Outage and Average Error Probability for UL-Massive MIMO Systems: Asymptotic Analysis

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***Abstract*—This paper investigates the asymptotic behaviour (error and outage probabilities) of a single cell multiple-input multiple-output (MIMO) system aided by a large scale antenna array. Specifically, the uplink transmission over composite fading channel with power-scaling scheme is considered. Where, most reported studies in this respect discuss the case of downlink scenario for convenience MIMO systems. Two assumptions are addressed: perfect channel information (CSI) and imperfect-CSI. In both cases, closed form expressions for error and outage probabilities in asymptotically large receive antenna environments are derived. Moreover, users' location impact on system performance is introduced. Numerical outcomes, validated by Monte-Carlo simulations, shed light on how different parameters and conditions can affect aforementioned performance metrics.**

***Index Terms*— Large scale antenna-number regimes, average bite error probability, rate outage-probability, composite fading channels.**

I. Introduction

Fading and shadowing are great challenges for reliable transmission in wireless environment [1]. In particular, the challenges are obvious for real-time applications, over slow fading environment, when the desired transmission-delay constraint is on the order of the channel coherent time. Two important metrics have been proposed in the literature to characterise system performance with different quality of service (QoS) and data rate limitations. These metrics are the average error and the outage probabilities. Where, outage-event occurs if instantaneous signal-to-noise-ratio (SNR) drops below the minimal desired-threshold value. The uplink of a multi-cell multi-user single-input multiple-output system (MU SIMO) has been considered in [2]. In their work, authors derived exact analytical-expressions for the symbol error rate and the outage probability.

In [3], authors investigated the multi-cellular uplink and downlink. Their adopted system model accounts for error in channel-estimation and antenna-correlation. The work in [4], addressed the uplink of a multi-cell MU-SIMO system, when the channel experiences small and large-scale fading. The detection is done by using linear ZF scheme and the base station has perfect CSI of all users in its own cell. L. Zhao et. al. in [5], derived both the outage probability and bit error rate expressions corresponding to the degrees of freedom (the difference between the number of user terminal and BS antennas) in downlink transmission of massive-MIMO. Power scaling law for uplink massive-MIMO is considered in [6]. Delayed-CSI due to user mobility is addressed in [7]. The implications of channel-aging on the uplink massive-MIMO transmission is investigated in [8]. In [9], the authors introduced tight closed-form lower-bounds for the rate performance over aged-CSI. Recently, the authors in [10] introduced approximate closed-form formulas for the uplink outage-probability of a user with maximal ratio combining (MRC) at the BS.

It is well known that the use of massive antenna arrays can significantly alleviate the effect of small scale fading and intra-cell interference. Consequently, increase the energy and spectral efficiency of wireless systems [11]. Motivated by this fact, we seek in this paper to address the potential benefits of massive-MIMO configuration on the uplink-error and outage performance of wireless systems. Different from the existing works on this aspect, in our analysis, we exploit the asymptotic results of random matrix theory introduced in [11] to further analysis the ZF-receiver's performance of large system focusing on shadowing effect of the uplink channel. Such analysis could play a key role in 5G networks designing, e.g, in mm-Wave systems which are vulnerable to path loss and shadow attenuation. The specific contributions of this paper can be summarized as follows,

1) We derive, in closed-form expression, the uplink's asymptotic error and outage probabilities when the BS deploys a large antenna-array with power scaling policy (unique to the best of our knowledge). The results enable us to explicitly study the impacts of the shadowing parameter, number of BS receive antennas and the transmit power on the system performance.
2) We evaluate the implications of channel and system parameters on the uplink performance via numerical analysis. The provided precise approximation results can replace the need for lengthy-Monte-Carlo simulations.

II. System and Channel Model

The system under study considers the uplink of single-cell with massive-antenna base station. Assuming that users equipped with a single transmitting antenna, the output of the uplink channel can be given as [12] $y = \sqrt{p_u} G x + n$, where, $G \in \mathbb{C}^{N_r \times K}$ is the complex channel matrix between the base station and its associated user terminals, N_r is the number of receive antennas at the BS, p_u is the average power transmitted by each user, x denotes the data signal and $n \sim \mathcal{CN}(0, 1)$ is the AWGN noise. For individual user k, the column vector of

G will be [11] [12, eq. 2] $g_k = h_k\sqrt{\beta_k}$, h_k is the fast fading component of g_k and β_k is the large-scale component which can be given as [6]

$$\beta_k = \mu_k / D_k^{\nu} \quad k = 1, 2, ...K \tag{1}$$

where, μ_k is the log-normal distributed shadowing, D_k is the distance between the BS and the k-th user and ν is the path-loss exponent. Its noteworthy that, real measurements in wireless channels have shown the distance dependence of the average received power[1].

