A Resilient Approach to Distributed Filter Design for Time-Varying Systems under Stochastic Nonlinearities and Sensor Degradation

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***Abstract*—This paper is concerned with the distributed filtering problem for a class of discrete time-varying systems with stochastic nonlinearities and sensor degradation over a finite horizon. A two-step distributed filter algorithm is proposed where the sensor nodes collaboratively estimate the states of the plant by exploiting the information from both the local and neighboring nodes. The goal of this paper is to design the distributed filters over a wireless sensor network subject to given sporadic communication topology. Moreover, a resilient operation is guaranteed to suppress random perturbations on the actually implemented filter gains. An upper bound is first derived for the filtering error covariance by utilizing an inductive method and such an upper bound is subsequently minimized via iteratively solving a quadratic optimization problem. To account for the topological information of the sensor networks, a novel matrix simplification technique is utilized to preserve the sparsity of the gain matrices in accordance with the given topology and the analytical parameterization is obtained for the gain matrices of the desired sub-optimal filter. Furthermore, a sufficient condition is established to guarantee the mean-square boundedness of the estimation errors. Numerical simulation is carried out to verify the effectiveness of the proposed filtering algorithm.**

***Index Terms*—Resilient filter, wireless sensor networks, distributed filtering, stochastic nonlinearity, sensor degradation.**

I. INTRODUCTION

The state estimation or filtering problem has proven to be one of the fundamental issues in signal processing and control engineering, and a number of algorithms have been proposed in the literature, see e.g. [2], [4], [11], [12], [16], [19], [20], [28], [30], [33], [34]. Accordingly, a core problem with the widespread applications of wireless sensor networks (WSNs) is to estimate the plant states based on noisy measurement outputs from distributed nodes. A seemingly natural way is to employ the traditional Kalman filters by establishing a fusion center in WSNs in order to collect *all* the measurements from the individual sensors and then process the measurements in a *global* sense. Unfortunately, due to the limited communication capability and energy supply, it might be impossible for the sensors to persistently forward the local messages to the fusion center. As such, the so-called *distributed* estimation scheme would be more preferable whose main idea is to estimate the plant states based on both the local and the neighboring information according to the topologies of WSNs. Recently, various types of consensus protocols have been proposed with aim to improve the efficiency of the distributed computation and a rich body of literature has been available on the consensus-based distributed filtering strategies, see e.g. the seminal work in [25].

Up to now, much research effort has been made to the distributed estimation problems over sensor networks [5], [13], [23] and there have been mainly two general approaches available in the literature. The *first* one is so-called Kalman-consensus filtering approach (see [1], [21], [24]) where the distributed and cooperative filters are implemented by two steps, that is, the local sensors first generate the optimal estimates by using Kalman Filter and then a one(multi)-step consensus is performed to spread the local information over sensor networks. The stability and performance analysis of this filtering approach has been addressed in [21], [24]. As individual sensors cannot access to all the measurements, the performance of a Kalman-consensus filter is naturally inferior to that of the centralized Kalman filter. Nevertheless, as pointed out in [1], the performance of this distributed algorithm will asymptotically converge to that of the centralized one after a sufficiently large number of consensus steps. The *second* consensus-based filtering scheme focuses on suppressing the influence of external disturbances through designing cooperative filtering schemes [29]. For example, in [6], the distributed H_∞ state estimation problem has been investigated for discrete-time Markovian jump nonlinear time-delay systems with incomplete statistics of transition probability.

As is well known, nonlinearities exist in almost all practical systems and the corresponding research on nonlinear control problems has served as one of the mainstream areas in systems and control communities. In certain noisy environments such as networked control systems, the nonlinearity disturbances may result from randomly fluctuated network

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conditions and/or communication constraints. In this case, the so-called stochastic uncertainties would become inevitable that might lead to serious degradation of system performance if not properly dealt with. So far, there has been a growing research interest in analysis and synthesis issues for the systems with stochastic nonlinearities. Some representative results have been reported in [10], [26], [38] and the references therein, where sophisticated models have been proposed to characterize the random occurrence of the nonlinearities through the statistics (typically the first and second-order moments). On the other hand, in engineering practice, the phenomenon of sensor degradation may occur randomly as well, which is caused by various factors ranging from sensors aging and sensor intermittent failure to transmission congestions, see [22]. Some research effort has been initiated on the estimation problem with sensor degradations, see e.g. [8], [27], [31]. However, when it comes to the distributed estimation problems, the corresponding results have been very few, not to mention the case where the stochastic nonlinearities are also a concern. Such a situation gives the primary motivation of the present investigation.

Another critical issue of practical significance in state estimation problems is the resilience of the filters, which can be understood as the insensitivity against possible parameter variations/drifts in implementing the designed filters. In most previous works, the filter algorithms have been developed based on an implicit assumption that the designed filter gains can be *accurately* realized. Such an assumption is, unfortunately, not always true in practice due primarily to the finite resolution instrumentation during filter implementation and the round-off errors resulting from the fixed word length in numerical computation. Since the performance of filters can be extremely fragile to even tiny gain variations, it is of vital importance to design resilient filters capable of tolerating possible gain variations. In the past decade, much work has been done to ensure the resilience (or non-fragility) of the filters/estimators/controllers, see e.g. [8], [9], [18], [31], [35]–[37]. For example, the problem of robust non-fragile Kalman filter design has been studied in [35] for a class of linear systems with norm-bounded uncertainties, and some new criteria have been provided to guarantee the mean-square stability in terms of the solutions to algebraic Riccati equations. The minimum variance state estimation problems have been considered in [8], [31] for linear and nonlinear systems with both sensor failures and gain perturbations in the case of centralized filtering.

Summarizing the above discussions, it can be concluded that there is a lack of systematic investigation on the distributed estimation problem for systems subject to stochastic nonlinearities, sensor degradation as well as filter gain perturbations over wireless sensor networks with a given topology. As such, the main purpose of this paper is to shorten such a gap by designing distributed filters that are resilient to filter implementation errors and robust to sensor degradations. The main contributions can be highlighted as follows: 1) the system under consideration is quite general that covers stochastic nonlinearities and sensor degradation; 2) a resilient distributed filtering algorithm is proposed so as to mitigate the adverse effects induced by filter gain variations; 3) a matrix simplification approach is exploited in the filter design algorithm to overcome the difficulties resulting from the sparsity of the sensor networks; and 4) a criterion is established for the mean-square boundedness of the estimator errors for the designed time-varying distributed resilient filter.

