of the long wave amplitude. A governing equation for the leading order long wave amplitude is obtained from the second order problem, as are higher order corrections for the infinitesimal displacement components and incremental pressure. These solutions are found in terms of both the leading order long wave amplitude and its second order correction. A higher order governing equation for the long wave amplitude is obtained from the third order problem. Similar results are given in respect of motion in the vicinity of the second family of cut-off frequencies.

Some interesting aspects of the governing equation for the long wave amplitude are especially noteworthy. The dispersion relations obtainable from both the leading order and second order 2-dimensional governing equation exactly match appropriate expansions derived from the exact dispersion relation, demonstrating a high level of consistency. Additionally, it is possible for the hyperbolic leading order governing equation to become non-wave-like, with time and one of the in-plane spatial variables swapping roles. Such a phenomenon is closely related to the possible existence of negative group velocity in the vicinity of the cut-off frequency. This point is illustrated with some numerical examples in respect of both a Mooney-Rivlin and a Varga material in Section 5.

2. Governing equations

Our concern in this paper is the propagation of infinitesimal waves in a finitely deformed layer, composed of incompressible elastic material, in particular deriving an asymptotic model for anti-symmetric (flexural) motion in the vicinity of the thickness shear resonance (cut-off) frequencies. In particular, this section is devoted to the derivation of equations governing wave propagation in an *un-*

bounded pre-stressed incompressible elastic media. We place the origin O of a Cartesian coordinate system $Ox_1x_2x_3$ in the mid-plane of the layer, and assume that two principal axes of the primary deformation lie in the plane of the layer along Ox_1 and Ox_3, with the third axis Ox_2 orthogonal to the layer. It can then be shown, see [11], that the appropriate equations of motion are

$$\begin{aligned} B_{1111}u_{1,11} + (B_{1122} + B_{2112})u_{2,12} + (B_{1133} + B_{3113})u_{3,13} \\ + B_{2121}u_{1,22} + B_{3131}u_{1,33} - p_{t,1} = \rho\ddot{u}_1 \,, \end{aligned} \tag{2.1}$$

$$\begin{aligned} (B_{2211} + B_{1221})u_{1,12} + B_{2222}u_{2,22} + (B_{2233} + B_{3223})u_{3,23} \\ + B_{1212}u_{2,11} + B_{3232}u_{2,33} - p_{t,2} = \rho\ddot{u}_2 \,, \end{aligned} \tag{2.2}$$

$$\begin{aligned} (B_{3311} + B_{1331})u_{1,13} + (B_{3322} + B_{2332})u_{2,23} + B_{3333}u_{3,33} \\ + B_{1313}u_{3,11} + B_{2323}u_{3,22} - p_{t,3} = \rho\ddot{u}_3 \,, \end{aligned} \tag{2.3}$$

where B_{ijkl} are the only non-zero components of the associated elasticity tensor, u_i, $i \in \{1,2,3\}$, are the infinitesimal displacement components, ρ is the material density and p_t is the incremental time-dependent part of the Lagrange multiplier $p = \bar{p} + p_t$, with $\bar{p}$ a static part associated with the primary deformation. This is essentially a measure of workless reaction stress brought into play by imposing incompressibility, usually interpreted as a pressure. Throughout this paper, unless otherwise stated, a comma subscript denotes differentiation with respect to x_1, x_2 or x_3 and a dot denotes time derivative. Under the assumption of incompressibility, the equations of motion (2.1)–(2.3) must be considered in conjunction with the linearised incompressibility condition

$$u_{1,1} + u_{2,2} + u_{3,3} = 0 \,. \tag{2.4}$$

For a detailed account of the background theory of incremental motions superimposed upon a pre-stressed elastic body the reader is referred to [12]. Detailed derivation of the governing equations for a pre-stressed incompressible elastic body is given in [13].

We seek the solutions of (2.1)–(2.4) in form of a travelling harmonic wave

$$(u_1, u_2, u_3, p_t) = (U_1, U_2, U_3, kP)e^{kqx_2}e^{ik(x_1c_\theta + x_3s_\theta - vt)} \,, \tag{2.5}$$

in which k is the wave number, v the wave speed, $(c_\theta, 0, s_\theta) = (\cos\theta, 0, \sin\theta)$ is the in-plane projection of the wave normal and q is to be determined from the governing equations. The analogous plane strain problem has been previously analysed in [14]. We will therefore concentrate on the three-dimensional case and tacitly assume that $c_\theta \neq 0$ and $s_\theta \neq 0$. Substituting the solution (2.5) into the system of equations (2.1)–(2.4), we may derive the equation

$$\gamma_{21}\gamma_{23}q^6 + ((\gamma_{21}+\gamma_{23})\bar{v}^2 - c_1)q^4 + (\bar{v}^4 - c_2\bar{v}^2 + c_3)q^2 - (\bar{v}^2 - c_4)(\bar{v}^2 - c_5) = 0 \,, \tag{2.6}$$

this being the criterion for existence of non-trivial solutions of the form (2.5) (this is true strictly for an unbounded media, the condition (2.6) being in general only necessary for waves propagating in a layer). The parameter $\bar{v} \equiv \sqrt{\rho}v$ will be referred to as the scaled wave speed, and

$$c_1 = (2\beta_{23}\gamma_{21} + \gamma_{23}\gamma_{31})s_\theta^2 + (2\beta_{12}\gamma_{23} + \gamma_{21}\gamma_{13})c_\theta^2 \,,$$

$$c_2 = (2\beta_{23} + \gamma_{21} + \gamma_{31})s_\theta^2 + (2\beta_{12} + \gamma_{23} + \gamma_{13})c_\theta^2\,,$$

$$\begin{aligned} c_3 = {} & (4\beta_{12}\beta_{23} + \gamma_{21}\gamma_{12} + \gamma_{23}\gamma_{32} + \gamma_{13}\gamma_{31} - \mu_{13}^2)s_\theta^2 c_\theta^2 \\ & + (2\beta_{23}\gamma_{31} + \gamma_{21}\gamma_{32})s_\theta^4 + (2\beta_{12}\gamma_{13} + \gamma_{23}\gamma_{12})c_\theta^4\,, \end{aligned}$$

$$c_4 = \gamma_{32}s_\theta^2 + \gamma_{12}c_\theta^2\,, \qquad c_5 = \gamma_{31}s_\theta^4 + 2\beta_{13}s_\theta^2 c_\theta^2 + \gamma_{13}c_\theta^4\,,$$

in which the material parameters γ_{ij}, β_{ij}, μ_{ij} are defined through the components of the elasticity tensor as follows:

$$\begin{gathered} \gamma_{ij} = B_{ijij}\,, \qquad b_{ij} = B_{iiii} - B_{iijj} - B_{ijji}\,, \qquad 2\beta_{ij} = b_{ij} + b_{ji}\,, \qquad i \neq j\,, \\ \mu_{ij} = \beta_{ij} - \beta_{ik} - \beta_{jk}\,, \quad i < j\,, \quad k \notin \{i,j\}\,, \qquad i,j,k \in \{1,2,3\}\,. \end{gathered}$$

Let q_1^2, q_2^2, q_3^2 denote three distinct non-zero roots of the equation (2.6). Then any solution for u_1, u_2, u_3 or p_t can be represented as a superposition of six linearly independent functions $\exp(kq_ix_2)$ and $\exp(-kq_ix_2)$, $i \in \{1,2,3\}$ (hereafter, we assume that each q_i has positive real part). In this paper we restrict attention to the case for which u_2 is the even function of the normal coordinate x_2. For this type of motion, usually referred to as flexural or anti-symmetric motion, solutions for u_1, u_2, u_3 or p_t can be represented as superpositions of only three linearly independent functions. The coefficients of these superpositions may all be expressed in terms of 3 disposable constants $U_2^{(m)}$, $m \in \{1,2,3\}$, as follows

$$\begin{aligned} u_1 &= \sum_{m=1}^{3} \frac{iq_m\mathcal{U}_1(q_m,\bar v)c_\theta}{\mathcal{V}(q_m,\bar v)} S_m(x_2)U_2^{(m)}\,, \quad & u_3 &= \sum_{m=1}^{3} \frac{iq_m\mathcal{U}_3(q_m,\bar v)s_\theta}{\mathcal{V}(q_m,\bar v)} S_m(x_2)U_2^{(m)}\,, \\ u_2 &= \sum_{m=1}^{3} C_m(x_2)U_2^{(m)}\,, & p_t &= \sum_{m=1}^{3} \frac{q_m\mathcal{P}(q_m,\rho v^2)}{\mathcal{V}(q_m,\bar v)} S_m(x_2)U_2^{(m)}\,, \end{aligned} \tag{2.7}$$

where $S_m(x_2) = \sinh(kq_mx_2)$, $C_m(x_2) = \cosh(kq_mx_2)$ and

$$\begin{aligned} \mathcal{U}_1(q_m,\bar v) &= \gamma_{23}q_m^2 + \mu_{12}s_\theta^2 - \gamma_{13}c_\theta^2 + \bar v^2\,, \\ \mathcal{U}_3(q_m,\bar v) &= \gamma_{21}q_m^2 - \gamma_{31}s_\theta^2 + \mu_{23}c_\theta^2 + \bar v^2\,, \\ \mathcal{P}(q_m,\bar v) &= \mathcal{U}_1(q_m,\bar v)\mathcal{U}_3(q_m,\bar v) + (b_{31} - b_{32})\mathcal{U}_1(q_m,\bar v)c_\theta^2 + (b_{13} - b_{12})\mathcal{U}_3(q_m,\bar v)s_\theta^2\,, \\ \mathcal{V}(q_m,\bar v) &= (\gamma_{21}s_\theta^2 + \gamma_{23}c_\theta^2)q_m^2 + \bar v^2 - c_5\,. \end{aligned}$$

The above representation of $\mathcal{P}(q_m,\bar v)$ has been derived with help of the equality

$$q_m^2\mathcal{U}_1(q_m,\bar v)\mathcal{U}_3(q_m,\bar v) = -(\mu_{13}q^2 + \gamma_{32}s_\theta^2 + \gamma_{12}c_\theta^2 - \bar v^2)\mathcal{V}(q_m,\bar v)\,,$$

which is a direct consequence of the equation (2.6).