In this work ZF-detector is adopted[2] and all results are derived based on this requisite. It is worth mentioning that in massive antenna environments, just small scale fading corresponding to micro diversity can be averaged out, while large scale fading corresponding to macro diversity environment stays active and affects system performance. As such, with ZF detector and perfect CSI, capitalizing on the outcomes of [11], asymptotic power scaling data-rate of the *k-th* user can be written as [11, [Eq. 21)]

$$R_k^{asy} = \log_2(1 + \beta_k p_u N_r) \qquad \text{bits/s/Hz} \tag{2}$$

LEMMA.1 *(Ergodic Rate); The upper bound on the uplink ergodic rate of k-th user can be derived as follows*

$$R_k^{asy}(\mu_k, D_k) = \mathbb{E}\left\{\log_2(1 + \beta_k p_u N_r)\right\} \overset{(a)}{\leqslant} \log_2(1 + \mathbb{E}\left\{\frac{\mu_k\, p_u}{D_k^{\nu}} N_r\right\}) \tag{3}$$

where, inequality (a) follows invoking (1) and Jensen's inequality since $\log_2(\cdot)$ is a concave function . In our analysis, μ_k and D_k are two random variables with distributions given in (4) and (5) respectively, where gamma distribution has shown a good fit to real-measurements and also it is analytically more tractable than the log-normal distribution. The probability density function (PDF) of gamma-distributed RVs can be written as [11, eq.26]

$$f(x) = \frac{x^{m_k - 1}}{\Gamma(m_k)\Omega_k^{m_k}} e^{-x/\Omega_k} U(0) \quad x, m_k, \Omega_k > 0, \tag{4}$$

here, $U(0)$ is the unit step function to insure that the probability defined over $0 \leq x < \infty$, parameters $0 \leq m_k$ and $0 \leq \Omega_k$ are the fading figure or shape and the average power or scale parameter of the gamma distribution, respectively. $\Gamma(\cdot)$ denotes the gamma function. The values of these parameters affected by the communication environments and the expectation $\Omega_k = \mathbb{E}\{\gamma_k\}/m_k$, is usually chosen to be one. Moreover, we assumed independent and uniform distribution of users on disc formed by two rings R_i and R_o in the cell coverage area. So, the corresponding PDF of the distance between users terminals and base station can be modelled as, [13], [5, eq.9]

$$f_d(x) = \frac{2x}{(R_o^2 - R_i^2)}, \qquad x \in (R_o\ R_i] \tag{5}$$

Next we investigate our performance metrics.

III. Error and Outage probability

In this section, closed-form expressions for the error probability and the outage probability are derived for both full and limited knowledge of channel information.

[1]This fact led to the well-known path loss law, where the typical values of ν ranging from 2 to 6 in urban areas.

[2]It is worth noting that ZF- detector tends to be sub-optimal detector [11].

A. Average error probability

For binary signals in AWGN-environments, the bit error probability (BEP) of coherent, differentially coherent, and non-coherent detection is given by the generic expression [1, eq.(8.100)]

$$\mathcal{P}_e(a, b, \gamma_k) = \frac{\Gamma(b, a\gamma_k)}{2\Gamma(b)} = \frac{1}{2} Q_b(0, \sqrt{2a\gamma_k}) \tag{6}$$

where parameters a and b are given in table I and the generalized Marcum-Q function $Q_b(\cdot,\cdot)$ is defined as [1, eq.(4.33)]

$$Q_b(q_1, q_2) = \int_{q_2}^{\infty} \frac{x^b}{a^{b-1}} \exp\left[-\frac{q_1^2 + x^2}{2}\right] \mathrm{I}_{b-1}(q_1 x)\, \mathrm{d}x \tag{7}$$

where $q_1 > 0$ and $q_2 \geq 0$, are real parameters and $\mathrm{I}_{b-1}(\cdot)$ is the *b-th* order modified-Bessel function of the first kind. The order-index b is an integer and typically $b \geq 0$. Employing scale law for the transmit power [11], then the approximated uplink SNR γ_k in eq.(3) for the *k-th* user at the receiver end (BS serving finite number of users K) with perfect CSI will achieve the following $\gamma_k - \dfrac{p_u N_r}{D^{\nu}} \xrightarrow[N_r \to \infty]{a.s} 0$, where notation a.s. means almost-sure convergence. Equation (6) can be written in terms of lower incomplete gamma function using the identity $\gamma(b, x) = \Gamma(b) - \Gamma(b, x)$, and consequently error probability can be given as

$$\mathcal{P}_e(a, b, \gamma_k) = \frac{1}{2}\left\{1 - \frac{\gamma(b, a\gamma_k)}{\Gamma(b)}\right\} \tag{8}$$