Notation. Except where otherwise stated, the notations used throughout the paper are standard. $\|\cdot\|$ is the Euclidian norm of real vectors or the spectral norm of real matrices and $\|\cdot\|_{\min}$ represents the smallest singular value of a matrix.. M' denotes the transpose of a matrix M, and I represents the identity matrix of appropriate dimensions. For $i = 1, 2, \cdots, n$, the notation $\mathrm{diag}_n\{A_i\}$ stands for a block-diagonal matrix where the square matrices A_i are in the corresponding main diagonal blocks. $\mathrm{col}_n\{x_i\}$ stacks the vectors as $[x_1', \cdots, x_N']'$, and $\{M_{ij}\}_{n\times n}$ represents a partitioned matrix, where M_{ij} is the (i, j)-th block submatrix. $\mathbb{N}$ represents the set of natural numbers and $\mathbb{E}\{.\}$ stands for the expectation of a stochastic variable $\{.\}$.

II. PROBLEM FORMULATION

A. Target plant and sensor network

In this paper, a sensor network consisting of n sensor nodes is exploited to measure the output of the target plant. We denote the topology of the network by a directed graph $G = (V, E, \mathcal{H})$ of order n with the set of nodes $V = \{1, 2, \cdots, n\}$, the set of edges $E \subseteq V \times V$, and the weighted adjacency matrix $\mathcal{H} = [a_{ij}]_{n\times n}$. The weighted adjacency matrix of the graph is a matrix with nonnegative elements a_{ij} satisfying the property $a_{ij} > 0 \iff (i, j) \in E$, which means that the ith node can receive the information from the jth node. All the neighbors of node i plus the node itself are denoted by the set as $N_i \triangleq \{j \in V | (i, j) \in E\}$.

Consider the following discrete time-varying target plant with stochastic nonlinearities:

$$x(k+1) = A(k)x(k) + f(k, x(k), \xi(k)) + w(k), \quad (1)$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector that cannot be measured directly, $f(k, x(k), \xi(k)) \in \mathbb{R}^{n_x}$ is the stochastic nonlinearities to be defined later, and $w(k) \in \mathbb{R}^{n_x}$ is a sequence of Gaussian random variables with zero mean value and covariance matrix $Q(k) > 0$. $A(k)$ is a known time-varying matrix of appropriate dimensions. The initial condition $x(0)$ is assumed to obey a Gaussian distribution with mean μ_0 and covariance matrix Σ_0.

For the ith ($i = 1, 2, \cdots, n$) sensor node, the measurement is described by:

$$y_i(k) = \gamma_i(k)C_i(k)x(k) + g_i(k, x(k), \zeta_i(k)) + v_i(k) \quad (2)$$

where $y_i(k) \in \mathbb{R}^{n_y}$ stands for the measurement information from sensor i and the measure noise $v_i(k) \in \mathbb{R}^{n_y}$ obeys a Gaussian distribution with zero mean value and covariance matrix $R_i(k) > 0$. The variable $\gamma_i(k)$ accounting for the sensor gain degradation has the probability density function $p_k^i(.)$ on the interval $[0, 1]$ with mean $\bar{\gamma}_i(k)$ and variance $\sigma_{\gamma_i}^2(k)$. $C_i(k)$ is a known time-varying matrix of appropriate dimensions.

The functions $f(k,x(k),\xi(k)) \in \mathbb{R}^{n_x}$ and $g_i(k,x(k),\zeta_i(k)) \in \mathbb{R}^{n_y}$ represent the stochastic nonlinearities satisfying $f(k,0,\xi(k)) = 0$, $g_i(k,0,\zeta_i(k)) = 0$ and the following statistics:

$$\begin{aligned} \mathbb{E}\{f(k,x(k),\xi(k))|x(k)\} = 0, \\ \mathbb{E}\{g_i(k,x(k),\zeta_i(k))|x(k)\} = 0, \end{aligned} \tag{3}$$

$$\mathbb{E}\left\{g_i(k,x(k),\zeta_i(k))g_j'(k,x(k),\zeta_j(k))|x(k)\right\} = 0, \quad i \neq j,$$

$$\mathbb{E}\left\{\begin{bmatrix} f(k,x(k),\xi(k)) \\ g_i(k,x(k),\zeta_i(k)) \end{bmatrix}\begin{bmatrix} f(s,x(s),\xi(s)) \\ g_i(s,x(s),\zeta_i(s)) \end{bmatrix}'\middle| x(k)\right\} = 0, \quad k \neq s,$$

$$\begin{aligned} &\mathbb{E}\left\{\begin{bmatrix} f(k,x(k),\xi(k)) \\ g_i(k,x(k),\zeta_i(k)) \end{bmatrix}\begin{bmatrix} f(k,x(k),\xi(k)) \\ g_i(k,x(k),\zeta_i(k)) \end{bmatrix}'\middle| x(k)\right\} \\ &\qquad = \sum_{s=1}^{m} \Pi_s(k)x'(k)\Gamma_s(k)x(k), \end{aligned} \tag{4}$$

where m is a given positive integer, and $\Pi_s(k) = \mathrm{diag}\left\{\Pi_{s1}(k), \Pi_{s2}^{(i)}(k)\right\}$. $\Pi_{s1}(k)$, $\Pi_{s2}^{(i)}(k)$ and $\Gamma_s(k)$ are known matrices with compatible dimensions for $s = 1, 2, \cdots, m$.

B. Distributed resilient filter

A fundamental issue in wireless sensor networks is to design the filters so as to restore the state vector in a cooperative behavior. Note that, in practical applications, gain variations often occur in the implementation of a filter due probably to computational or tuning uncertainties. Since the performance of the filter may be susceptible to the perturbations in gain parameters, the design of resilient filters capable of tolerating some level of gain variations is of engineering significance.