3. The dispersion relation

The coordinate system specified previously is such that the layer surfaces are defined by the outward unit normals $\boldsymbol{n}_u = (0,1,0)$ and $\boldsymbol{n}_l = (0,-1,0)$ for the upper and lower surface, respectively. In order to formulate zero incremental surface traction boundary conditions, an appropriate measure of the surface traction is chosen in the following component form

$$\tau_1 = B_{2121} u_{1,2} + (B_{2112} + \bar{p}) u_{2,1} \,, \tag{3.1}$$
$$\tau_2 = B_{2211} u_{1,1} + (B_{2222} + \bar{p}) u_{2,2} + B_{2233} u_{3,3} - p_t \,, \tag{3.2}$$
$$\tau_3 = (B_{2332} + \bar{p}) u_{2,3} + B_{2323} u_{3,2} \,, \tag{3.3}$$

see [11]. In the subsequent analysis we eliminate $\bar{p}$ in favour of the normal Cauchy stress component σ_2. Since for the present case the coordinate axes are coincident with the principal axes of static pre-stress, $\bar{p}$ and σ_2 are related through

$$\bar{p} = \gamma_{21} - B_{1221} - \sigma_2 = \gamma_{23} - B_{2332} - \sigma_2 \,.$$

Inserting the displacement and pressure representations (2.7) into (3.1)–(3.3) and imposing zero surface traction boundary conditions, a homogeneous system of six linear equations is derived. For flexural motion three of these are satisfied identically, the remaining three may be written as

$$\begin{aligned}
\sum_{m=1}^{3} \frac{\mathcal{T}_1(q_m,\bar{v})}{\mathcal{V}(q_m,\bar{v})} C_m(h) U_2^{(m)} &= 0 \,, \\
\sum_{m=1}^{3} \frac{q_m \mathcal{T}_2(q_m,\bar{v})}{\mathcal{V}(q_m,\bar{v})} S_m(h) U_2^{(m)} &= 0 \,, \\
\sum_{m=1}^{3} \frac{\mathcal{T}_3(q_m,\bar{v})}{\mathcal{V}(q_m,\bar{v})} C_m(h) U_2^{(m)} &= 0 \,,
\end{aligned} \tag{3.4}$$

where h denotes the half-thickness of the layer and

$$\begin{aligned}
&\mathcal{T}_1(q_m,\bar{v}) = \gamma_{21} \mathcal{U}_1(q_m,\bar{v}) q_m^2 + g_1 \mathcal{V}(q_m,\bar{v}) \,, \\
&\mathcal{T}_2(q_m,\bar{v}) = (g_1 - \mu_{13}) \mathcal{U}_1(q_m,\bar{v}) c_\theta^2 + (g_3 - \mu_{13}) \mathcal{U}_3(q_m,\bar{v}) s_\theta^2 - \mathcal{U}_1(q_m,\bar{v}) \mathcal{U}_3(q_m,\bar{v}) \,, \\
&\mathcal{T}_3(q_m,\bar{v}) = \gamma_{23} \mathcal{U}_3(q_m,\bar{v}) q_m^2 + g_3 \mathcal{V}(q_m,\bar{v}) \,,
\end{aligned}$$

in which $g_i = \gamma_{2i} - \sigma_2, G_i = 2\gamma_{2i} - \sigma_2, i \in \{1,3\}$.

The homogeneous system of three linear equations (3.4) possesses a non-trivial solution provided its determinant is equal to zero, so we require

$$\begin{vmatrix}
\mathcal{T}_1(q_1,\bar{v}) C_1(h) & \mathcal{T}_1(q_2,\bar{v}) C_2(h) & \mathcal{T}_1(q_3,\bar{v}) C_3(h) \\
q_1 \mathcal{T}_2(q_1,\bar{v}) S_1(h) & q_2 \mathcal{T}_2(q_2,\bar{v}) S_2(h) & q_3 \mathcal{T}_2(q_3,\bar{v}) S_3(h) \\
\mathcal{T}_3(q_1,\bar{v}) C_1(h) & \mathcal{T}_3(q_2,\bar{v}) C_2(h) & \mathcal{T}_3(q_3,\bar{v}) C_3(h)
\end{vmatrix} = 0 \,. \tag{3.5}$$

Note, that several non-dispersive factors of equation (3.5) have been omitted. Evaluating the determinant, and introducing a new function $\mathcal{H}(q_i, q_j, \bar{v})$, we obtain the dispersion relation

$$\begin{aligned}
(q_2^2 - q_3^2) \mathcal{T}_2(q_1,\bar{v}) \mathcal{H}(q_2,q_3,\bar{v}) q_1 T_1(h) - (q_1^2 - q_3^2) \mathcal{T}_2(q_2,\bar{v}) \mathcal{H}(q_1,q_3,\bar{v}) q_2 T_2(h) \\
+ (q_1^2 - q_2^2) \mathcal{T}_2(q_3,\bar{v}) \mathcal{H}(q_1,q_2,\bar{v}) q_3 T_3(h) = 0 \,,
\end{aligned} \tag{3.6}$$

which was seemingly first derived, in slightly different notation, in [11]. In (3.6) we have denoted $T_m(h) = \tanh(kq_m h)$, $m \in \{1,2,3\}$ and

$$\begin{aligned}
\mathcal{H}(q_i, q_j, \bar{v}) &= \gamma_{21}\gamma_{23}\mathcal{H}_1(\bar{v})q_i^2 q_j^2 + (\bar{v}^2 - c_5)(\gamma_{21}\gamma_{23}(\gamma_{23}-\gamma_{21})(q_i^2+q_j^2) - \mathcal{H}_2(\bar{v}))\,,\\
\mathcal{H}_1(\bar{v}) &= (\gamma_{23}-\gamma_{21})\bar{v}^2 - (\gamma_{21}(\mu_{12}-\gamma_{23}+\gamma_{21}) + \gamma_{23}\gamma_{31})s_\theta^2\\
&\quad + (\gamma_{23}(\mu_{23}+\gamma_{23}-\gamma_{21}) + \gamma_{21}\gamma_{13})c_\theta^2\,,\\
\mathcal{H}_2(\bar{v}) &= \gamma_{23}g_1(\mu_{23}c_\theta^2 - \gamma_{31}s_\theta^2 + \bar{v}^2) - \gamma_{21}g_3(\mu_{12}s_\theta^2 - \gamma_{13}c_\theta^2 + \bar{v}^2)\,,
\end{aligned}$$

with $\mathcal{T}_2(q_i, \bar{v})$ given immediately after the system of equations (3.4).

3.1. Long wave high frequency approximations

In order to establish a consistent lower-dimensional theory it is necessary to have deep insight into asymptotic structure of the associated solutions. We therefore begin our investigation by deriving appropriate approximations of the dispersion relation to study dynamic response of three-dimensional theory and, subsequently, to verify the consistency of lower-dimensional model. It is well-known that as the scaled wave number $kh \to 0$, $\bar{v} \to \infty$ for all harmonics of the dispersion relation (3.6), see for example [13], with the corresponding limiting (cut-off) frequencies finite and non-zero. Following [2], we term this type of motion as *long wave high frequency.* Thus, to find appropriate asymptotic approximations of the dispersion relation, we assume that $\bar{v} \to \infty$ as $kh \to 0$. Analysis of the coefficients of the cubic (in q^2) equation (2.6) suggests that two of its roots (q_1^2 and q_3^2) are $O(\bar{v}^2)$, whereas the third root q_2^2 is $O(1)$. Specifying these roots as power series in $\bar{v}^2$, and substituting them into the equation (2.6), the following approximations may be derived

$$\begin{aligned}
q_1^2 &= -\frac{\bar{v}^2}{\gamma_{21}} + \frac{\mathcal{Q}_{1s}^{(0)}s_\theta^2 + \mathcal{Q}_{1c}^{(0)}c_\theta^2}{\gamma_{21}} - \left(\mathcal{Q}_{1s}^{(-2)}s_\theta^2 + \mathcal{Q}_{1c}^{(-2)}c_\theta^2\right)\frac{c_\theta^2}{\bar{v}^2} + O(\bar{v}^{-4})\,,\\
q_2^2 &= 1 + \frac{\mathcal{Q}_{2s}^{(-2)}s_\theta^2 + \mathcal{Q}_{2c}^{(-2)}c_\theta^2 - c_5}{\bar{v}^2} + O(\bar{v}^{-4})\,,\\
q_3^2 &= -\frac{\bar{v}^2}{\gamma_{23}} + \frac{\mathcal{Q}_{3s}^{(0)}s_\theta^2 + \mathcal{Q}_{3c}^{(0)}c_\theta^2}{\gamma_{23}} - \left(\mathcal{Q}_{3s}^{(-2)}s_\theta^2 + \mathcal{Q}_{3c}^{(-2)}c_\theta^2\right)\frac{s_\theta^2}{\bar{v}^2} + O(\bar{v}^{-4})\,,
\end{aligned}\tag{3.7}$$

in which

$$\begin{aligned}
&\mathcal{Q}_{1c}^{(0)} = 2\beta_{12} - \gamma_{21}\,, \quad \mathcal{Q}_{1s}^{(0)} = \gamma_{31}\,, \quad \mathcal{Q}_{3c}^{(0)} = \gamma_{13}\,, \quad \mathcal{Q}_{3s}^{(0)} = 2\beta_{23} - \gamma_{23}\,,\\
&\mathcal{Q}_{1s}^{(-2)} = \gamma_{31} - \gamma_{32} + \frac{(\mu_{13}+\gamma_{21})^2}{\gamma_{23}-\gamma_{21}}\,, \quad \mathcal{Q}_{3c}^{(-2)} = \gamma_{13} - \gamma_{12} - \frac{(\mu_{13}+\gamma_{23})^2}{\gamma_{23}-\gamma_{21}}\,,\\
&\mathcal{Q}_{1c}^{(-2)} = 2\beta_{12} - \gamma_{21} - \gamma_{12}\,, \quad \mathcal{Q}_{3s}^{(-2)} = 2\beta_{23} - \gamma_{23} - \gamma_{32}\,,\\
&\mathcal{Q}_{2c}^{(-2)} = \mathcal{Q}_{1c}^{(0)} + \mathcal{Q}_{3c}^{(0)} - \gamma_{12}\,, \quad \mathcal{Q}_{2s}^{(-2)} = \mathcal{Q}_{1s}^{(0)} + \mathcal{Q}_{3s}^{(0)} - \gamma_{32}\,.
\end{aligned}$$

Corresponding expansions for q_1, q_2 and q_3 are given by

$$
\begin{aligned}
q_1 &= \frac{i\bar{v}}{\sqrt{\gamma_{21}}} - \frac{i\left(\mathcal{Q}_{1s}^{(0)} s_\theta^2 + \mathcal{Q}_{1c}^{(0)} c_\theta^2\right)}{2\sqrt{\gamma_{21}}\,\bar{v}} + O(\bar{v}^{-3})\,, \\
q_2 &= 1 + \frac{\mathcal{Q}_{2s}^{(-2)} s_\theta^2 + \mathcal{Q}_{2c}^{(-2)} c_\theta^2 - c_5}{2\bar{v}^2} + O(\bar{v}^{-4})\,, \\
q_3 &= \frac{i\bar{v}}{\sqrt{\gamma_{23}}} - \frac{i\left(\mathcal{Q}_{3s}^{(0)} s_\theta^2 + \mathcal{Q}_{3c}^{(0)} c_\theta^2\right)}{2\sqrt{\gamma_{23}}\,\bar{v}} + O(\bar{v}^{-3})\,.
\end{aligned} \tag{3.8}
$$

The equation (2.6) may be treated as quadratic in $\bar{v}^2$. For long wave high frequency motion the wave speed must tend to infinity as $kh \to 0$ and since $\bar{v}^2$ is $O(q_1^2)$ (or $O(q_3^2)$), only the wave speeds associated with q_1^2 and q_3^2 are of interest, appropriate expansions given by

$$
\bar{v}_1^2 = -\gamma_{21} q_1^2 + \mathcal{Q}_{1s}^{(0)} s_\theta^2 + \mathcal{Q}_{1c}^{(0)} c_\theta^2 + \left(\mathcal{Q}_{1s}^{(-2)} s_\theta^2 + \mathcal{Q}_{1c}^{(-2)} c_\theta^2\right) \frac{c_\theta^2}{q_1^2} + O(q_1^{-4})\,, \tag{3.9}
$$

$$
\bar{v}_3^2 = -\gamma_{23} q_3^2 + \mathcal{Q}_{3s}^{(0)} s_\theta^2 + \mathcal{Q}_{3c}^{(0)} c_\theta^2 + \left(\mathcal{Q}_{3s}^{(-2)} s_\theta^2 + \mathcal{Q}_{3c}^{(-2)} c_\theta^2\right) \frac{s_\theta^2}{q_3^2} + O(q_3^{-4})\,. \tag{3.10}
$$

The two wave speeds associated with q_2^2, are of $O(1)$ and are therefore not relevant for high frequency motion. Lack of a third large wave speed associated with q_2 is a direct consequence of imposing the incompressibility constraint, which disabled propagation of any longitudinal wave, see [15], and associated thickness stretch resonance. In an unconstrained material there is a third possible large speed associated with q_2, see for example [16] in respect of a compressible transversely isotropic elastic plate.