PROPOSITION A.1 (PERFECT CSI)*: In the regime of large N_r with linear ZF receiver, the asymptotic uplink average bit error probability of k-th user under generalised-k channels, with full channel information, can be expressed in the following compact form*

$$\mathcal{P}_{e_k}^{(P)} = \tfrac{1}{2} - \tfrac{(ap_u N_r \Omega/D^{\nu})^b \Gamma(m+b)}{2\Gamma(b+1)\Gamma(m)} \times {}_2F_1\left[\begin{smallmatrix} b, b+m \\ b+1 \end{smallmatrix}\middle| -\frac{ap_u N_r \Omega}{D^{\nu}}\right] \tag{9}$$

where, ${}_2F_1(\cdot)$ stands for qauss-hypergeometric-function [15, eq.(9.14.2)].

proof: A detailed proof is given in Appendix A.1 □

REMARK 1 : The asymptotic uplink average bit error probability can be expressed in terms of the upper incomplete beta function, $\mathrm{B}_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1}dt$, which, in many cases, can be calculated more efficiently than the gauss-hypergeometric-function (using e.g., MATLAB® software). Then, with aid of the identity $\mathrm{B}_x(a, b) = \frac{x^a}{a} F(a, 1-b; a+1; x)$, eq.(9) can be rewritten as follows

$$\mathcal{P}_{e_k}^{(P)}(D, N_r, p_u, m) = \tfrac{1}{2} - \tfrac{(-1)^b\, b\, \Gamma(m+b)}{2\Gamma(b+1)\Gamma(m)} \quad \mathrm{B}_{\frac{-ap_u N_r \Omega}{D^{\nu}}}(b, 1-b-m) \tag{10}$$

Table I. Values of parameters a and b for different combinations of Modulation/Detection.

Modulation or Detection type	Parameters values	
	a	**b**
Orthogonal-coherent BFSK	$1/2$	$1/2$
Antipodal-coherent BPSK	1	$1/2$
Differentially-coherent DPSK	1	1
Orthogonal-noncoherent BFSK	$1/2$	1

SPACIAL AVERAGE*:* Next, we pursue an error probability analysis derived in the previous proposition for more practical case where users are randomly-located within cell-area, and consequently experiencing different path loss. The following theorem corresponds to this significant fact by using distance marginal distribution function PDF given by eq.(5) to evaluate the average error probability over users of entire-cell area.

THEOREM A.1 (PERFECT CSI-SPACIAL AVERAGE)*: The asymptotic average error probability over all the users in uplink of large N_r regimes for generalised-k channels is given by eq.(11) at the top of the next page*

proof:

The proof of this theorem is shown in appendix A.2 □

IMPERFECT CSI*:* In realistic wireless systems, training pilots or sequences of limited length say e.g. τ symbols, usually used in acquiring CSI. Because of limited length of training-pilots and the time-varying characteristic of the fading channels, estimation-errors are unavoidable and this effect is termed as channel-information imperfection. Hence imperfect-CSI performance is of key point in real systems analysis. For simplicity, we assume no power control policy or equal power scale transmitting for all users [14], $p_u = E_u/\sqrt{M}$, where E_u is the total available power. In this case, invoking the law of large numbers, the large system asymptotic uplink rate can be written as [11, eq.37]

$$R_k^{asy} \overset{N_r \to \infty}{\approx} \frac{T-\tau}{T} \log_2(1 + \tau\beta_k^2 E_u^2) \overset{T \gg \tau}{\approx} \log_2(1 + \tau\beta_k^2 E_u^2) \quad (12)$$

where T is the total number of symbols in one time frame. Accordingly, the approximated uplink SNR γ_k for k-th user at the receiver end (BS serving finite number of users K) for imperfect CSI with power scaling law [11], will satisfy the following $\gamma_k - \dfrac{\tau p_u^2 N_r}{D^{2\nu}} \overset{N_r \to \infty}{\longrightarrow} 0$.

PROPOSITION A.2 (IMPERFECT CSI)*: For large scale MIMO systems with linear ZF receiver, the asymptotic uplink average bit error probability of k-th user under generalised-k channels with perfect CSI can be expressed as following*

$$\mathcal{P}_{e_k}^{(IP)} = \tfrac{1}{2} - \frac{(a\rho)^b \Gamma(b+\frac{m}{2})\Gamma(b+\frac{(m+1)}{2})}{2^{2-m}\sqrt{\pi}\Gamma(b+1)\Gamma(m)} \times {}_3F_1\left[\begin{matrix} b, b+\frac{m}{2}, b+\frac{m+1}{2} \\ b+1 \end{matrix} \middle| -a\rho\right] \quad (13)$$

where $\rho = 4\tau p_u^2 N_r \Omega / D^{2\nu}$.

proof: A detailed proof is given in Appendix A.3 □

SPACIAL AVERAGE*:* Now, we pursue an error probability analysis, which is limited by the performance of the worst user, by taking into account the impacts of randomness of users' location as well as the imperfect CSI. To this end, we invoking user distance distribution function PDF given by eq.(5) once more in the following theorem.

THEOREM A.2 (SPACIAL AVERAGE)*: The average asymptotic error probability over all the users in uplink of massive MIMO systems over generalised-k channels and imperfect channel information is given by eq.(14) on the next page.*

proof: Using the same methodology used in appendix A.2, the theorem can be easily proofed. □

Intuitively, error probability ($\mathcal{P}_e$) increases when it's averaged over entire cell area, due to the worst-case corresponding to cell-edge users.