To observe the target plant through a network of interconnected sensors, a two-step distributed estimator is proposed as follows:

$$\hat{x}_i(k|k-1) = A(k-1)\hat{x}_i(k-1|k-1), \tag{5a}$$

$$\hat{x}_i(k|k) = \hat{x}_i(k|k-1) + \sum_{j \in N_i} a_{ij}(G_{ij}(k) + \Delta_{ij}(k))\tilde{y}_j(k), \tag{5b}$$

with the initial value $\hat{x}_i(0|0) = \mathbb{E}[x(0)] = \mu_0$, for $i \in V$. Note that $\hat{x}_i(k|k-1)$ and $\hat{x}_i(k|k)$ are the one-step prediction and the estimate of state vector $x(k)$, respectively. $\tilde{y}_i(k) = y_i(k) - \bar{\gamma}_i(k)C_i(k)\hat{x}_i(k|k-1)$ is the innovation sequence exchanged via the network. The matrix $G_{ij}(k) \in \mathbb{R}^{n_x \times n_y}$ represents the gain coefficients of the filters to be designed. The term $\Delta_{ij}(k) \in \mathbb{R}^{n_x \times n_y}$ models the computational or implementation error associated with the estimator gain, and is assumed to have zero mean and a bounded second moment, i.e.,

$$E[\Delta_{ij}(k)] = 0, \quad E[\Delta_{ij}(k)\Delta_{ij}'(k)] \leq \delta_{ij}I, \tag{6}$$

where δ_{ij} is a positive scalar. Moreover, throughout the paper, we assume that all the stochastic variables, i.e., $\Delta_{ij}(k)$, $\gamma_i(k)$, $\xi(k)$, $\zeta_i(k)$, $x(0)$, $w(k)$ and $v_i(k)$, are white and mutually independent.

For the convenience of later development, let us define the local state prediction and local state estimation error vectors, respectively, as follows:

$$\begin{aligned} e_i(k|k-1) \triangleq x(k) - \hat{x}_i(k|k-1) \\ e_i(k|k) \triangleq x(k) - \hat{x}_i(k|k). \end{aligned}$$

Substituting (5a) into the state prediction error equation yields

$$e_i(k+1|k) = A(k)e_i(k|k) + f(k,x(k),\xi(k)) + w(k), \tag{7}$$

and it can then be seen from (5b) that the dynamics of the estimation errors evolves according to

$$\begin{aligned} e_i(k|k) =& e_i(k|k-1) - \sum_{j \in N_i} a_{ij}(G_{ij}(k) + \Delta_{ij}(k))\Big\{v_j(k) \\ &+ g_j(k,x(k),\zeta_j(k)) + \tilde{\gamma}_j(k)C_j(k)x(k) \\ &+ \bar{\gamma}_j(k)C_j(k)e_j(k|k-1)\Big\}, \end{aligned} \tag{8}$$

where $\tilde{\gamma}_i(k) = \gamma_i(k) - \bar{\gamma}_i(k)$. For the sake of simplicity, we denote

$$\begin{aligned} e(k|k-1) \triangleq \mathrm{col}_n\{e_i(k|k-1)\}, \\ e(k|k) \triangleq \mathrm{col}_n\{e_i(k|k)\}, \end{aligned}$$

and then (7)-(8) can be rearranged into a more compact form as follows:

$$\begin{aligned} e(k+1|k) =& \mathcal{A}(k)e(k|k) + \vec{f}(k) + \vec{w}(k), \\ e(k|k) =& e(k|k-1) - \sum_{i=1}^{n} E_i(G(k) + \Delta(k))H_i\Big\{\vec{g}(k) + \vec{v}(k) \\ &+ \tilde{\Gamma}(k)\mathcal{C}(k)\vec{x}(k) + \bar{\Gamma}(k)\mathcal{C}(k)e(k|k-1)\Big\}, \end{aligned}$$

where

$$\begin{aligned} &\mathcal{A}(k) \triangleq \mathrm{diag}_n\{A(k)\}, \ \vec{f}(k) \triangleq \mathrm{col}_n\{f(k,x(k),\xi(k))\}, \\ &\mathcal{C}(k) \triangleq \mathrm{diag}_n\{C_i(k)\}, \ \vec{g}(k) \triangleq \mathrm{col}_n\{g_i(k,x(k),\zeta_i(k))\} \\ &\vec{w}(k) \triangleq \mathrm{col}_n(w(k)), \ H_i = \mathrm{diag}\{a_{i1}I, \cdots, a_{in}I\} \\ &\Delta(k) \triangleq \{\Delta_{ij}(k)\}_{n \times n}, \ G(k) \triangleq \{G_{ij}(k)\}_{n \times n} \\ &\vec{v}(k) \triangleq \mathrm{col}_n\{v_i(k)\}, \ \tilde{\Gamma}(k) \triangleq \mathrm{diag}_n\{\tilde{\gamma}_i(k)I\}, \\ &\vec{x}(k) \triangleq \mathrm{col}_n(x(k)), \ \bar{\Gamma}(k) \triangleq \mathrm{diag}_n\{\bar{\gamma}_i(k)I\} \\ &E_i \triangleq \mathrm{diag}\{\underbrace{0, \cdots, 0}_{i-1}, I, \underbrace{0, \cdots, 0}_{n-i}\}. \end{aligned}$$

Furthermore, by letting $K(k) = -\sum_{i=1}^{n} E_i(G(k)+\Delta(k))H_i$, we have

$$\begin{aligned} e(k|k) =& (I + K(k)\bar{\Gamma}(k)\mathcal{C}(k))e(k|k-1) \\ &+ K(k)\Big\{\vec{g}(k) + \vec{v}(k) + \tilde{\Gamma}(k)\mathcal{C}(k)\vec{x}(k)\Big\} \end{aligned} \tag{9}$$

To quantify the transient performance of the proposed distributed resilient filter, a finite horizon quadratic filtering cost function is introduced for the wireless sensor networks as follows:

$$J_T(\mathbf{G}(T)) = \sum_{k=0}^{T}\sum_{i=1}^{n} \mathbb{E}[e_i'(k|k)e_i(k|k)] \tag{10}$$

where the set $\mathbf{G}(T)=\{G(k),\ k=1,2,\cdots,T-1\}$ gathers the filter coefficients in all the T steps. Define the error covariances as $P_{k|k-1}\triangleq\mathbb{E}[e(k|k-1)e'(k|k-1)]$ and $P_{k|k}\triangleq\mathbb{E}[e(k|k)e'(k|k)]$. Obviously, the above quadratic filtering cost function can be rewritten as $J_T(\mathbf{G}(T))=\sum_{k=0}^{T}\operatorname{tr}\left\{P_{k|k}\right\}$. In the paper, we aim to design the optimal distributed filters by solving the following optimization problem

$$J_T=\arg\min_{G(T-1)} J_T(\mathbf{G}(T)) \tag{11}$$

III. PRELIMINARY

In this section, some preliminary knowledge is derived for preparation. At the very beginning, the following lemmas are introduced, which will be used to establish our main results.