To begin asymptotic analysis of the dispersion relation (3.6) we first introduce a small parameter η, the ratio of plate half-thickness h and typical wave length l, hence $\eta = h/l = kh$. Recalling that $\bar{v} \to \infty$ as $\eta \to 0$, we insert expansions (3.7) and (3.8) into the dispersion relation (3.6) to obtain

$$
i\left(A_1^{(2)} \bar{v}^2 + A_1^{(0)}\right) T_1(h) + \bar{v}^3 \left(A_2^{(5)} \bar{v}^2 + A_2^{(3)}\right) T_2(h) + i\left(A_3^{(2)} \bar{v}^2 + A_3^{(0)}\right) T_3(h) \sim 0\,, \tag{3.11}
$$

in which the leading order coefficients $A_1^{(2)}$, $A_2^{(5)}$ and $A_3^{(2)}$ are given by

$$
A_1^{(2)} = \frac{G_1^2(\gamma_{23} - \gamma_{21}) c_\theta^2}{\sqrt{\gamma_{21}}}\,, \qquad A_2^{(5)} = -(\gamma_{23} - \gamma_{21})\,, \qquad A_3^{(2)} = \frac{G_3^2(\gamma_{23} - \gamma_{21}) s_\theta^2}{\sqrt{\gamma_{23}}}\,,
$$

and the second order coefficients of (3.11) $A_1^{(0)}$, $A_2^{(3)}$ and $A_3^{(0)}$ have form

$$
\begin{aligned}
A_1^{(0)} = \frac{G_1 c_\theta^2}{2\sqrt{\gamma_{21}}} \Big\{ &\Big(G_1(\gamma_{21}(2\gamma_{21} - 6\mu_{12} + \gamma_{31}) - \gamma_{23}(4\beta_{23} - 4\gamma_{23} + 4\gamma_{21} + 5\gamma_{31})) \\
&- 4\gamma_{21}(\gamma_{23} - \gamma_{21})(\gamma_{32} - \gamma_{31} - \gamma_{21} - \mu_{13})\Big) s_\theta^2 - 2G_1(\gamma_{23} - \gamma_{21}) c_5 \\
&+ \Big(G_1(2\gamma_{23}(\mu_{23} + \gamma_{23} - 2(\beta_{12} - \gamma_{13})) - (\gamma_{23} - \gamma_{21})(2\beta_{12} + 6\gamma_{13} - \gamma_{21})) \\
&+ 4\gamma_{21}(\gamma_{23} - \gamma_{21}) \mathcal{Q}_{1c}^{(-2)}\Big) c_\theta^2 \Big\}\,,
\end{aligned}
$$

$$
\begin{aligned}
A_2^{(3)} &= \frac{1}{2}\Big\{(\gamma_{23}-\gamma_{21})(c_4+c_5-4\sigma_2) \\
&+ (2\gamma_{21}(\mu_{12}+\gamma_{21}-2(\beta_{23}-\gamma_{31})) + (\gamma_{23}-\gamma_{21})(6\beta_{23}+\gamma_{23}+7\gamma_{31}))s_\theta^2 \\
&- (2\gamma_{23}(\mu_{23}+\gamma_{23}-2(\beta_{12}-\gamma_{13})) - (\gamma_{23}-\gamma_{21})(6\beta_{12}+\gamma_{21}+7\gamma_{13}))c_\theta^2\Big\}\,,
\end{aligned}
$$

$$
\begin{aligned}
A_3^{(0)} &= \frac{G_3 s_\theta^2}{2\sqrt{\gamma_{23}}}\Big\{\Big(G_3(\gamma_{23}(6\mu_{23}-\gamma_{13}-2\gamma_{23}) + \gamma_{21}(4\gamma_{23}-4\gamma_{21}+4\beta_{12}+5\gamma_{13})) \\
&- 4\gamma_{23}(\gamma_{23}-\gamma_{21})(\gamma_{12}-\mu_{13}-\gamma_{13}-\gamma_{23})\Big)c_\theta^2 - 2G_3(\gamma_{23}-\gamma_{21})c_5 \\
&- \Big(G_3(2\gamma_{21}(\mu_{12}+\gamma_{21}-2(\beta_{23}-\gamma_{31})) + (\gamma_{23}-\gamma_{21})(2\beta_{23}+6\gamma_{31}-\gamma_{23})) \\
&- 4\gamma_{23}(\gamma_{23}-\gamma_{21})\mathcal{Q}_{3s}^{(-2)}\Big)s_\theta^2\Big\}\,,
\end{aligned}
$$

with parameters g_m and G_m, $m \in \{1,2,3\}$, defined after relations (3.4). We presume that all of $A_1^{(2)}$, $A_1^{(0)}$, $A_2^{(5)}$, $A_2^{(3)}$, $A_3^{(2)}$ and $A_3^{(0)}$ are generally of $O(1)$. Since q_2 is $O(1)$, see (3.8), $T_2(h) = O(\bar{v}^{-1})$ and consequently the asymptotic equality (3.11) implies $T_1(h) = O(\bar{v}^2)$ or, alternatively, $T_3(h) = O(\bar{v}^2)$.

Suppose $T_1(h) = O(\bar{v}^2)$, hence the argument of this hyperbolic tangent must be imaginary and to the leading order equal to $i(\frac{1}{2}+n)\pi$. Thus we expand the argument in a power series in small η as follows:

$$
kq_1 h = i\left(\left(\frac{1}{2}+n\right)\pi + \phi_1^{f(2)}\eta^2 + \phi_1^{f(4)}\eta^4 + O(\eta^6)\right)\,, \tag{3.12}
$$

in which the $O(1)$ parameters $\phi_1^{f(2)}$ and $\phi_1^{f(4)}$ are to be determined. The associated expansion for $T_1(h)$ is given by

$$
T_1(h) = -\frac{i}{\phi_1^{f(2)}\eta^2} + \frac{i\phi_1^{f(4)}}{(\phi_1^{f(2)})^2} + O(\eta^2)\,. \tag{3.13}
$$

At this point it is convenient to introduce the parameter $\Lambda_1^f = \sqrt{\gamma_{21}}(\frac{1}{2}+n)\pi$, $n = 1,2,3\ldots$, the physical interpretation of which is deferred until later. We may now utilise (3.8) and (3.9) to obtain

$$
q_1 = \frac{i\Lambda_1^f}{\sqrt{\gamma_{21}}\eta} + i\phi_1^{f(2)}\eta + O(\eta^3)\,, \tag{3.14}
$$

$$
\bar{v} = \frac{\Lambda_1^f}{\eta} + \left(\sqrt{\gamma_{21}}\phi_1^{f(2)} + \frac{\mathcal{Q}_{1s}^{(0)}s_\theta^2 + \mathcal{Q}_{1c}^{(0)}c_\theta^2}{2\Lambda_1^f}\right)\eta + O(\eta^3)\,. \tag{3.15}
$$

These may now be inserted into (3.8), which allows us to approximate corresponding hyperbolic tangents, thus

$$
q_2 = 1 + \frac{\mathcal{Q}_{2s}^{(-2)}s_\theta^2 + \mathcal{Q}_{2c}^{(-2)}c_\theta^2 - c_5}{2(\Lambda_1^f)^2}\eta^2 + O(\eta^4)\,, \tag{3.16}
$$

$$T_2(h) = \eta + \left(\frac{\mathcal{Q}_{2s}^{(-2)} s_\theta^2 + \mathcal{Q}_{2c}^{(-2)} c_\theta^2 - c_5}{2(\Lambda_1^f)^2} - \frac{1}{3} \right) \eta^3 + O(\eta^5)\,, \tag{3.17}$$

$$q_3 = \frac{i\Lambda_1^f}{\sqrt{\gamma_{23}}\eta} + O(\eta)\,, \qquad T_3(h) = i \tan \left(\frac{\Lambda_1^f}{\sqrt{\gamma_{23}}} \right) + O(\eta^2)\,. \tag{3.18}$$

Substituting expansions (3.12)–(3.18) back into the approximation of the dispersion relation (3.11), we obtain expressions for $\phi_1^{f(2)}$ and $\phi_1^{f(4)}$ in the form

$$\phi_1^{f(2)} = \frac{G_1^2 c_\theta^2}{\sqrt{\gamma_{21}}(\Lambda_1^f)^3}\,, \tag{3.19}$$

$$\begin{aligned} \phi_1^{f(4)} = \phi_1^{f(2)} \Bigg\{ & 5 \frac{\sqrt{\gamma_{21}}(\Lambda_1^f)^2 A_2^{(5)}}{A_1^{(2)}} (\phi_1^{f(2)})^2 + \frac{A_1^{(2)} \left(\mathcal{Q}_{1s}^{(0)} s_\theta^2 + \mathcal{Q}_{1c}^{(0)} c_\theta^2 \right) + A_1^{(0)}}{(\Lambda_1^f)^2 A_1^{(2)}} \\ & + \Lambda_1^f \left(\frac{1}{2} \left(\left(6\mathcal{Q}_{1s}^{(0)} + \mathcal{Q}_{3s}^{(0)} - \gamma_{32} \right) s_\theta^2 + \left(6\mathcal{Q}_{1c}^{(0)} + \mathcal{Q}_{3c}^{(0)} - \gamma_{12} \right) c_\theta^2 - c_5 \right) A_2^{(5)} \right. \\ & \left. + 2 \frac{\sqrt{\gamma_{21}} A_1^{(2)}}{(\Lambda_1^f)^2} - \frac{A_3^{(2)}}{\Lambda_1^f} \tan \left(\frac{\Lambda_1^f}{\sqrt{\gamma_{23}}} \right) - \frac{A_2^{(5)}}{3} (\Lambda_1^f)^2 + A_2^{(3)} \right) \frac{\phi_1^{f(2)}}{A_1^{(2)}} \Bigg\}\,. \end{aligned} \tag{3.20}$$