B. RATE OUTAGE PROBABILITY

The rate outage probability can be defined as the probability that the user rate R_k (as a random variable) drops under particular rate threshold R_{th} which is the achievable or desired transmission data-rate for the specific *k-th* user. Hence the outage probability is given by

$$\mathcal{P}_{out_k} \triangleq Pr\big(R_k < R_{th}\big) = Pr\big(\gamma_k < (2^{R_{th}} - 1)\big), \quad (15)$$

where the second line follows from Shannon-capacity definition i.e. the tight upper-bound on information rate of the channel.

PROPOSITION B.1 (PERFECT CSI)*: For large scale MIMO systems with linear ZF receiver, the asymptotic uplink rate outage probability of k-th user under generalised-k channels with perfect CSI can be expressed in the following compact form*

$$\mathcal{P}_{out_{-k}}^{(P)} = \frac{(D^\nu\,(2^{R_{th}}-1)/p_u N_r \Omega)^m}{m\,\Gamma(m)} \times {}_1F_1\left[\begin{matrix} m \\ m+1 \end{matrix} \middle| -\frac{D^\nu\,(2^{R_{th}}-1)}{p_u N_r \Omega}\right] \quad (16)$$

proof: see appendix B.1 □

Next, we pursue an outage analysis for case with limited knowledge of CSI.

PROPOSITION B.2 (IMPERFECT CSI)*: For large scale MIMO systems with linear ZF receiver, the asymptotic uplink rate outage probability of k-th user under generalised-k channels with Imperfect CSI can be expressed as following*

$$\mathcal{P}_{out_{-k}}^{(IP)} = \frac{(D^{2\nu}\,(2^{R_{th}}-1)/\tau p_u^2 N_r \Omega)^{(m+2)}}{(m+2)\,\Gamma(m+2)} \; {}_1F_1\left[\begin{matrix} m+2 \\ m+3 \end{matrix} \middle| -\frac{D^{2\nu}\,(2^{R_{th}}-1)}{\tau p_u^2 N_r \Omega}\right] \quad (17)$$

proof: see appendix B.2 □

IV. NUMERICAL RESULTS

In this section, we present simulated performance-results, corresponding to the uplink of a single cell with coverage area ranges between two rings of radii $R_i = 100$ m and $R_o = 1$ Km from the BS. In our simulation, power is normalised to the distance of R_i for all scenarios. Gamma distribution is used for large scale fading channel with scale value of $\Omega = 1/m$. We used some specific parameters, e.g. path loss exponent is set to $\nu = 2.0$ and length of pilot symbols is set to $\tau = 4$, unless otherwise specified.

a) Impact of Transmit Power: In fig.1, two popular modulation schemes, namely, Binary Phase Shift King (BPSK) and Binary deferential Phase Shift Keying (BDPSK) are considered. For both techniques, bit error probability versus transmit power via Monte Carlo simulations and analytical expression provided using equations (20), (22) for single user and spacial averaging respectively. The analysis is carried out for perfect and imperfect CSI with fixed shadowing parameter m. The outputs of a Monte Carlo is obtained through generation of 10^4 gamma random realizations for the large scale fading matrix. The validation of the derived closed form expressions can be

$$\mathcal{P}_e^{(P)} = \tfrac{1}{2} - \frac{\Gamma(m+b)}{(2-b\nu)\Gamma(b+1)\Gamma(m)} \left\{ \frac{a^b R_o^2 (p_u N_r \Omega / R_o^{\nu})^b}{(R_o^2 - R_i^2)} \, {}_3F_2\left[\begin{matrix} b, b+m, b-\frac{2}{\nu} \\ b+1, b-\frac{2}{\nu}+1 \end{matrix} \middle| \frac{-p_u N_r \Omega}{R_o^{\nu}} \right] - \frac{a^b R_i^2 (p_u N_r \Omega / R_i^{\nu})^b}{(R_o^2 - R_i^2)} \, {}_3F_2\left[\begin{matrix} b, b+m, b-\frac{2}{\nu} \\ b+1, b-\frac{2}{\nu}+1 \end{matrix} \middle| -\frac{a p_u N_r \Omega}{R_i^{\nu}} \right] \right\} \quad (11)$$