Lemma 1 ([7]): Let $D=[d_{ij}]_{p\times p}$ be a real-valued matrix and $B=\operatorname{diag}\{b_1,b_2,\cdots,b_p\}$ be a diagonal random matrix. Then

$$\mathbb{E}[BDB']=\begin{pmatrix}\mathbb{E}[b_1^2] & \mathbb{E}[b_1b_2] & \cdots & \mathbb{E}[b_1b_p]\\ \mathbb{E}[b_2b_1] & \mathbb{E}[b_2^2] & \cdots & \mathbb{E}[b_2b_p]\\ \vdots & \vdots & \ddots & \vdots\\ \mathbb{E}[b_pb_1] & \mathbb{E}[b_pb_2] & \cdots & \mathbb{E}[b_p^2]\end{pmatrix}\odot D,$$

where $\odot$ is the Hadamard product.

Lemma 2: Consider a discrete time-varying process with stochastic nonlinearities given by (1). The system state covariance $X(k)\triangleq E[x(k)x'(k)]$ satisfies the following recursive equation:

$$\begin{aligned}X(k+1)=&A(k)X(k)A'(k)\\&+\sum_{s=1}^{m}\Pi_{s1}(k)\operatorname{tr}\left\{X(k)\Gamma_s(k)\right\}+Q(k).\end{aligned}$$

Proof: Substituting (1) into $X(k+1)$ yields

$$\begin{aligned}&X(k+1)\\&=A(k)X(k)A'(k)+\mathbb{E}[f(k)f'(k)]+\mathbb{E}[w(k)w'(k)],\end{aligned}$$

where the cross terms vanish because $x(k)$, $f(k)$ and $w(k)$ are mutually independent. From (4), it can be seen that

$$\mathbb{E}[f(k)f'(k)]=\sum_{s=1}^{m}\Pi_{s1}(k)\mathbb{E}[x'(k)\Gamma_s(k)x(k)].$$

Note that $\mathbb{E}[x'(k)\Gamma_s(k)x(k)]$ is a scalar, so its value will not be changed by taking its trace as follows:

$$\mathbb{E}[x'(k)\Gamma_s(k)x(k)]=\operatorname{tr}\left\{X(k)\Gamma_s(k)\right\},$$

which concludes the proof. ■

The following lemma gives the dynamic evolution of the prediction error covariance.

Lemma 3: Given the error covariance $P_{k|k}>0$ at step k, the prediction error covariance satisfies the following dynamic equation:

$$P_{k+1|k}=\mathcal{A}(k)P_{k|k}\mathcal{A}'(k)+\mathcal{F}(k)+\mathcal{W}(k), \tag{12}$$

where

$$\mathcal{F}(k)=\mathbf{1}_n\otimes\sum_{s=1}^{m}\Pi_{s1}(k)\operatorname{tr}\left\{X(k)\Gamma_s(k)\right\},$$

$$\mathcal{W}(k)=\mathbf{1}_n\otimes Q(k).$$

Here, $\mathbf{1}_n\in\mathbb{R}^{n\times n}$ is a square matrix with all the elements equal to one and $\otimes$ is the Kronecker product.

Proof: The proof of this lemma is similar to Lemma 2, and thus is omitted here for sake of brevity. ■

Lemma 4: The two-step distributed filters given by (5a)-(5b) are unbiased and the corresponding estimation error covariance can be given as follows:

$$\begin{aligned}P_{k|k}=&\mathbb{E}[(I+K(k)\bar{\Gamma}(k)\mathcal{C}(k))P_{k|k-1}(I+K(k)\bar{\Gamma}(k)\mathcal{C}(k))']\\&+\mathbb{E}[K(k)\vec{g}(k)\vec{g}'(k)K'(k)]+\mathbb{E}[K(k)\vec{x}(k)\vec{x}'(k)K(k)]\\&+\mathbb{E}[K(k)\tilde{\Gamma}(k)\mathcal{C}(k)\vec{x}(k)\vec{x}'(k)\mathcal{C}'(k)\tilde{\Gamma}'(k)K'(k)]\end{aligned} \tag{13}$$

Proof: In view of the initial estimate $\hat{x}_i(0|0)=\mathbb{E}\{x(0)\}$, it is obvious that $\mathbb{E}\{e(0|0)\}=0$. Using the fact that the stochastic nonlinearities and measurement noises are of zero means, we obtain $\mathbb{E}\{e(1|0)\}=0$. Furthermore, we have $\mathbb{E}\{e(1|1)\}=0$ since $\mathbb{E}[\tilde{\gamma}_i(k)]=0$. By repeating such a procedure, it can be concluded that $\mathbb{E}[e(k|k-1)]=0,\ \ \mathbb{E}[e(k|k)]=0$. Thus, the unbiasedness of the proposed distributed filters can be guaranteed. As for the error covariance, by applying (9), we arrive at the following equation

$$\begin{aligned}P_{k|k}=&\mathbb{E}\left[(I+K(k)\bar{\Gamma}(k)\mathcal{C}(k))P_{k|k-1}(I+K(k)\bar{\Gamma}(k)\mathcal{C}(k))'\right]\\&+\mathscr{P}+\mathscr{P}'+\mathbb{E}\left[K(k)\vec{g}(k)\vec{g}'(k)K'(k)\right]+\mathscr{L}+\mathscr{L}'\\&+\mathbb{E}\left[K(k)\vec{v}(k)\vec{v}'(k)K(k)\right]+\mathscr{R}+\mathscr{R}'\\&+\mathbb{E}\left[K(k)\tilde{\Gamma}(k)\mathcal{C}(k)\vec{x}(k)\vec{x}'(k)\mathcal{C}'(k)\tilde{\Gamma}'(k)K'(k)\right],\end{aligned}$$