It is now possible to employ (3.9) to obtain the appropriate long wave high frequency approximation of the dispersion relation, in a form of scaled frequency $\bar{\omega} \equiv \bar{v}\eta$ as function of η for each n (note that Λ_1^f is a function of n, $n = 1, 2, 3 \dots$), given by

$$\bar{\omega}^2 = (\Lambda_1^f)^2 + \left(\mathcal{F}_{1c}^{(2)} c_\theta^2 + \mathcal{F}_{1s}^{(2)} s_\theta^2 \right) \eta^2 - \left(\mathcal{F}_{1c}^{(4)} c_\theta^2 + \mathcal{F}_{1s}^{(4)} s_\theta^2 \right) c_\theta^2 \eta^4 + O(\eta^6)\,, \tag{3.21}$$

$$\mathcal{F}_{1c}^{(2)} = \frac{2G_1^2}{(\Lambda_1^f)^2} + \mathcal{Q}_{1c}^{(0)}\,, \qquad \mathcal{F}_{1s}^{(2)} = \mathcal{Q}_{1s}^{(0)}\,,$$

$$\mathcal{F}_{1c}^{(4)} = \frac{G_1^2}{(\Lambda_1^f)^4} \left(\frac{5G_1^2}{(\Lambda_1^f)^2} - 4g_1 - \frac{2}{3} (\Lambda_1^f)^2 \right) - \frac{\mathcal{Q}_{1c}^{(-2)}}{(\Lambda_1^f)^4} \left(2\sigma_2 G_1 - \gamma_{21} (\Lambda_1^f)^2 \right)\,,$$

$$\begin{aligned} \mathcal{F}_{1s}^{(4)} = & \frac{2G_1^2 G_3^2}{\sqrt{\gamma_{23}}(\Lambda_1^f)^5} \tan \left(\frac{\Lambda_1^f}{\sqrt{\gamma_{23}}} \right) - \frac{\frac{2}{3} G_1^2 - \gamma_{21} \mathcal{Q}_{1s}^{(-2)}}{(\Lambda_1^f)^2} \\ & - \frac{2G_1}{(\Lambda_1^f)^4} \left(G_1 (2g_3 + \gamma_{23} - \gamma_{31} + \gamma_{32}) + 2\gamma_{21} D_1^f \right)\,, \end{aligned}$$

in which

$$D_1^f = \frac{(\gamma_{21} + \gamma_{23} - \sigma_2)(\mu_{13} + \gamma_{21})}{\gamma_{23} - \gamma_{21}} + \gamma_{31} - \gamma_{32}\,.$$

The analogous analysis, applied to the case $T_3(h) = O(\bar{v}^2)$, delivers another set of frequency approximations, which may be written as

$$\bar{\omega}^2 = (\Lambda_3^f)^2 + \left(\mathcal{F}_{3c}^{(2)} c_\theta^2 + \mathcal{F}_{3s}^{(2)} s_\theta^2 \right) \eta^2 - \left(\mathcal{F}_{3c}^{(4)} c_\theta^2 + \mathcal{F}_{3s}^{(4)} s_\theta^2 \right) s_\theta^2 \eta^4 + O(\eta^6)\,, \tag{3.22}$$

$$\mathcal{F}_{3c}^{(2)} = \mathcal{Q}_{3c}^{(0)}\,, \qquad \mathcal{F}_{3s}^{(2)} = \frac{2G_3^2}{(\Lambda_3^f)^2} + \mathcal{Q}_{3s}^{(0)}\,,$$

$$\mathcal{F}_{3s}^{(4)} = \frac{G_3^2}{(\Lambda_3^f)^4}\left(\frac{5G_3^2}{(\Lambda_3^f)^2} - 4g_3 - \frac{2}{3}(\Lambda_3^f)^2\right) - \frac{\mathcal{Q}_{3s}^{(-2)}}{(\Lambda_3^f)^4}\left(2\sigma_2 G_3 - \gamma_{23}(\Lambda_3^f)^2\right),$$

$$\mathcal{F}_{3c}^{(4)} = \frac{2G_1^2 G_3^2}{\sqrt{\gamma_{21}}(\Lambda_3^f)^5}\tan\left(\frac{\Lambda_3^f}{\sqrt{\gamma_{21}}}\right) - \frac{\frac{2}{3}G_3^2 - \gamma_{23}\mathcal{Q}_{3c}^{(-2)}}{(\Lambda_3^f)^2}$$
$$- \frac{2G_3}{(\Lambda_3^f)^4}\left(G_3(2g_1 + \gamma_{21} - \gamma_{13} + \gamma_{12}) - 2\gamma_{23}D_3^f\right),$$

where $\Lambda_3^f = \sqrt{\gamma_{23}}(\frac{1}{2} + n)\pi$, $n = 1, 2, 3\ldots$, and

$$D_3^f = \frac{(\gamma_{21} + \gamma_{23} - \sigma_2)(\mu_{13} + \gamma_{23})}{\gamma_{23} - \gamma_{21}} + \gamma_{12} - \gamma_{13}\,.$$

3.2. Relative orders of displacements

Comparison of the relative orders of the particle displacements not only gives us a clear physical picture of the structure of this type of motion, but also provides the basis for building a lower-dimensional asymptotically consistent model. To obtain the relative orders of displacement components and pressure increment for long wave high frequency motion we utilise the approximations (3.6). When the dispersion relation is satisfied, the system of boundary conditions (3.4) possesses non-trivial solutions, for which the coefficients $U_2^{(k)}$, $k \in \{1, 2, 3\}$, may be represented in terms of the single constant $U_2^{(0)}$ as follows

$$U_2^{(k)} = (-1)^k \frac{(q_i^2 - q_j^2)\mathcal{H}(q_i, q_j, \bar{v})\mathcal{V}(q_k, \bar{v})}{C_k(h)} U_2^{(0)}, \qquad i < j, \quad k \notin \{i, j\}, \quad i, j, k \in \{1, 2, 3\}. \tag{3.23}$$

We may use (2.7) to find displacements and pressure in terms of $U_2^{(0)}$. In order to compare their asymptotic orders we determine the orders of the functions occurring in (2.7) and (3.23). First note that it is the consequence of (3.8) that

$$S_2(x_2) = \eta\frac{x_2}{h} + O(\eta^3), \qquad C_2(x_2) = 1 + O(\eta^2), \tag{3.24}$$

in which we assume that x_2/h is $O(1)$. Additionally, our expansions for the first case of asymptotic balance of the dispersion relation ($T_1(h) = O(\bar{v}^2)$) also imply

$$S_1(h) = i(-1)^n \phi_1^{f(4)}\eta^4 + O(\eta^6), \quad S_m(x_2) = i\sin\left(\frac{\Lambda_1^f x_2}{\sqrt{\gamma_{2m}}\,h}\right) + O(\eta^2),$$
$$C_m(x_2) = i\cos\left(\frac{\Lambda_1^f x_2}{\sqrt{\gamma_{2m}}\,h}\right) + O(\eta^2), \qquad m \in \{1, 3\},$$

which together with the approximations (3.14)–(3.18) yields

$$u_1 \sim O(p_t), \qquad u_2 \sim \eta\, O(p_t), \qquad u_3 \sim \eta^2\, O(p_t). \tag{3.25}$$

Repeating the procedure for the second case ($T_3(h) = O(\bar{v}^2)$) one may obtain the following distribution of relative orders of displacements

$$u_1 \sim \eta^2 \, O(p_t) \,, \qquad u_2 \sim \eta \, O(p_t) \,, \qquad u_3 \sim O(p_t) \,. \tag{3.26}$$

3.3. Physical interpretation

The asymptotic expansions derived in previous sections show that in both cases ($T_1(h) = O(\bar{v}^2)$ or $T_3(h) = O(\bar{v}^2)$) to leading order the wave normal is given by a non-normalised vector of the form $(c_\theta, O(\eta^{-1}), s_\theta)$, see (3.14). The second component of this vector is large, so the leading order direction of wave propagation is normal to plate. The polarisation directions for each case are given by (non-normalised) vectors of the form $(O(1), O(\eta), O(\eta^2))$ and $(O(\eta^2), O(\eta), O(1))$ respectively, which is suggested by the relative orders of the displacements (3.25) and (3.26). As $kh \to 0$, to leading order waves travel along the normal direction and are polarised along one of the in-plane axes of primary deformation. This description is essentially that of two shear waves, concurring with the previously mentioned fact that only two shear waves propagate in any direction in an incompressible elastic solid, see [15], with the longitudinal motion prohibited by the incompressibility constraint. The parameters $\Lambda_m^f = \sqrt{\gamma_{2m}}(\frac{1}{2} + n)\pi$, $m \in \{1, 3\}$, $n = 1, 2, 3 \ldots$, are the leading orders of the scaled frequency expansions (3.21) and (3.22) respectively. They define two infinite families of so-called cut-off frequencies, frequency limits as $\eta \to 0$. These are in fact natural thickness shear resonance frequencies of an infinitesimally thin transverse fibre of the layer, which satisfy one of the eigen-value problems

$$\gamma_{2m} u_{m,22} + \omega^2 u_m = 0 \,, \qquad u_{m,2}|_{x_2 = \pm h} = 0 \,, \qquad m \in \{1, 3\} \,. \tag{3.27}$$

4. Asymptotically approximate equations

The information obtained in the previous sections may be used to build a lower-dimensional model for long wave high frequency motion. In order to set up the necessary perturbation scheme we first need to introduce appropriate scales of space and time. Recalling that l denotes a typical wavelength, and keeping in mind that $\eta = h/l$, we may choose the following spatial scalings

$$x_1 = l\xi_1 \,, \qquad x_2 = h\zeta = l\eta\zeta \,, \qquad x_3 = l\xi_3 \,, \tag{4.1}$$

where ξ_1, ζ and ξ_3 are new non-dimensional spatial variables. Let us now focus on the first family of the shear resonance frequencies ($\bar{\omega} = \Lambda_1^f$). As expansion (3.15) shows, a typical (long) wave propagates with scaled speed Λ_1^f/η and therefore travels the distance of one wave length in time $l\eta\sqrt{\rho}/\Lambda_1^f$. Hence, it is appropriate to rescale time as

$$t = l\eta\sqrt{\frac{\rho}{\gamma_{21}}}\tau \,. \tag{4.2}$$

According to the distribution of the relative orders (3.25), the displacement components and pressure increment must have the following asymptotic structure

$$
\begin{aligned}
u_m(x_1,x_2,x_3,t) &= l\eta^{m-1}u_m^*(\xi_1,\zeta,\xi_3,\tau)\,, \qquad m\in\{1,2,3\}\,,\\
p_t(x_1,x_2,x_3,t) &= \gamma_{21}p_t^*(\xi_1,\zeta,\xi_3,\tau)\,,
\end{aligned}
\tag{4.3}
$$

in which * denotes non-dimensional quantities of a same asymptotic order and γ_{21} is introduced purely for algebraic convenience.