$$\mathcal{P}_e^{(IP)} = \tfrac{1}{2} - \frac{\Gamma(b+\frac{m}{2})\Gamma(b+\frac{(m+1)}{2})}{2^{2-m}\sqrt{\pi}\Gamma(b+1)\Gamma(m)(b-1/\nu)} \left\{ \frac{a^b R_o^2 (4\tau p_u^2 N_r \Omega^2 / R_o^{2\nu})^b}{(R_o^2 - R_i^2)} \, {}_3F_2\left[\begin{matrix} b, b+\frac{m}{2}, b+\frac{m+1}{2}, b-\frac{1}{\nu} \\ b+1, b+1-\frac{1}{\nu} \end{matrix} \middle| -\frac{4a\tau p_u^2 N_r \Omega^2}{R_o^{2\nu}} \right] - \frac{a^b R_i^2 (4\tau p_u^2 N_r \Omega^2 / R_i^{2\nu})^b}{(R_o^2 - R_i^2)} \, {}_3F_2\left[\begin{matrix} b, b+\frac{m}{2}, b+\frac{m+1}{2}, b-\frac{1}{\nu} \\ b+1, b+1-\frac{1}{\nu} \end{matrix} \middle| -\frac{4a\tau p_u^2 N_r \Omega^2}{R_i^{2\nu}} \right] \right\} \quad (14)$$

observed where the simulation results agrees very well with the analytical results. As expected, channel imperfection increases error probability compared to full channel knowledge. Also, the figure shows the impact of user location on the uplink bit error probability. It can be seen that spacial averaging (red curves) causes in performance degradation compared to single user at distance of $600m$ i.e., cell-interior user (blue curves). Where spacial averaging takes into account the worst-case i.e., cell-edge users.

b) Impact of Pilot Length: Fig.2, demonstrates the impact of the pilot length used in channel estimation on the rate outage probability of user located at distance of $R = 600m$ from the base station with two values of channel shape parameter and two modulation schemes. It is seen that the performance gets better with high shadowing parameter m as well as with increasing number of symbols used for channel estimation τ.

c) Impact of Adding more Receiver Antenna: Fig.3, reveals the interesting implications of increasing the number of receive-antenna on the error performance with fixed and normalised transmit power p_u. In addition, the figure compares results for two shadowing parameter values m and two modulation techniques (dashed curves for BPSK, solid curves for BDPSK). Note that any increase in the number of antennas N_r, tends to increase the performance logarithmically, for both cases, perfect CSI and imperfect CSI. The relative difference between the curves gets steadily larger because of squaring effect, see (eq. 12) and this can quantify the total information loss due to imperfect CSI.

d) Impact of large-scale Fading: Fig.4, jointly compares the functionality among error probability, the shadowing shape parameter m and the size of the receive antenna array

e) Rate-Profile: Fig.5, considers the uplink outage performance. It is interesting to observe the effect of adding more receive antennas on rate CDF. Moreover, the figure depicts impact of channel imperfection, especially at the high outage probability regime, on the achievable user data rate threshold.

f) Impact of User Distance: Finally, fig.6, clearly shows the impact of user distance on the rate outage-probability for different values of shadowing parameter m and path-loss exponent ν at rate-threshold of $R_{th} = 2.5bit/s/Hz$. As expected, increasing ν causes curves divergence with distance increasing. On the other hand, decreasing m causes constant outage increasing for all user radii.

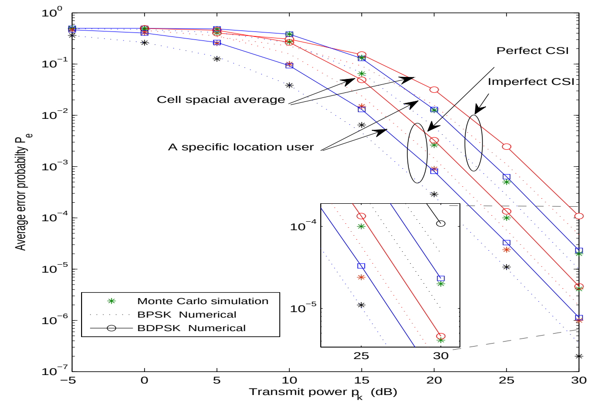

Fig. 1. Average bit error probability for single user located at $R_k = 600m$ compared with spacial averaged user (entire cell) with Perfect/Imperfect CSI and different modulation ($\nu = 3.3, m = 3.0, \tau = 4, N_r = 250$).

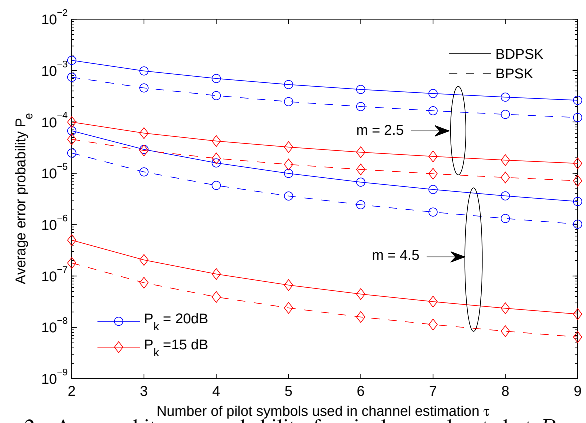

Fig. 2. Average bit error probability for single user located at $R_k = 600m$ versa number of symbols used in training sequence of channel estimation ($Nr = 150, \nu = 2.0$).