where

$$\begin{aligned}\mathscr{P}=&\mathbb{E}[(I+K(k)\bar{\Gamma}(k)\mathcal{C}(k))e(k|k-1)(\vec{g}(k)+\vec{v}(k),\\&+\tilde{\Gamma}(k)\mathcal{C}(k)\vec{x}(k))'K'(k)],\\\mathscr{L}=&\mathbb{E}[K(k)\vec{g}(k)\vec{v}'(k)K'(k)]+\mathbb{E}[K(k)\vec{g}(k)\vec{x}'(k)\mathcal{C}'(k)\tilde{\Gamma}'(k)]\\\mathscr{R}=&\mathbb{E}[K(k)\vec{v}(k)\vec{x}'(k)\mathcal{C}'(k)\tilde{\Gamma}'(k)K'(k)].\end{aligned}$$

Noting that the prediction error vector $e(k|k-1)$ is uncorrelated with $\vec{g}(k)$, $\vec{v}(k)$ and $\tilde{\Gamma}(k)$, we have the term $\mathscr{P}$ vanished. Also, exploiting the fact that the noises $\vec{g}(k)$ and $\vec{v}(k)$ are independent with each other and the initial state is $x(0)$, one can derive that $\mathscr{L}=0$ and $\mathscr{R}=0$, which ends the proof. ■

In the above lemma, the dynamics of the error covariance is presented, which turns out to be dependent on determined by the network topology, the state transition matrix, the measurement matrices and the statistics of stochastic nonlinearities, sensor gain degradations and external disturbances. Unfortunately, due primarily to the existence of gain variations in this paper, it is impossible to solve the error covariance analytically, not to mention the further design of the optimal gain coefficients. An alternative yet effective way for designing the filters is to establish an upper bound on the estimation error covariance.

Suppose that, for all $k\geq 0$, there exist positive definite matrices $M_{k|k}\geq P_{k|k}$. Define a finite horizon quadratic cost function as follows:

$$\bar{J}_T(\mathbf{G}(T))=\sum_{k=0}^{T}\operatorname{tr}\left\{M_{k|k}\right\}.$$

It is clear that $J_T(\mathbf{G}(T)) \le \bar{J}_T(\mathbf{G}(T))$ which implies that $\bar{J}_T(\mathbf{G}(T))$ is an upper bound on the original cost function. As a result, we can focus on minimizing this upper bound by appropriately choosing the filter parameters, namely,

$$\bar{J}_T = \arg\min_{G(T-1)} \bar{J}_T(\mathbf{G}(T)).$$

A distinguished advantage of the above proposed scheme lies in that it can solve some sort of optimization problems where the analytical expression of the objective function is unavailable. By constructing an analytical upper bound, one can provide an alternative, feasible, yet sub-optimal option for the filter design under guaranteed performance.

IV. Sub-optimal distributed resilient filter design

In this section, let us first derive an analytical upper bound for the estimation error covariance of the system (9), and then design the gain $G(k)$ of the sub-optimal filter in order to minimize the upper bound at each time-step k.

For presentation convenience, some notations are introduced as follows

$$\begin{aligned}&\bar{K}(k) \triangleq -\sum_{i=1}^{n} E_i G(k) H_i,\ \ \tilde{K}(k) \triangleq -\sum_{i=1}^{n} E_i \Delta(k) H_i,\\ &\Upsilon(k) \triangleq \vec{g}(k)\vec{g}'(k) + \vec{v}(k)\vec{v}'(k)\\ &\qquad\qquad + \tilde{\Gamma}(k)\mathcal{C}(k)\vec{x}(k)\vec{x}'(k)\mathcal{C}'(k)\tilde{\Gamma}'(k).\end{aligned}$$

Moreover, from the definition of the stochastic nonlinearity $g_i(k, x(k), \zeta_i(k))$, we have

$$\aleph(k) \triangleq \mathbb{E}[\vec{g}(k)\vec{g}'(k)] = \sum_{s=1}^{m} \Pi_{s2}(k)\mathrm{tr}[X(k)\Gamma_s(k)],$$

where $\Pi_{s2}(k) \triangleq \mathrm{diag}\left\{\Pi_{s2}^{(1)}(k), \cdots, \Pi_{s2}^{(n)}(k)\right\}$. Additionally, in light of Lemma 1, we have

$$\begin{aligned}&\mathbb{E}\Big[\tilde{\Gamma}(k)\mathcal{C}(k)\vec{x}(k)\vec{x}'(k)\mathcal{C}'(k)\tilde{\Gamma}'(k)\Big]\\ &\qquad\qquad = \Xi(k) \odot \big(\mathcal{C}(k)\mathcal{X}(k)\mathcal{C}'(k)\big),\end{aligned}$$

where $\mathcal{X}(k) \triangleq \mathbf{1}_n \otimes X(k)$ and $\Xi(k) \triangleq \mathrm{diag}\{\sigma_{\gamma_1}^2(k)\mathbf{1}_n, \cdots, \sigma_{\gamma_n}^2(k)\mathbf{1}_n\}$. Therefore, the mean value $\bar{\Upsilon}(k) \triangleq \mathbb{E}[\Upsilon(k)]$ can be computed by

$$\bar{\Upsilon}(k) = \aleph(k) + \mathrm{diag}_n\{R_i(k)\} + \Xi(k) \odot \big(\mathcal{C}(k)\mathcal{X}(k)\mathcal{C}'(k)\big).$$

Now, we can derive the upper bound of $P_{k|k}$ in the following theorem.