The system of equations of motion (2.1)–(2.3) can now be recast in terms of the non-dimensional variables, yielding

$$
\begin{aligned}
&\gamma_{21}u^*_{1,\zeta\zeta} + (\Lambda_1^f)^2u_1^* - \{(\Lambda_1^f)^2u_1^* + \gamma_{21}u^*_{1,\tau\tau}\} + \eta^2\big(B_{1111}u^*_{1,\xi_1\xi_1} + \gamma_{31}u^*_{1,\xi_3\xi_3}\\
&\qquad + (B_{1122}+B_{1221})u^*_{2,\xi_1\zeta} - \gamma_{21}p^*_{t,\xi_1}\big) + \eta^4(B_{1133}+B_{1331})u^*_{3,\xi_1\xi_3} = 0\,, \qquad (4.4)\\
&B_{2222}u^*_{2,\zeta\zeta} + (\Lambda_1^f)^2u_2^* - \{(\Lambda_1^f)^2u_2^* + \gamma_{21}u^*_{2,\tau\tau}\} + (B_{1122}+B_{1221})u^*_{1,\xi_1\zeta}\\
&\qquad - \gamma_{21}p^*_{t,\zeta} + \eta^2\big(\gamma_{12}u^*_{2,\xi_1\xi_1} + \gamma_{32}u^*_{2,\xi_3\xi_3} + (B_{2233}+B_{2332})u^*_{3,\xi_3\zeta}\big) = 0\,, \qquad (4.5)\\
&\gamma_{23}u^*_{3,\zeta\zeta} + (\Lambda_1^f)^2u_3^* - \{(\Lambda_1^f)^2u_3^* + \gamma_{21}u^*_{3,\tau\tau}\} + (B_{1133}+B_{1331})u^*_{1,\xi_1\xi_3} - \gamma_{21}p^*_{t,\xi_3}\\
&\qquad + (B_{2233}+B_{2332})u^*_{2,\xi_3\zeta} + \eta^2\big(\gamma_{13}u^*_{3,\xi_1\xi_1} + B_{3333}u^*_{3,\xi_3\xi_3}\big) = 0\,, \qquad (4.6)
\end{aligned}
$$

here comma subscripts denote differentiation with respect to the indicated scaled (space or time) variable. These equations must be solved in conjunction with the appropriately rescaled incompressibility condition

$$
u^*_{1,\xi_1} + u^*_{2,\zeta} + \eta^2u^*_{3,\xi_3} = 0\,,
\tag{4.7}
$$

and solved subject to the zero surface traction boundary conditions

$$
\begin{aligned}
\gamma_{21}u^*_{1,\zeta} + \eta^2(B_{1221}+\bar p)u^*_{2,\xi_1} &= 0 \qquad \text{at}\ \ \zeta=\pm1\,, \qquad (4.8)\\
B_{1122}u^*_{1,\xi_1} + (B_{2222}+\bar p)u^*_{2,\zeta} - \gamma_{21}p_t^* + \eta^2B_{2233}u^*_{3,\xi_3} &= 0 \qquad \text{at}\ \ \zeta=\pm1\,, \qquad (4.9)\\
\gamma_{23}u^*_{3,\zeta} + (B_{2332}+\bar p)u^*_{2,\xi_3} &= 0 \qquad \text{at}\ \ \zeta=\pm1\,. \qquad (4.10)
\end{aligned}
$$

A glance at the boundary value problem (4.4)–(4.10) exposes, that in order to ensure response compatible with (3.27), and in view of the approximation (3.21), we require

$$
\gamma_{21}u^*_{m,\tau\tau} + (\Lambda_1^f)^2u_m^* \sim \eta^2u_m^*\,, \qquad m\in\{1,2,3\}\,,
\tag{4.11}
$$

which can also be verified by direct substitution of the travelling wave solution (2.5). Note, that one of the implications of imposing (4.11) is that all values in braces in system (4.4)–(4.6) must be considered as $O(\eta^2)$. We now seek the solutions in a form of the power series expansions

$$
(u_1^*,u_2^*,u_3^*,p_t^*) = \sum_{n=0}^{m}\eta^{2n}\left(u_1^{*(2n)},u_2^{*(2n)},u_3^{*(2n)},p_t^{*(2n)}\right) + O(\eta^{2m+2})\,.
\tag{4.12}
$$

It must be remarked that although in general the remainder estimate will be of order indicated in (4.12), for certain combinations of material and pre-stress parameters it is possible that order of this correction term is modified. In practical applications care should be taken to ensure that all of the requirements inherent in the model are satisfied or to adjust the model accordingly. Not withstanding these comments, the situation indicated in solutions (4.12) is the most likely to occur. Inserting these solutions into the equations (4.4)–(4.10) we obtain hierarchical systems of essentially ordinary differential equations at various orders.

4.1. Leading order problem

The leading order equations of motion

$$\gamma_{21} u^{*(0)}_{1,\zeta\zeta} + (\Lambda^f_1)^2 u^{*(0)}_1 = 0\,, \tag{4.13}$$

$$B_{2222} u^{*(0)}_{2,\zeta\zeta} + (\Lambda^f_1)^2 u^{*(0)}_2 + (B_{1122} + B_{1221}) u^{*(0)}_{1,\xi_1\zeta} - \gamma_{21} p^{*(0)}_{t,\zeta} = 0\,, \tag{4.14}$$

$$\begin{aligned}\gamma_{23} u^{*(0)}_{3,\zeta\zeta} + (\Lambda^f_1)^2 u^{*(0)}_3 + (B_{1133} + B_{1331}) u^{*(0)}_{1,\xi_1\xi_3}&\\ + (B_{2233} + B_{2332}) u^{*(0)}_{2,\xi_3\zeta} - \gamma_{21} p^{*(0)}_{t,\xi_3} &= 0\,, \end{aligned}\tag{4.15}$$

must be solved in conjunction with the leading order incompressibility condition

$$u^{*(0)}_{1,\xi_1} + u^{*(0)}_{2,\zeta} = 0\,, \tag{4.16}$$

subject to the leading order boundary conditions

$$\gamma_{21} u^{*(0)}_{1,\zeta} = 0 \qquad \text{at} \ \ \zeta = \pm 1\,, \tag{4.17}$$

$$(B_{2222} + \bar{p}\,) u^{*(0)}_{2,\zeta} + B_{1122} u^{*(0)}_{1,\xi_1} - \gamma_{21} p^{*(0)}_t = 0 \qquad \text{at} \ \ \zeta = \pm 1\,, \tag{4.18}$$

$$\gamma_{23} u^{*(0)}_{3,\zeta} + (B_{2332} + \bar{p}\,) u^{*(0)}_{2,\xi_3} = 0 \qquad \text{at} \ \ \zeta = \pm 1\,. \tag{4.19}$$

The solution of the boundary value problem (4.13), (4.17) is given by

$$u^{*(0)}_1 = u^{*(0,0)}_1 \sin\left(\frac{\Lambda^f_1 \zeta}{\sqrt{\gamma_{21}}}\right)\,, \tag{4.20}$$

in which the function with double superscript does not depend on ζ. Hereafter, double superscripts will denote functions independent of ζ, with first superscript referring to the order of the approximation and second denoting the power of any possible ζ^m multiplier. Note, that our choice of scaling ensures that the displacement components and pressure vary appropriately for flexural motion, i.e. as an anti-symmetric (u^*_1, u^*_3 and p^*_t) or a symmetric (u^*_2) function of the normal coordinate ζ. Thus, for the sake of brevity, we will always omit terms of the solutions which do not comply to this requirement.

Substituting the solution (4.20) into the incompressibility condition (4.16) we may establish the form of the expression for $u^{*(0)}_2$

$$u^{*(0)}_2 = u^{*(0,0)}_2 \cos\left(\frac{\Lambda^f_1 \zeta}{\sqrt{\gamma_{21}}}\right) + U^{*(0,0)}_2\,, \qquad u^{*(0,0)}_2 = \frac{\gamma_{21}}{\Lambda^f_1} u^{*(0,0)}_{1,\xi_1}\,. \tag{4.21}$$

Solutions (4.20) and (4.21) suggest, that the solution of the second equation of motion (4.14) should be sought in the following form

$$\begin{gathered}\gamma_{21} p^{*(0)}_t = p^{*(0,0)}_t \sin\left(\frac{\Lambda^f_1 \zeta}{\sqrt{\gamma_{21}}}\right) + P^{*(0,1)}_t \zeta\,, \\ p^{*(0,0)}_t = (\gamma_{21} - b_{21}) u^{*(0,0)}_{1,\xi_1}\,, \qquad P^{*(0,1)}_t = (\Lambda^f_1)^2 U^{*(0,0)}_2\,,\end{gathered} \tag{4.22}$$

whereas the boundary condition (4.18) yields

$$U_2^{*(0,0)} = -\frac{G_1}{(\Lambda_1^f)^2} u_{1,\xi_1}^{*(0,0)} \sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}}\right).$$

The leading order problem for u_3^* is given by (4.15), (4.19). The result of substitution of the previously established solutions (4.20), (4.21) and (4.22) into the equation (4.15) indicates that $u_3^{*(0)}$ may be represented as

$$u_3^{*(0)} = u_3^{*(0,0)} \sin\left(\frac{\Lambda_1^f \zeta}{\sqrt{\gamma_{21}}}\right) + v_3^{*(0,0)} \sin\left(\frac{\Lambda_1^f \zeta}{\sqrt{\gamma_{23}}}\right) + U_3^{*(0,1)} \zeta\,, \tag{4.23}$$

$$U_3^{*(0,1)} = -\frac{G_1}{(\Lambda_1^f)^2} u_{1,\xi_1\xi_3}^{*(0,0)} \sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}}\right).$$

Finally, we utilise appropriate boundary condition (4.19) to find

$$u_3^{*(0,0)} = -\frac{\gamma_{21}(\mu_{13}+\gamma_{21})}{(\gamma_{23}-\gamma_{21})(\Lambda_1^f)^2} u_{1,\xi_1\xi_3}^{*(0,0)}\,,$$

$$v_3^{*(0,0)} = \frac{G_1 G_3}{\sqrt{\gamma_{23}}\,(\Lambda_1^f)^3} u_{1,\xi_1\xi_3}^{*(0,0)} \sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}}\right) \sec\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{23}}}\right).$$

Thus, the leading order solutions for the displacement components u_i^*, $i \in \{1,2,3\}$ and pressure p_t^* are obtained in terms of a function $u_1^{*(0,0)} = u_1^{*(0,0)}(\xi_1, \xi_3, \tau)$ and its derivatives. We remark that $u_1^{*(0,0)}$ alone specifies the long wave high frequency motion at the leading order, and term it the leading order *long wave amplitude*. This function can not be determined without resorting to the higher order.

4.2. Second order problem

At second order we only consider the first two second order equations of motion

$$\begin{aligned}\gamma_{21} u_{1,\zeta\zeta}^{*(2)} + (\Lambda_1^f)^2 u_1^{*(2)} &= -B_{1111} u_{1,\xi_1\xi_1}^{*(0)} - \gamma_{31} u_{1,\xi_3\xi_3}^{*(0)} \\ &\quad - (B_{1122}+B_{1221}) u_{2,\xi_1\zeta}^{*(0)} + \gamma_{21} p_{t,\xi_1}^{*(0)} + \eta^{-2}\left(\gamma_{21} u_{1,\tau\tau}^{*(0)} + (\Lambda_1^f)^2 u_1^{*(0)}\right), \end{aligned} \tag{4.24}$$

$$\begin{aligned}B_{2222} u_{2,\zeta\zeta}^{*(2)} + (\Lambda_1^f)^2 u_2^{*(2)} + (B_{1122}+B_{1221}) u_{1,\xi_1\zeta}^{*(2)} - \gamma_{21} p_{t,\zeta}^{*(2)} &= -\gamma_{12} u_{2,\xi_1\xi_1}^{*(0)} \\ - \gamma_{32} u_{2,\xi_3\xi_3}^{*(0)} - (B_{2233}+B_{2332}) u_{3,\xi_3\zeta}^{*(0)} + \eta^{-2}\left(\gamma_{21} u_{2,\tau\tau}^{*(0)} + (\Lambda_1^f)^2 u_2^{*(0)}\right), \end{aligned} \tag{4.25}$$

which must be solved in association with the second order incompressibility condition

$$u_{1,\xi_1}^{*(2)} + u_{2,\zeta}^{*(2)} = -u_{3,\xi_3}^{*(0)}\,, \tag{4.26}$$

and the appropriate boundary conditions

$$\gamma_{21} u_{1,\zeta}^{*(2)} = -(B_{1221}+\bar{p}) u_{2,\xi_1}^{*(0)} \qquad \text{at } \zeta = \pm 1\,, \tag{4.27}$$

$$(B_{2222}+\bar{p}) u_{2,\zeta}^{*(2)} + B_{1122} u_{1,\xi_1}^{*(2)} - \gamma_{21} p_t^{*(2)} = -B_{2233} u_{3,\xi_3}^{*(0)} \qquad \text{at } \zeta = \pm 1\,. \tag{4.28}$$

Substituting the leading order displacements and pressure, see (4.20), (4.21) and (4.22), into the first second order equation of motion (4.24) and satisfying the corresponding boundary condition (4.27), immediately yields

$$u_1^{*(2)} = u_1^{*(2,0)} \sin\left(\frac{\Lambda_1^f \zeta}{\sqrt{\gamma_{21}}}\right) + u_1^{*(2,1)} \zeta \cos\left(\frac{\Lambda_1^f \zeta}{\sqrt{\gamma_{21}}}\right) + U_1^{*(2,1)} \zeta \,, \tag{4.29}$$

$$u_1^{*(2,1)} = -\frac{G_1^2}{\sqrt{\gamma_{21}}\,(\Lambda_1^f)^3} u^{*(0,0)}_{1,\xi_1\xi_1} \,, \quad U_1^{*(2,1)} = -\frac{G_1}{(\Lambda_1^f)^2} u^{*(0,0)}_{1,\xi_1\xi_1} \sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}}\right) .$$

The solution (4.29) is valid provided

$$\gamma_{21} u^{*(0,0)}_{1,\tau\tau} + (\Lambda_1^f)^2 u_1^{*(0,0)} - \eta^2 \left(\mathcal{F}^{(2)}_{1c} u^{*(0,0)}_{1,\xi_1\xi_1} + \mathcal{F}^{(2)}_{1s} u^{*(0,0)}_{1,\xi_3\xi_3}\right) = 0 \,. \tag{4.30}$$

The functions of a material parameters and pre-stress $\mathcal{F}^{(2)}_{1c}$ and $\mathcal{F}^{(2)}_{1s}$ were defined previously and are given directly after the first scaled frequency expansion (3.21).