V. Conclusion

This paper studies the asymptotic power-scaling performance (average error probability and rate outage) of uplink transmission in large antennas regime. Specifically, closed form formulas are derived for the aforementioned performance metrics when BS uses linear ZF detector. In addition, impact of user location on the average error probability is characterised, where the derived formulas take into consideration the inevitable statistical-spatial-randomness of users distribution. The findings of this paper point out that imperfect CSI degrades both the corresponding error probability and rate outage. However, obtained results reveal that increasing the number of received antennas at the base station can significantly compensate for this deterioration in system performance.

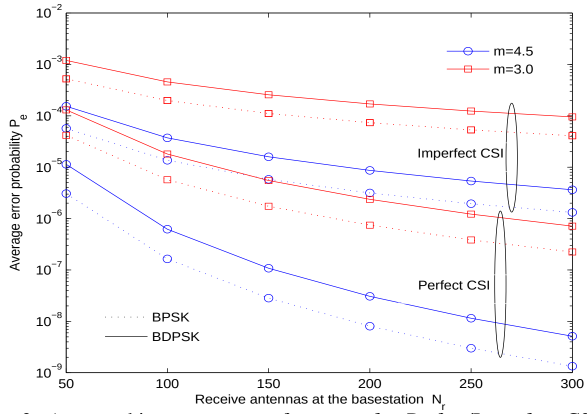

Fig. 3. Average bit error rate performance for Perfect/Imperfect CSI with different values of channel parameter m and two modulation schemes for single user located at $R_k = 600m$ ($P = 15dB, Nr = 128, \nu = 2.0, m = 2.5,\ \tau = 4$).

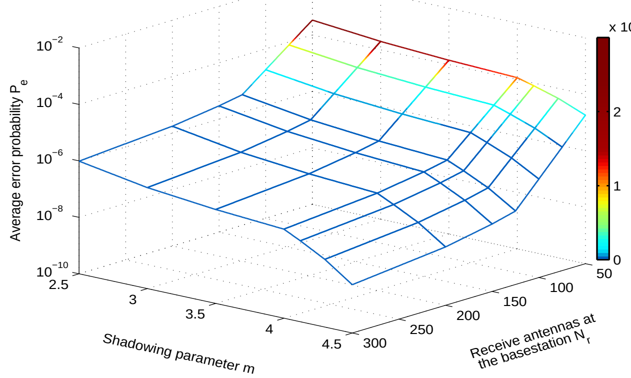

Fig. 4. The functional comparison among average bit error rate performance, channel parameter m and number of receive antennas for perfect CSI and differentially coherent DPSK detection ($R_k = 600m, P = 15dB, \nu = 2.0,\ a = 1, b = 1$)).

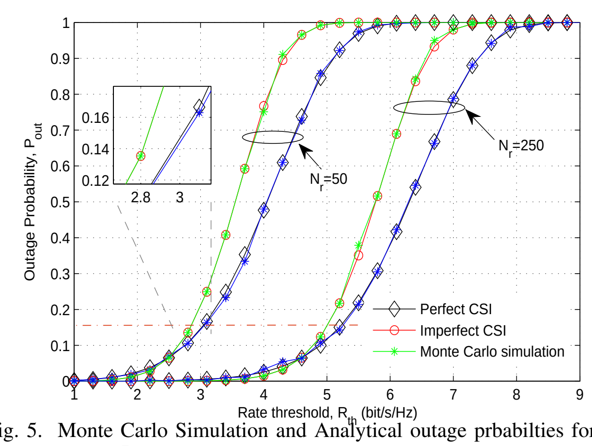

Fig. 5. Monte Carlo Simulation and Analytical outage prbabilties for single user located at $R_k = 600m$ from the base station with perfect/imperfect CSI and different number of base station antennas ($P = 15dB$, $\nu = 2.5$, $m = 2.5, \tau = 1$).

Appendix A.1

The unconditional average bit probability (BEP) can be obtained through averaging conditional error probability eq.(8) over the statistical distribution of the SNR that means finding

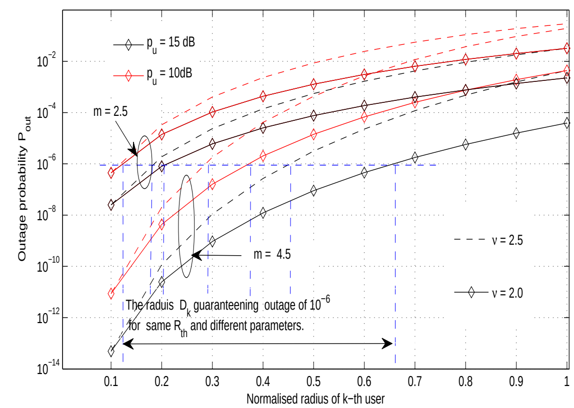

Fig. 6. Outage prbabilties versa normalised user locatition with perfect CSI and different system parameters ($N_r = 250,\ R_{th} = 2.5b/s/Hz$).