Theorem 1: Consider the following difference equations

$$M_{k+1|k} = \mathcal{A}(k)M_{k|k}\mathcal{A}'(k) + \mathcal{F}(k) + \mathcal{W}(k), \tag{14a}$$

$$\begin{aligned}M_{k|k} =&(I + \bar{K}(k)\bar{\Gamma}(k)\mathcal{C}(k))M_{k|k-1}(I + \bar{K}(k)\bar{\Gamma}(k)\mathcal{C}(k))'\\ &+ \lambda_{max}(\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k) + \bar{\Upsilon}(k))\Lambda\\ &+ \bar{K}(k)\bar{\Upsilon}(k)\bar{K}'(k)\end{aligned} \tag{14b}$$

with the initial condition $M_{0|0} = P_{0|0} = \Sigma_0$, where $\Lambda \triangleq \mathrm{diag}\left\{\sum_{s=1}^{n} a_{1s}^2\delta_{1s}I, \cdots, \sum_{s=1}^{n} a_{ns}^2\delta_{ns}I\right\}$. Then, the inequalities $P_{k|k} \le M_{k|k}$ and $P_{k+1|k} \le M_{k+1|k}$ always hold for all $k \ge 0$.

Proof: Since the uncertainty $\Delta(k)$ is of zero mean and independent with other stochastic variables, (13) can be rewritten in the following form:

$$\begin{aligned}P_{k|k} = &(I + \bar{K}(k)\bar{\Gamma}(k)\mathcal{C}(k))P_{k|k-1}(I + \bar{K}(k)\bar{\Gamma}(k)\mathcal{C}(k))'\\ &+ \mathbb{E}[\tilde{K}(k)\bar{\Gamma}(k)\mathcal{C}(k)P_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k)'\tilde{K}'(k)]\\ &+ \mathbb{E}[\tilde{K}(k)\Upsilon(k)\tilde{K}'(k)] + \bar{K}(k)\bar{\Upsilon}(k)\bar{K}'(k).\end{aligned} \tag{15}$$

Subsequently, let us prove this theorem by induction. Assume, inductively, that $P_{k-1|k-1} \le M_{k-1|k-1}$. Applying (12) and (14a), we have

$$\begin{aligned}&P_{k|k-1} - M_{k|k-1}\\ &= \mathcal{A}(k-1)(P_{k-1|k-1} - M_{k-1|k-1})\mathcal{A}'(k-1) \le 0,\end{aligned}$$

which implies $P_{k|k-1} \le M_{k|k-1}$. The difference $P_{k|k} - M_{k|k}$ can be written as

$$\begin{aligned}&P_{k|k} - M_{k|k}\\ &\le \mathbb{E}\Big[\tilde{K}(k)(\bar{\Gamma}(k)\mathcal{C}(k)P_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k)' + \Upsilon(k))\tilde{K}'(k)\Big]\\ &\qquad - \lambda_{max}\Big(\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k) + \bar{\Upsilon}(k)\Big)\Lambda.\end{aligned} \tag{16}$$

Moreover, since $P_{k|k-1} \le M_{k|k-1}$, it follows that

$$\begin{aligned}&\mathbb{E}\Big[\tilde{K}(k)(\bar{\Gamma}(k)\mathcal{C}(k)P_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k)' + \Upsilon(k))\tilde{K}'(k)\Big]\\ \le&\lambda_{\max}\Big(\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k) + \bar{\Upsilon}(k)\Big)\mathbb{E}\left[\tilde{K}(k)\tilde{K}'(k)\right].\end{aligned}$$

Now, we are in the position to tackle the term in the right-hand side of the above equation. Utilizing algebraic transformations, it is not difficult to verify that

$$\begin{aligned}\tilde{K}(k)\tilde{K}'(k) &= \left(\sum_{i=1}^{n} E_i\Delta(k)H_i\right)\left(\sum_{i=1}^{n} E_i\Delta(k)H_i\right)'\\ &= \left\{\left(\sum_{s=1}^{n} a_{is}a_{js}\Delta_{is}(k)\Delta_{js}'(k)\right)_{ij}\right\}_{n\times n}.\end{aligned}$$

Based on (6) and the fact that $\Delta_{ij}(k)$ are mutually independent for $i, j = 1, \cdots, n$, one has

$$\mathbb{E}\left[\tilde{K}(k)\tilde{K}'(k)\right] \le \Lambda. \tag{17}$$

Together with (16)-(17), we can see $P_{k|k} \le M_{k|k}$. The inductive hypothesis implies that $P_{k|k} \le M_{k|k}$, which completes the proof. ∎

In the next step, we will design the optimal filter gains such that the upper bound $P_{k|k}$ can be minimized at each step. Before proceeding further, let us define $G^{(i)}(k)$ to be the ith row of the block matrix $G(k)$, i.e.,

$$G^{(i)}(k) \triangleq \left[G_{i1}(k), \cdots, G_{in}(k)\right]$$

and $M_{k|k-1}^{(i)}$ to be the ith row of the block matrix $M_{k|k-1}$. Moreover, define

$$\begin{aligned}&\mathcal{M}_i(k) \triangleq H_i\Big[\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k) + \bar{\Upsilon}(k)\Big]H_i,\\ &\mathcal{N}_i(k) \triangleq M_{k|k-1}^{(i)}\mathcal{C}'(k)\bar{\Gamma}(k)H_i.\end{aligned}$$

By removing the b-th ($b \notin N_i$) column block from the matrices $\mathcal{N}_i(k)$ and $G^{(i)}(k)$, one can obtain $\bar{\mathcal{N}}_i(k)$ and $\bar{G}^{(i)}(k)$, respectively. In addition, we let $\bar{\mathcal{M}}_i(k)$ be a simplified matrix by removing both the b-th row and b-th column block from $\mathcal{M}_i(k)$ when $b \notin N_i$.

Theorem 2: Consider the time-varying system (1)-(2) with distributed resilient filters given by (5a)-(5b). The upper bound of the error covariance (14a)-(14b) can be minimized at each step by choosing the parameters of filters as follows

$$G_{ij}(k) = \begin{cases} 0 & a_{ij} = 0 \\ \left(\bar{\mathcal{N}}_i(k)\bar{\mathcal{M}}_i(k)^{-1}\right)^\sharp, & a_{ij} \neq 0 \end{cases} \tag{18}$$

where $(*)^\sharp$ extracts the corresponding submatrix from the matrix '$*$' associated with the parameter $G_{ij}(k)$.