The incompressibility condition (4.26) is then considered to obtain the form of the solution for $u_2^{*(2)}$, which may be expressed as

$$u_2^{*(2)} = u_2^{*(2,0)} \cos\left(\frac{\Lambda_1^f \zeta}{\sqrt{\gamma_{21}}}\right) + u_2^{*(2,1)} \zeta \sin\left(\frac{\Lambda_1^f \zeta}{\sqrt{\gamma_{21}}}\right) + v_2^{*(2,0)} \cos\left(\frac{\Lambda_1^f \zeta}{\sqrt{\gamma_{23}}}\right) + U_2^{*(2,2)} \zeta^2 + U_2^{*(2,0)} \,, \tag{4.31}$$

where

$$u_2^{*(2,0)} = \frac{\sqrt{\gamma_{21}}}{\Lambda_1^f} u^{*(2,0)}_{1,\xi_1} - \frac{\gamma_{21}^2(\mu_{13}+\gamma_{21})}{\sqrt{\gamma_{21}}(\gamma_{23}-\gamma_{21})(\Lambda_1^f)^3} u^{*(0,0)}_{1,\xi_1\xi_3\xi_3} + \frac{\sqrt{\gamma_{21}}G_1^2}{(\Lambda_1^f)^5} u^{*(0,0)}_{1,\xi_1\xi_1\xi_1} \,,$$

$$u_2^{*(2,1)} = \frac{G_1^2}{(\Lambda_1^f)^4} u^{*(0,0)}_{1,\xi_1\xi_1\xi_1} \,, \qquad v_2^{*(2,0)} = \frac{G_1 G_3}{(\Lambda_1^f)^4} u^{*(0,0)}_{1,\xi_1\xi_3\xi_3} \sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}}\right) \sec\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{23}}}\right) ,$$

$$U_2^{*(2,2)} = \frac{G_1}{2(\Lambda_1^f)^2} \left(u^{*(0,0)}_{1,\xi_1\xi_1\xi_1} + u^{*(0,0)}_{1,\xi_1\xi_3\xi_3}\right) \sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}}\right) .$$

We mention that in fact the leading order of every displacement component and pressure can be expressed as a linear function of the leading order long wave amplitude $u_1^{*(0,0)}$ and its derivatives. As a consequence the equality (4.30) is also valid for u_i^*, $i \in \{1,2,3\}$, and p_t^*, hence the $O(\eta^{-2})$ term in the equation (4.25) can be represented without time derivatives, which enables us to determine the solution for $p_t^{*(2)}$ in the form

$$\gamma_{21} p_t^{*(2)} = p_t^{*(2,0)} \sin\left(\frac{\Lambda_1^f \zeta}{\sqrt{\gamma_{21}}}\right) + p_t^{*(2,1)} \zeta \cos\left(\frac{\Lambda_1^f \zeta}{\sqrt{\gamma_{21}}}\right) + \tilde{p}_t^{*(2,0)} \sin\left(\frac{\Lambda_1^f \zeta}{\sqrt{\gamma_{23}}}\right) + P_t^{*(2,3)} \zeta^3 + P_t^{*(2,1)} \zeta \,. \tag{4.32}$$

The functions $p_t^{*(2,0)}$, $p_t^{*(2,1)}$, $\tilde{p}_t^{*(2,0)}$ and $P_t^{*(2,3)}$ can now be obtained by inserting (4.32) into the second equation of motion (4.25), yielding

$$p_t^{*(2,0)} = (\gamma_{21} - b_{21}) u_{1,\xi_1}^{*(2,0)} - \frac{\gamma_{21}}{(\Lambda_1^f)^2} \left(\frac{(\mu_{13} + \gamma_{21})(\gamma_{21} - b_{23})}{\gamma_{23} - \gamma_{21}} + \gamma_{31} - \gamma_{32} \right) u_{1,\xi_1\xi_3\xi_3}^{*(0,0)}$$
$$- \frac{\gamma_{21} \mathcal{Q}_{1c}^{(-2)}}{(\Lambda_1^f)^2} u_{1,\xi_1\xi_1\xi_1}^{*(0,0)} , \qquad p_t^{*(2,1)} = - \frac{G_1^2 (\gamma_{21} - b_{21})}{\sqrt{\gamma_{21}}\, (\Lambda_1^f)^3} u_{1,\xi_1\xi_1\xi_1}^{*(0,0)} ,$$
$$\tilde{p}_t^{*(2,0)} = \frac{G_1 G_3 (\gamma_{23} - b_{23})}{\sqrt{\gamma_{23}}\, (\Lambda_1^f)^3} u_{1,\xi_1\xi_3\xi_3}^{*(0,0)} \sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}} \right) \sec\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{23}}} \right) ,$$
$$P_t^{*(2,3)} = \frac{G_1}{6} \left(u_{1,\xi_1\xi_1\xi_1}^{*(0,0)} + u_{1,\xi_1\xi_3\xi_3}^{*(0,0)} \right) \sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}} \right) .$$

Satisfying the boundary condition (4.28) gives

$$P_t^{*(2,1)} = \sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}} \right) \left\{ \left(G_1 \left(\frac{g_1 + b_{21}}{(\Lambda_1^f)^2} - \frac{1}{6} \right) + \frac{\gamma_{21} \mathcal{Q}_{1c}^{(-2)}}{(\Lambda_1^f)^2} \right) u_{1,\xi_1\xi_1\xi_1}^{*(0,0)} - G_1 u_{1,\xi_1}^{*(2,0)} \right.$$
$$\left. + \left(G_1 \left(\frac{g_3 + b_{23}}{(\Lambda_1^f)^2} - \frac{1}{6} \right) - \frac{G_1 G_3^2}{\sqrt{\gamma_{23}}\, (\Lambda_1^f)^3} \tan\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{23}}} \right) + \frac{\gamma_{21} D_1^f}{(\Lambda_1^f)^2} \right) u_{1,\xi_1\xi_3\xi_3}^{*(0,0)} \right\} ,$$

which after resorting back to (4.25) returns the last unknown function of the $u_2^{*(2)}$ representation, expressed here as

$$U_2^{*(2,0)} = \frac{1}{(\Lambda_1^f)^2} \left\{ \left(G_1 \left(\frac{g_1}{(\Lambda_1^f)^2} - \frac{2G_1^2}{(\Lambda_1^f)^4} - \frac{1}{6} \right) - \frac{g_1 \mathcal{Q}_{1c}^{(-2)}}{(\Lambda_1^f)^2} \right) u_{1,\xi_1\xi_1\xi_1}^{*(0,0)} \right.$$
$$+ \left(G_1 \left(\frac{g_3 - \gamma_{31} + \gamma_{32}}{(\Lambda_1^f)^2} - \frac{1}{6} \right) - \frac{G_1 G_3^2}{\sqrt{\gamma_{23}} (\Lambda_1^f)^3} \tan\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{23}}} \right) \right.$$
$$\left. \left. + \frac{\gamma_{21} D_1^f}{(\Lambda_1^f)^2} \right) u_{1,\xi_1\xi_3\xi_3}^{*(0,0)} - G_1 u_{1,\xi_1}^{*(2,0)} \right\} \sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}} \right) .$$

Let us take a closer look at the equation (4.30). Its satisfaction ensures the existence of the solution for the second order problem and its solution $u_1^{*(0,0)}$ completely determines the leading order stressed state, see (4.20), (4.21), (4.23) and (4.22). We will refer to equation (4.30) as the leading order governing equation for the long wave amplitude. It can also be rewritten in terms of the original non-scaled variables as

$$\left[\rho h^2 \frac{\partial^2}{\partial t^2} + (\Lambda_1^f)^2 \right] u_1^{(0,0)} - h^2 \left(\mathcal{F}_{1c}^{(2)} \frac{\partial^2 u_1^{(0,0)}}{\partial x_1^2} + \mathcal{F}_{1s}^{(2)} \frac{\partial^2 u_1^{(0,0)}}{\partial x_3^2} \right) = 0 . \qquad (4.33)$$

within which $u_1^{(0,0)}(x_1, x_3, t) \equiv u_1^{*(0,0)}(\xi_1, \xi_3, \tau)$. The solution (2.5) when substituted into (4.33) yields a dispersion relation which matches the expansion (3.21), thus demonstrating a high level of consistency.

When $\mathcal{F}_{1c}^{(2)}$ and $\mathcal{F}_{1s}^{(2)}$ are positive, the leading order governing equation for the long wave amplitude (4.33) is hyperbolic. However, it is possible (and is demonstrated later numerically) to choose such combinations of the material and pre-stress parameters that $\mathcal{F}_{1c}^{(2)}$ will become negative. However, although (4.33) remains hyperbolic, it will certainly be non-wave-like, with time and one of the in-plane spatial variables swapping their roles. This behaviour is closely related to the phenomenon of negative group velocity. In the present case this phenomenon is a necessary, but not sufficient, condition for a non-wave-like hyperbolic equation. It has previously been remarked that in the plane strain case the existence of negative group velocity is both necessary and sufficient to lose hyperbolicity, see [16].