the expectation over shadowing distribution eq.(4) as following

$$\mathcal{P}_{e_k}^{(P)} = \mathbb{E}_\mu\left\{\frac{1}{2} - \frac{(a\gamma_k)^b}{2\Gamma(b+1)}\,{}_1F_1\left[\begin{matrix} b \\ b+1\end{matrix}\middle| -a\gamma_k\right]\right\}$$
$$= \int_0^\infty \frac{1}{2} - \frac{(axp_uN_r/D^\nu)^b}{2\Gamma(b+1)}\,{}_1F_1\left[\begin{matrix} b \\ b+1\end{matrix}\middle| -\frac{axp_uN_r}{D^\nu}\right]\frac{x^{m-1}e^{-x/\Omega}}{\Omega^m\Gamma(m)}\mathrm{d}x \qquad (18)$$

where first line of eq.(18) follows by using hypergeometric identity of incomplete gamma function [15, eq.(3.351)]. And $\mathbb{E}(\cdot)$ is the expectation operator. Which becomes in terms of pochhammer-symbol expression

$$\mathcal{P}_{e_k}^{(P)} = \int_0^\infty \frac{1}{2} - \frac{(axp_uN_r/D^\nu)^b}{2\Gamma(b+1)}\sum_{n=0}^\infty \frac{(b)_n}{n!\ (b+1)_n}\left(-\frac{axp_uN_r}{D^\nu}\right)^n \frac{x^{m-1}e^{-x/\Omega}}{\Omega^m\Gamma(m)}\mathrm{d}x \qquad (19)$$

Interchanging summation symbol with integration and re-arranging terms yields

$$\mathcal{P}_{e_k}^{(P)} = \frac{\Omega m}{2} - \frac{(ap_uN_r/D^\nu)^b}{2\Omega^m\Gamma(m)\Gamma(b+1)}\sum_{n=0}^\infty \frac{(b)_n}{n!\ (b+1)_n}$$
$$\times\left(-\frac{ap_uN_r}{D^\nu}\right)^n \underbrace{\int_0^\infty x^{n+b+m-1}e^{-x/\Omega}\mathrm{d}x}_{\mathcal{I}} \qquad (20)$$

Since we set $\Omega = 1/m$, so the mean Ωm is equal to one (a unit-mean gamma RV) and henceforth, we will use this normalisation. Now, with the aid of the primary definition of gamma function [15, eq.(3.326.2)] and pochhammer-symbol, the integration $\mathcal{I}$ can be evaluated as $\mathcal{I} = \Omega^{n+b+m}\Gamma(n+b+m) = \Omega^{n+b+m}\,\Gamma(b+m)\,(b+m)_n$
Next, substitution for $\mathcal{I}$ in eq.(20) gives

$$\mathcal{P}_{e_k}^{(P)} = \frac{1}{2} - \frac{(ap_uN_r/D^\nu)^b\Gamma(b+m)}{2\Omega^m\Gamma(m)\Gamma(b+1)}\ \sum_{n=0}^\infty \frac{(b)_n,(b+m)_n}{n!\ (b+1)_n}\ \left(-\frac{ap_uN_r}{D^\nu}\right)^n \qquad (21)$$

Then, invoking the definition of the generalized hypergeometric function will complete the proof.

Appendix A.2

Assuming uniform distributed users given in eq.(5), then spacial averaging of asymptotical error probability eq.(19) can be expressed as

$$\mathcal{P}_e^{(P)} = \frac{1}{2} - \frac{1}{(R_o^2 - R_i^2)}\int_{R_i}^{R_o} r\left\{\frac{(ap_uN_r/r^\nu)^b\Gamma(b+m)}{\Omega^m\Gamma(m)\Gamma(b+1)}\right.$$
$$\left.\times\sum_{n=0}^\infty \frac{(b)_n,(b+m)_n}{n!\ (b+1)_n}\ \left(-\frac{ap_uN_r}{r^\nu}\right)^n\right\}dr \qquad (22)$$

Using the basic definite integral identity $\int r^a dr = \dfrac{r^{a+1}}{a+1}$ result in

$$\mathcal{P}_e^{(P)} = \frac{1}{2} - \frac{1}{(R_o^2 - R_i^2)}\left| r^2\frac{(ap_uN_r/r^\nu)^b\Gamma(b+m)}{\Omega^m\Gamma(m)\Gamma(b+1)}\right.$$

$$\times \sum_{n=0}^{\infty} \frac{(b)_n, (b+m)_n}{n!\ (b+1)_n} \ (-\frac{ap_u N_r}{r^\nu})^n \Big|_{R_i}^{R_o} \quad (23)$$

Then, after the substitution of integration limits we conclude the proof using the definition of generalized hypergeometric function.