Proof: Taking the trace for the both sides of (14b) yields that

$$\begin{aligned} &\operatorname{tr}\{M_{k|k}\} \\ =&\operatorname{tr}\Big\{(I+\bar{K}(k)\bar{\Gamma}(k)\mathcal{C}(k))M_{k|k-1}(I+\bar{K}(k)\bar{\Gamma}(k)\mathcal{C}(k))'\Big\} \\ &+\operatorname{tr}\big\{\bar{K}(k)\bar{\Upsilon}(k)\bar{K}'(k)\big\}+\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^2\delta_{ij}n_x \\ &\times\lambda_{max}\Big(\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k)+\bar{\Upsilon}(k)\Big). \end{aligned} \tag{19}$$

The first term in the right-hand side of (19) can be rewritten into the following expression

$$\begin{aligned} &\operatorname{tr}\Big\{(I+\bar{K}(k)\bar{\Gamma}(k)\mathcal{C}(k))M_{k|k-1}(I+\bar{K}(k)\bar{\Gamma}(k)\mathcal{C}(k))'\Big\} \\ &=\operatorname{tr}\{M_{k|k-1}\}+2\operatorname{tr}\{\bar{K}(k)\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\} \\ &\quad+\operatorname{tr}\{\bar{K}(k)\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k)'\bar{K}'(k)\}. \end{aligned}$$

Resorting to the properties of trace, we have

$$\begin{aligned} &\operatorname{tr}\{E_iG(k)H_iMH_iG'(k)E_j\} \\ &=\operatorname{tr}\{E_jE_iG(k)H_iMH_iG'(k)\}=0, \quad \text{for } i\neq j \end{aligned} \tag{20}$$

for an arbitrary matrix M with appropriate dimensions. Noticing the definition of $\bar{K}(k)$ and exploiting (20), it is obvious that

$$\begin{aligned} &\operatorname{tr}\big\{\bar{K}(k)\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k)\bar{K}'(k)\big\} \\ &=\operatorname{tr}\left\{\sum_{i=1}^{n}E_iG(k)H_i\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k)H_iG'(k)\right\}. \end{aligned}$$

As for the second term in the right-hand side of (19), one can derive that

$$\begin{aligned} &\operatorname{tr}\{\bar{K}(k)\bar{\Upsilon}(k)\bar{K}'(k)\} \\ &\qquad=\operatorname{tr}\left\{\sum_{i=1}^{n}E_iG(k)H_i\bar{\Upsilon}(k)H_iG'(k)\right\}. \end{aligned}$$

Moreover, taking the partial derivation of the trace of the matrix $M_{k|k}$ with respect to the gain parameters $G(k)$, we have

$$\begin{aligned} &\frac{\partial\operatorname{tr}\{M_{k|k}\}}{\partial G(k)} \\ &=-2\sum_{i=1}^{n}E_iM_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k)H_i+2\sum_{i=1}^{n}E_iG(k)H_i \\ &\quad\times\Big[\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k)+\bar{\Upsilon}(k)\Big]H_i, \end{aligned}$$

Since (19) is in a positive semi-definite quadratic form with respect to the matrix $G(k)$, in order to minimize $\operatorname{tr}\{M_{k|k}\}$, we let its partial derivative be zero. As such, we have

$$\begin{aligned} &\sum_{i=1}^{n}E_iG(k)H_i\Big[\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k)+\bar{\Upsilon}(k)\Big]H_i \\ &=\sum_{i=1}^{n}E_iM_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k)H_i, \end{aligned} \tag{21}$$

which is equal to the following equations containing sparse matrices:

$$G^{(i)}(k)\mathcal{M}_i(k)=\mathcal{N}_i(k)$$

for $i=1,\cdots,n$. Now, it remains to obtain $G^{(i)}(k)$ by solving the above equations. Note that

$$H_i=\operatorname{diag}\{a_{i1}I,\ a_{i2}I,\ \cdots,\ a_{in}I\}$$

where the element $a_{ij}=0$ when $j\notin N_i$. Removing the corresponding zero rows and zero columns from $\mathcal{M}_i(k)$ and the corresponding zero columns from $\mathcal{N}_i(k)$, we obtain the simplified matrices $\bar{\mathcal{M}}_i(k)$ and $\bar{\mathcal{N}}_i(k)$. Subsequently, we can partition the matrices by $\mathcal{M}_i(k)=\{M_{ab}(k)\}_{n\times n}$ and $\mathcal{N}_i(k)=\{N_b(k)\}_{1\times n}$, and thus have

$$\sum_{j=1}^{n}G_{ij}(k)M_{jb}(k)=N_b(k), \quad \text{for } b\in N_i$$

As for $b\notin N_i$, we have $M_{jb}(k)=N_b(k)=0$ and therefore the above equation always holds. Moreover, for $b\notin N_i$, we can choose the coupling filter gains $G_{ib}(k)=0$ because the local sensor cannot receive any message from its non-neighbor nodes. As a result, it follows that

$$\bar{G}^{(i)}(k)\bar{\mathcal{M}}_i(k)=\bar{\mathcal{N}}_i(k), \quad \text{for } i=1,2,\cdots,n$$

Noticing that the matrix $\bar{\mathcal{M}}_i(k)$ is positive definite, we derive that $\bar{G}^{(i)}(k)=\bar{\mathcal{N}}_i(k)\bar{\mathcal{M}}_i(k)^{-1}$ and, consequently, the parameter $G_{ij}(k)$ can be obtained by selecting the corresponding column block matrix in the matrix $\bar{\mathcal{N}}_i(k)\bar{\mathcal{M}}_i(k)^{-1}$, which ends the proof. ■

Remark 1: A crucial step for designing the filter gain is to solve the equality (21). However, due to the sparsity of the communication topology, there is a remarkable difficulty to obtain $G(k)$ directly. Actually, the diagonal entries of matrix H_i are nonzero when the corresponding sensor is in the neighboring set of sensor i. In other words, $a_{ij}>0$ if only if $j\in N_i$, and therefore H_i is likely to be rank deficient, which means that $\mathcal{M}_i(k)$ is also rank deficient. By employing the matrix simplification technique proposed in the above proof, we remove the zero columns and rows to guarantee the positive definiteness of the simplified matrix $\bar{\mathcal{M}}_i(k)$, which renders the explicit expression of $G(k)$ possible.

V. Boundedness analysis

In this section, we will discuss the mean-square boundedness of the estimation errors for the proposed distributed resilient filter.