4.3. Third order problem

The third order problem will be solved only for the first third order equation of motion

$$\begin{aligned}\gamma_{21}u_{1,\zeta\zeta}^{*(4)}+(\Lambda_1^f)^2u_1^{*(4)}&=-(B_{1133}+B_{1331})u_{3,\xi_1\xi_3}^{*(0)}-B_{1111}u_{1,\xi_1\xi_1}^{*(2)}-\gamma_{31}u_{1,\xi_3\xi_3}^{*(2)}\\&-(B_{1122}+B_{1221})u_{2,\xi_1\zeta}^{*(2)}+\gamma_{21}p_{t,\xi_1}^{*(2)}+\eta^{-2}\left(\gamma_{21}u_{1,\tau\tau}^{*(2)}+(\Lambda_1^f)^2u_1^{*(2)}\right),\end{aligned}\tag{4.34}$$

and the associated boundary condition

$$\gamma_{21}u_{1,\zeta}^{*(4)}+(B_{1221}+\bar{p}\,)u_{2,\xi_1}^{*(2)}=0\,.\tag{4.35}$$

In view of the results obtained at previous orders, the solution of (4.34) may be sought in the form

$$\begin{aligned}u_1^{*(4)}=u_1^{*(4,0)}\sin\left(\frac{\Lambda_1^f\zeta}{\sqrt{\gamma_{21}}}\right)+u_1^{*(4,1)}\zeta\cos\left(\frac{\Lambda_1^f\zeta}{\sqrt{\gamma_{21}}}\right)+u_1^{*(4,2)}\zeta^2\sin\left(\frac{\Lambda_1^f\zeta}{\sqrt{\gamma_{21}}}\right)\\+v_1^{*(4,0)}\sin\left(\frac{\Lambda_1^f\zeta}{\sqrt{\gamma_{23}}}\right)+U_1^{*(4,3)}\zeta^3+U_1^{*(4,1)}\zeta\,.\end{aligned}\tag{4.36}$$

Substituting (4.36) into the first equation of motion (4.34) and comparing the coefficients of linearly independent terms it is possible to obtain

$$u_1^{*(4,2)}=-\frac{G_1^4}{2\gamma_{21}(\Lambda_1^f)^6}u_{1,\xi_1\xi_1\xi_1\xi_1}^{*(0,0)},$$

$$v_1^{*(4,0)}=\frac{\sqrt{\gamma_{23}}G_1G_3(\mu_{13}+\gamma_{23})}{(\gamma_{23}-\gamma_{21})(\Lambda_1^f)^5}u_{1,\xi_1\xi_1\xi_3\xi_3}^{*(0,0)}\sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}}\right)\sec\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{23}}}\right),$$

$$\begin{aligned}U_1^{*(4,1)}=\frac{1}{(\Lambda_1^f)^2}\Bigg\{&\left(G_1\left(\frac{g_1}{(\Lambda_1^f)^2}-\frac{2G_1^2}{(\Lambda_1^f)^4}-\frac{1}{6}\right)+\frac{g_1\mathcal{Q}_{1c}^{(-2)}}{(\Lambda_1^f)^2}\right)u_{1,\xi_1\xi_1\xi_1\xi_1}^{*(0,0)}\\&+\left(G_1\left(\frac{g_3-\mu_{13}-\gamma_{21}}{(\Lambda_1^f)^2}-\frac{1}{6}\right)-\frac{G_1G_3^2}{\sqrt{\gamma_{23}}(\Lambda_1^f)^3}\tan\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{23}}}\right)\right.\\&\left.+\frac{\gamma_{21}D_1^f}{(\Lambda_1^f)^2}\right)u_{1,\xi_1\xi_3\xi_3}^{*(0,0)}-G_1u_{1,\xi_1\xi_1}^{*(2,0)}\Bigg\}\sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}}\right),\end{aligned}$$

$$U_1^{*(4,3)} = \frac{G_1}{6(\Lambda_1^f)^2}\left(u^{*(0,0)}_{1,\xi_1\xi_1\xi_1\xi_1} + u^{*(0,0)}_{1,\xi_1\xi_1\xi_3\xi_3}\right)\sin\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{21}}}\right).$$

And the boundary condition (4.35) is then used to establish

$$\begin{aligned} u_1^{*(4,1)} = \frac{G_1}{\sqrt{\gamma_{21}}(\Lambda_1^f)^3}\Bigg\{&\left(G_1\left(\frac{2g_1}{(\Lambda_1^f)^2} - 3\frac{G_1^2}{(\Lambda_1^f)^4} + \frac{1}{3}\right) + \frac{\sigma_2 \mathcal{Q}_{1c}^{(-2)}}{(\Lambda_1^f)^2}\right) u^{*(0,0)}_{1,\xi_1\xi_1\xi_1\xi_1} \\ &+ \left(G_1\left(\frac{2g_3+\gamma_{23}-\gamma_{31}+\gamma_{32}}{(\Lambda_1^f)^2} + \frac{1}{3}\right) - \frac{G_1 G_3^2}{\sqrt{\gamma_{23}}(\Lambda_1^f)^3}\tan\left(\frac{\Lambda_1^f}{\sqrt{\gamma_{23}}}\right)\right. \\ &\left. + \frac{2\gamma_{21}D_1^f}{(\Lambda_1^f)^2}\right) u^{*(0,0)}_{1,\xi_1\xi_1\xi_3\xi_3} - G_1 u^{*(2,0)}_{1,\xi_1\xi_1}\Bigg\}. \end{aligned}$$

As with the second order problem, the solution for $u_1^{*(4)}$ given by (4.36) is only valid provided an additional condition is satisfied, which is

$$\begin{aligned} \gamma_{21}u^{*(2,0)}_{1,\tau\tau} + (\Lambda_1^f)^2 u_1^{*(2,0)} - \eta^2\Big(&\mathcal{F}_{1c}^{(2)} u^{*(2,0)}_{1,\xi_1\xi_1} + \mathcal{F}_{1s}^{(2)} u^{*(2,0)}_{1,\xi_3\xi_3} \\ &+ \mathcal{F}_{1c}^{(4)} u^{*(0,0)}_{1,\xi_1\xi_1\xi_1\xi_1} + \mathcal{F}_{1s}^{(4)} u^{*(0,0)}_{1,\xi_1\xi_1\xi_3\xi_3}\Big) = 0\,, \end{aligned} \tag{4.37}$$

Let us introduce a new function

$$u = u^{(0)} + u^{(2)}\eta^2 + O(\eta^4)\,, \qquad u^* = u^{*(0)} + u^{*(2)}\eta^2 + O(\eta^4)\,, \tag{4.38}$$

in which $u(x_1, x_3, t) = u^*(\xi_1, \xi_3, \tau)$ and for the first family of the shear resonance frequencies we assume $u^{(2m)} = u_1^{*(2m,0)}$. The fact that the function $u_1^{*(2m,0)}$ is essentially the solution of all boundary value problems posed for $u_1^{*(2m-2,0)}$, $m = 1, 2, 3\dots$, means that every displacement component and pressure is the linear function of u and its derivatives. Therefore, we will term u as the long wave amplitude. Now if we add the leading order governing equation (4.30) to the product of η^2 and (4.37) we obtain a second order governing equation, which is given in terms of original variables and long wave amplitude as

$$\begin{aligned} &\left[\rho h^2 \frac{\partial^2}{\partial t^2} + (\Lambda_1^f)^2\right] u \\ &\qquad - h^2\left(\mathcal{F}_{1c}^{(2)}\frac{\partial^2 u}{\partial x_1^2} + \mathcal{F}_{1s}^{(2)}\frac{\partial^2 u}{\partial x_3^2}\right) - h^4\left(\mathcal{F}_{1c}^{(4)}\frac{\partial^4 u}{\partial x_1^4} + \mathcal{F}_{1s}^{(4)}\frac{\partial^4 u}{\partial x_1^2 x_3^2}\right) = 0\,. \end{aligned} \tag{4.39}$$

The dispersion relation associated with this equation matches the third order approximation of the exact dispersion relation (3.21) exactly.

4.4. Second family of shear resonance frequencies

The analysis of the asymptotic behaviour of a plate in the vicinity of the second family of shear resonance frequencies is very similar to the case just discussed. The space and time coordinates are to be re-scaled according to (4.1) and

$$t = l\eta\sqrt{\frac{\rho}{\gamma_{23}}}\tau\,, \tag{4.40}$$

with displacements, whose scalings are chosen to coincide with (3.26), thus

$$\begin{aligned} u_m(x_1,x_2,x_3,t) &= l\eta^{3-m}u_m^*(\xi_1,\zeta,\xi_3,\tau)\,, \qquad m\in\{1,2,3\}\,,\\ p_t(x_1,x_2,x_3,t) &= \gamma_{23}p_t^*(\xi_1,\zeta,\xi_3,\tau)\,. \end{aligned} \tag{4.41}$$

The consequent derivation yields the leading and second order governing equations for a long wave amplitude, given by

$$\left[\rho h^2\frac{\partial^2}{\partial t^2}+(\Lambda_3^f)^2\right]u_3^{(0,0)}-h^2\left(\mathcal{F}_{3c}^{(2)}\frac{\partial^2 u_3^{(0,0)}}{\partial x_1^2}+\mathcal{F}_{3s}^{(2)}\frac{\partial^2 u_3^{(0,0)}}{\partial x_3^2}\right)=0\,, \tag{4.42}$$

$$\begin{aligned} &\left[\rho h^2\frac{\partial^2}{\partial t^2}+(\Lambda_3^f)^2\right]u\\ &\quad -h^2\left(\mathcal{F}_{3c}^{(2)}\frac{\partial^2 u}{\partial x_1^2}+\mathcal{F}_{3s}^{(2)}\frac{\partial^2 u}{\partial x_3^2}\right)-h^4\left(\mathcal{F}_{3c}^{(4)}\frac{\partial^4 u}{\partial x_1^4}+\mathcal{F}_{3s}^{(4)}\frac{\partial^4 u}{\partial x_1^2 x_3^2}\right)=0\,, \end{aligned} \tag{4.43}$$

where $\mathcal{F}_{3c}^{(2)}$, $\mathcal{F}_{3s}^{(2)}$, $\mathcal{F}_{3c}^{(4)}$ and $\mathcal{F}_{3s}^{(4)}$ were given immediately after the expansion (3.22) and we assume $u^{(2m)}=u_3^{*(2m,0)}$ in the definition (4.38). The dispersion relation associated with this equation is consistent with the exact dispersion relation (3.6) in the sense that it matches all three orders of the frequency expansion (3.22) exactly (first two orders in case of the leading order governing equation (4.42)). As for the existence of negative group velocity, it may also occur for the second family of shear resonance frequencies, with the necessary condition given by $\mathcal{F}_{3s}^{(2)}<0$.