APPENDIX A.3

Invoking eq.(12) for the SNR and eq.(4) for the channel distribution, and with help of eq.([15, eq.(3.351)]), the average bit probability (BEP) in eq.(8) can be expressed as following

$$\mathcal{P}_{e_k}^{(IP)} = \int_0^\infty \frac{1}{2} - \frac{(a\tau x^2 p_u^2 N_r/D^{2\nu})^b}{2\Gamma(b+1)} \times {}_1F_1\left[\begin{smallmatrix} b \\ b+1 \end{smallmatrix}\Big| -\frac{a\tau x^2 p_u^2 N_r}{D^{2\nu}}\right] \frac{x^{m-1}e^{-x/\Omega}}{\Omega^m \Gamma(m)} \mathrm{d}x \quad (24)$$

similar to Appendix A.1, we have

$$\mathcal{P}_{e_k}^{(IP)} = \frac{1}{2} - \frac{(a\tau p_u^2 N_r/D^{2\nu})^b}{2\Omega^m \Gamma(m)\Gamma(b+1} \sum_{n=0}^{\infty} \frac{(b)_n}{n!\ (b+1)_n} \times (-\frac{a\tau p_u^2 N_r}{D^{2\nu}})^n \underbrace{\int_0^\infty x^{2n+2b+m-1} e^{-x/\Omega} \mathrm{d}x}_{\mathcal{I}_2} \quad (25)$$

With the aid of the identity [15, eq.(3.326.2)], the integration $\mathcal{I}_2$ can be evaluated as $\mathcal{I}_2 = \Omega^{2n+2b+m}\Gamma(2n+2b+m)$ Now, using multiplication theorem of gamma function [15, eq.(8.335)]

$$\begin{aligned}\Gamma(2n+2b+m) &= \Gamma(2(\frac{2b+m}{2}+n)) \\ &= \frac{2^{2b+m-1/2}}{\sqrt{2\pi}} 2^{2n}\Gamma(\frac{2b+m}{2}+n)\Gamma(\frac{2b+m+1}{2}+n) \end{aligned} \quad (26)$$

Plugging it again into eq.(25) yields

$$\mathcal{P}_{e_k}^{(IP)} = \frac{1}{2} - \frac{(\frac{4a\tau p_u^2 N_r}{D^{2\nu}})^b \Gamma(\frac{2b+m}{2})}{\Omega^m \Gamma(m)\Gamma(b+1)} \frac{\Gamma(\frac{2b+m+1}{2})}{2^{2-m}\sqrt{\pi}} \times \sum_{n=0}^{\infty} \frac{(b)_n\ (\frac{2b+m}{2})_n\ (\frac{2b+m+1}{2})_n}{n!\ (b+1)_n} (-\frac{4a\tau p_u^2 N_r}{D^{2\nu}})^n \quad (27)$$

Finally, with the aid of the generalized hypergeometric function definition we conclude the proof.

APPENDIX B.1

Simple observation of equations (3) and (13), one may deduce the following

$$\mathcal{P}_{out_k} = Pr\left(\frac{p_u N_r \mu_k}{D^\nu} < 2^{R_{th}} - 1\right) \quad (28)$$

Since μ_k is random variable (channel is RV with gamma distribution), therefore, the probability of this inequality is simply the commutative distribution function (CDF) for this RV

$$\mathcal{P}_{out_k} = \int_0^\chi \frac{x p_u N_r}{D^\nu} \frac{x^{m-1}e^{-x/\Omega}}{\Omega^m \Gamma(m)} \mathrm{d}x, \quad (29)$$

where the upper limit of the integration is $\chi = (2^{R_{th}} - 1)$. Then, setting new variable $z = x/\Omega$, we obtain

$$\mathcal{P}_{out_k} = \int_0^\chi \frac{z\Omega p_u N_r}{D^\nu} \frac{z^{m-1}e^{-z}}{\Gamma(m)} \mathrm{d}z \quad (30)$$

Now, invoking the lower incomplete gamma function identity and then express gamma function in term of confluent-hypergeometric-function [15, eq.(3.351)] $\gamma(a,z) = \frac{z^a}{a}\ {}_1F_1\left[\begin{smallmatrix} a \\ a+1 \end{smallmatrix}\Big| -z\right]$. And this concludes the proof after simple parameters and coefficients mapping.

APPENDIX B.2

From equations (11) and (13) we obtain

$$\mathcal{P}_{out_k} = Pr\left(\frac{\tau p_u^2 \mu_k^2 N_r}{D^{2\nu}} < 2^{R_{th}} - 1\right) \quad (31)$$

Employing similar procedure used in Appendix-C.1, we get the following

$$\mathcal{P}_{out_k}^{(IP)} = \int_0^\chi \frac{\tau(z\Omega p_u)^2 N_r}{D^{2\nu}} \frac{z^{m-1}e^{-z}}{\Gamma(m)} \mathrm{d}z \quad (32)$$

The proof can be completed by exploiting once more the identity [15, eq.(3.351)] with some straightforward algebraic manipulation.

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