5. Numerical results and discussion

Some illustrative numerical results are now presented in respect of the Mooney-Rivlin strain-energy function

$$W=\frac{\mu_1}{2}(\lambda_1^2+\lambda_2^2+\lambda_3^2-3)+\frac{\mu_2}{2}(\lambda_1^{-2}+\lambda_2^{-2}+\lambda_3^{-2}-3)\,, \tag{5.1}$$

in which μ_1 is the shear modulus, μ_2 characterises the departure from the symmetric neo-Hookean model and λ_m, $m\in\{1,2,3\}$, are the principal stretches of primary deformation. The components of the elasticity tensor associated with the strain energy (5.1) are given by

$$\begin{aligned} B_{iiii}&=(\mu_1+\mu_2(\lambda_j^2+\lambda_k^2))\lambda_i^2\,, \qquad B_{iijj}=2\mu_2\lambda_i^2\lambda_j^2\,, \qquad B_{ijji}=-\mu_2\lambda_i^2\lambda_j^2\,,\\ B_{ijij}&=(\mu_1+\mu_2\lambda_k^2)\lambda_i^2\,, \qquad i\neq j\neq k\neq i\,, \qquad i,j,k\in\{1,2,3\}\,. \end{aligned}$$

In Figure 1, $\bar{\omega}$ is shown as a function of kh. Specifically, the numerical solution and both the second and third order approximations are presented in respect of the first three harmonics associated with first family of shear resonances. It is easy to see that the accuracy of the approximation in general increases considerably for the second (and further) harmonic. This is because the third order term in the frequency expansion is divided by $(\Lambda_1^f)^2$, which is $O(n^2)$. It is worth noting

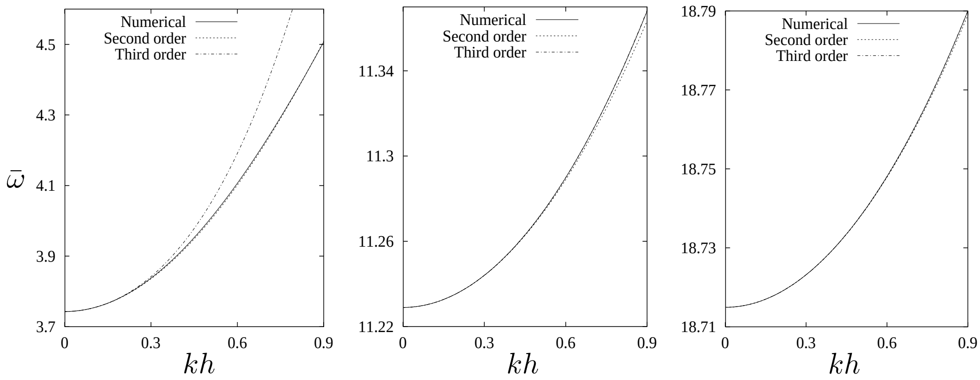

Figure 1. Scaled frequency of the first three harmonics associated with the first family of shear resonances, shown against scaled wave number kh together with their second and third order approximations. Waves propagate in the Mooney-Rivlin material with $\mu_1 = 3.0$, $\mu_2 = 1.1$, $\lambda_1 = 0.9$, $\lambda_2 = 1.2$, $\sigma_2 = 5.5$ and $\theta = 15°$.

that the left-most plot in Figure 1 depicts a situation when the second order approximation is apparently better then the third order. However, direct comparing of absolute errors for both second and third order approximations can be used to demonstrate that the third order approximation is better for all $kh \leqslant 0.2$ and therefore for all $kh \ll 1$. It is worth reiterating that all our approximations were obtained for $kh \ll 1$ and they, as any asymptotic expansions, may, but should not be expected to, provide good approximation outside their destined domain of validity.

In view of the fact that the dynamic response of a Mooney-Rivlin is somewhat limited, see e.g. [7], we also demonstrate some typical plots in respect of the Varga strain energy function

$$W(\lambda_1, \lambda_2, \lambda_3) = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3)\,, \tag{5.2}$$

where μ is shear modulus. The associated non-zero components of B_{milk} may be obtained as follows

$$B_{ijij} = \frac{2\mu\lambda_i^2}{\lambda_i + \lambda_j}\,, \qquad B_{ijji} = -\frac{2\mu\lambda_i\lambda_j}{\lambda_i + \lambda_j}\,, \qquad i \neq j\,, \qquad i, j \in \{1, 2, 3\}\,. \tag{5.3}$$

In Figure 2 the numerical solution and both the second and third order approximations are presented for the three harmonics associated with first family of shear resonance frequencies. We mentioned previously that for the certain combinations of the pre-stress and material parameters it is possible to obtain negative group velocity at low wave number, corresponding to $\bar{\omega}$ being a decreasing function of wave number for $kh \sim 0$. The phenomenon of negative group velocity is well-known in many areas of physics dealing with dispersive waves, in particular in optics where it is associated with so called *anomalous dispersion.* Seemingly the first mentioning of possible negative group velocity at low wave number for some high frequency modes in an isotropic elastic plate was given in [17]. There is also some experimental work claiming to observe it for ultrasound waves, see [18]. For the first family of shear resonance frequencies, existence of negative group velocity will arise for small kh whenever

$$\mathcal{F}_1^{(2)} \equiv \mathcal{F}_{1c}^{(2)} c_\theta^2 + \mathcal{F}_{1s}^{(2)} s_\theta^2 < 0\,, \tag{5.4}$$

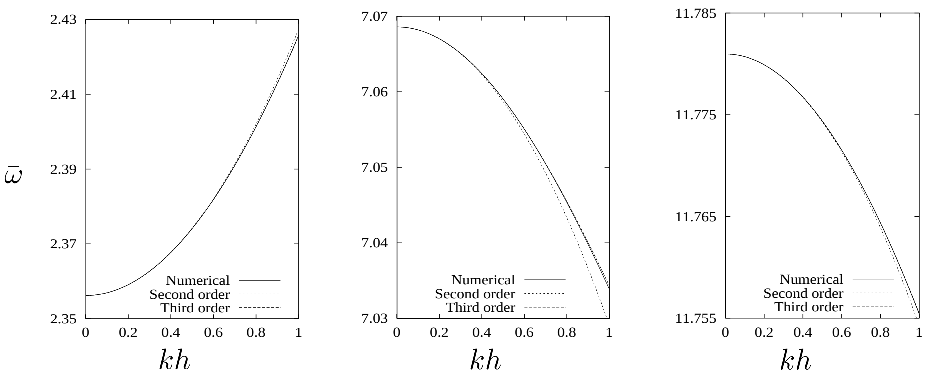

Figure 2. Scaled frequency of the first three harmonics associated with the first family of shear resonances, shown against scaled wave number kh together with their second and third order approximations. Waves propagate in the Varga material with $\mu = 1.0$, $\lambda_1 = 0.5$, $\lambda_2 = 1.5$, $\sigma_2 = 2.8$ and $\theta = 10°$.

see (3.21). Note, that for physically realistic response $\mathcal{F}_{1s}^{(2)}$ is always positive, which can be shown by taking $q = 0$ and $s_\theta = 0$ in equation (2.6). Hence, $\mathcal{F}_{1c}^{(2)} < 0$ is the necessary condition for the existence of negative group velocity. It is important to keep in mind that $\mathcal{F}_{1c}^{(2)} < 0$ also indicates that equation (4.33) is not wave-like. In order to illustrate possible scenarios for which $\mathcal{F}_{1c}^{(2)} < 0$, Figure 3 shows $\mathcal{F}_{1}^{(2)}$ and $\mathcal{F}_{1c}^{(2)}$ against kh for a variety of angles of propagation and normal stretches, respectively. We remark that in [14] it is shown that in the

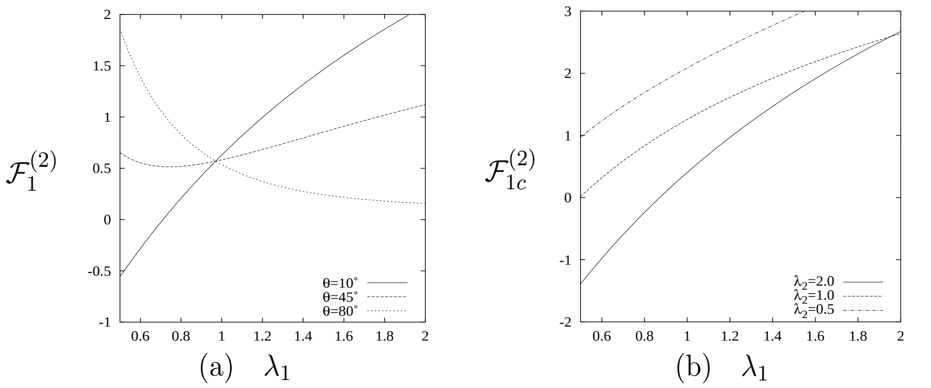

Figure 3. The coefficients of the second order scaled frequency expansion, associated with the first family of shear resonances, shown for a variety of (a) angles of propagation (b) normal stretches. Waves propagate in the Varga material with the same parameters as in Figure 2 if not specified otherwise.

analogous plain strain case the existence of negative group velocity also changes the associated governing equation for the long wave amplitude from hyperbolic to elliptic. It should be stressed that the governing equation (4.33) is only valid in the vicinity of the cut-off frequencies. Further work is therefore required to fully elucidate the implications of its change in type on dynamic response.

6. Concluding remarks

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References

1. A. E. Green, On the linear theory of thin elastic shells. *Proc. R. Soc. Lond., Ser. A*, 266, pp. 143–160, (1962).
2. J. D. Kaplunov, L. Y. Kossovich, and E. V. Nolde, *Dynamics of Thin Walled Elastic Bodies.* New York: Academic Press, (1998).
3. M. A. Dowaikh and R. W. Ogden, On surface waves and deformations in a pre-stressed incompressible elastic solid. *IMA Jl. Appl. Math.*, 44, pp. 261–284, (1990).
4. M. A. Dowaikh and R. W. Ogden, Interfacial waves and deformations in pre-stressed elastic media. *Proc. R. Soc. Lond., Ser. A*, 433, pp. 313–328, (1991).
5. R. W. Ogden and D. A. Sotiropoulos, On interfacial waves in pre-stressed layered incompressible solids. *Proc. R. Soc. Lond., Ser. A*, 450, pp. 319–341, (1995).
6. D. A. Sotiropoulos and C. G. Sifniotopoulos, Interfacial waves in pre-stressed incompressible elastic interlayers. *J. Mech. Phys. Solids*, 43, pp. 365–387, (1995).
7. G. A. Rogerson, Some asymptotic expansions of the dispersion relation for an incompressible elastic plate. *Int. Jl. Solid. Struct.*, 34, pp. 2785–2802, (1997).
8. P. M. Sheridan, F. O. James, and T. S. Miller, Design of components. in *Engineering with Rubber* (A. N. Gent, ed.), pp. 209–235, Munich: Hanser, (1992).
9. J. D. Kaplunov and D. G. Markushevich, Plane vibrations and radiation of an elastic layer lying on a liquid half-space. *Wave Motion*, 17, pp. 199–211, (1993).
10. J. D. Kaplunov, Long wave vibrations of a thin body with fixed faces. *Quart. Jl. Mech. Appl. Math.*, 48, pp. 311–327, (1995).
11. G. A. Rogerson and K. J. Sandiford, Harmonic wave propagation along a non-principal direction in a pre-stressed elastic plate. *Int. Jl. Eng. Sci.*, 37, pp. 1663–1691, (1999).
12. R. W. Ogden, *Non-linear Elastic Deformations.* New York: Ellis Horwood, (1984).
13. G. A. Rogerson and Y. B. Fu, An asymptotic analysis of the dispersion relation of a pre-stressed incompressible elastic plate. *Acta Mechanica*, 111, pp. 59–77, (1995).
14. J. D. Kaplunov, E. V. Nolde, and G. A. Rogerson, An asymptotically consistent model for long wave high frequency motion in a pre-stressed elastic plate. *Mech. Math. Solids*, (2002). To appear.
15. J. L. Ericksen, On the propagation of waves in isotropic, incompressible, perfectly elastic materials. *Jl. Rational Mech. Analysis*, 2, pp. 329–337, (1953).
16. J. D. Kaplunov, L. Y. Kossovich, and G. A. Rogerson, Direct asymptotic integration of the equations of transversely isotropic elasticity for a plate near cut-off frequencies. *Quart. J. Mech. Appl. Math.*, 53, pp. 323–341, (2000).
17. I. Tolstoy and E. Usdin, Wave propagation in elastic plates: low and high mode dispersion. *J. Acoust. Soc. Am.*, 29, pp. 37–42, (1957).
18. J. Wolf, T. D. K. Ngoc, R. Kille, and W. G. Mayer, Investigation of Lamb waves having a negative group velocity. *J. Acoust. Soc. Am.*, 83, pp. 122–126, (1988).