For convenience of discussion, without loss of generality, we set the weights $a_{ij}=1$ for $j\in N_i$. Moreover, an assumption is introduced to place some constraints on the system parameters:

Assumption 1: There exist positive real numbers $\bar{a}, \underline{c}, \bar{c}, \bar{\tau}, \underline{\lambda}, \bar{\lambda}, \bar{q}, \bar{r}, \kappa_{si}, \kappa_s, \bar{\sigma}^2$, such that the following bounds on matrices are fulfilled for all $i = 1, \cdots, n$, $j = 1, 2$, and $s = 1, \cdots, m$:

$$\|A(k)\| \le \bar{a},\ \underline{c} \le \|C_i(k)\|_{\min},\ \|C_i(k)\| \le \bar{c},\ \mathrm{tr}\{X(k)\} \le \bar{\tau},$$
$$\|Q(k)\| \le \bar{q},\ \underline{\lambda} \le \|\bar{\Gamma}(k)\|_{\min},\ \ \|\bar{\Gamma}(k)\| \le \bar{\lambda},\ \|R_i(k)\| \le \bar{r},$$
$$\|\Pi_{sj}(k)\| \le \kappa_{sj},\ \|\Gamma_s(k)\| \le \kappa_s,\ \sigma_{\gamma_i}^2 \le \bar{\sigma}^2.$$

Denote $\bar{k} \triangleq n\bar{\lambda}\bar{c}/\underline{\lambda}^2\underline{c}^2$ and $\bar{\zeta} \triangleq \max_i\{\sum_{s=1}^n \delta_{is}\}$. With Assumption 1, we are able to establish a sufficient condition for the mean-square boundedness of the estimation errors as follows.

Theorem 3: Consider the time-varying system (1)-(2) with the distributed resilient filters given by (5a)-(5b) whose gain parameters are provided in Theorem 2. Under Assumption 1, the filtering error dynamics is mean-square bounded, i.e.,

$$\sup_{k\in\mathbb{N}} \sum_{i=1}^n \mathbb{E}[e_i'(k|k)e_i(k|k)] < \infty,$$

if the following inequality holds

$$\bar{a}^2((1+\bar{k}\bar{\lambda}\bar{c})^2 + \bar{\lambda}^2\bar{c}^2\bar{\zeta}) < 1. \tag{22}$$

Proof: It follows from (14a) and Assumption 1 that

$$\|M_{k|k-1}\| \le \bar{a}^2\|M_{k-1|k-1}\| + \|\mathcal{F}(k)\| + \|\mathcal{W}(k)\|.$$

Noting that

$$\mathrm{tr}\{X(k)\Gamma_s(k)\} = \mathrm{tr}\big\{\mathbb{E}[x'(k)\Gamma_s(k)x(k)]\big\} \le \kappa_s\bar{\tau},$$

one has

$$\|\mathcal{F}(k)\| \le n\sum_{s=1}^m \kappa_{s1}\kappa_s\bar{\tau}.$$

In addition, it can be seen that $\|\mathcal{W}(k)\| \le n\bar{q}$. Therefore, we can obtain the following inequality

$$\|M_{k|k-1}\| \le \bar{a}^2\|M_{k-1|k-1}\| + n\sum_{s=1}^m \kappa_{s1}\kappa_s\bar{\tau} + n\bar{q} \tag{23}$$

Since we only care about the non-sparse part of $G(k)$, it is not difficult to verify that (21) results in the following equation

$$\sum_{i=1}^n E_iG(k)H_i = \sum_{i=1}^n E_iU(k)(Z(k))^{-1}H_i \tag{24}$$

where

$$U(k) \triangleq M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k),$$
$$Z(k) \triangleq \bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k) + \bar{\Upsilon}(k).$$

Taking the norm for the both sides of the equation (24) yields that

$$\|\bar{K}(k)\| \le n\|U(k)(Z(k))^{-1}\| \le n\frac{\bar{\lambda}\bar{c}}{\underline{\lambda}^2\underline{c}^2} = \bar{k}$$

Thus, it is clear that

$$\|I + \bar{K}(k)\bar{\Gamma}(k)\mathcal{C}(k)\| \le 1 + \bar{k}\bar{\lambda}\bar{c} \triangleq \bar{b}.$$

Moreover, we have

$$\|\aleph(k)\| \le \sum_{s=1}^m \|\Pi_{s2}(k)\|\|\mathrm{tr}[X(k)\Gamma_s(k)]\| \le \sum_{s=1}^m \kappa_{s2}\kappa_s\bar{\tau}$$

and

$$\begin{aligned}&\|\Xi \odot \mathcal{C}'(k)\mathcal{X}(k)\mathcal{C}'(k)\| \\ &\quad= \big\|\mathrm{diag}_n\{\sigma_{\gamma_i}^2 C_i'(k)X(k)C_i(k)\}\big\| \le \bar{\sigma}^2\bar{c}^2\bar{\tau}\end{aligned}$$

Therefore, it is obvious that

$$\begin{aligned}&\|\bar{\Upsilon}(k)\| \\ &\le \|\aleph(k)\| + \|\mathrm{diag}_n\{R_i(k)\}\| + \|\Xi \odot \mathcal{C}'(k)\mathcal{X}(k)\mathcal{C}'(k)\| \\ &\le \sum_{s=1}^m \kappa_{s2}\kappa_2\bar{\tau} + \bar{\sigma}^2\bar{c}^2\bar{\tau} + \bar{r} \triangleq \bar{h}\end{aligned} \tag{25}$$

By letting $\bar{\zeta} = \max_i\{\sum_{s=1}^n \delta_{is}\}$, we have

$$\begin{aligned}&\|\lambda_{max}(\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k) + \bar{\Upsilon}(k))\Lambda\| \\ &\le (\bar{\lambda}^2\bar{c}^2\|M_{k|k-1}\| + \bar{h})\bar{\zeta}.\end{aligned}$$

In light of (14b), it is straightforward to see that

$$\begin{aligned}\|M_{k|k}\| &\le (\bar{b}^2 + \bar{\lambda}^2\bar{c}^2\bar{\zeta})\|M_{k|k-1}\| + \bar{k}^2\bar{h} + \bar{h}\bar{\zeta} \\ &\le \bar{a}^2(\bar{b}^2 + \bar{\lambda}^2\bar{c}^2\bar{\zeta})\|M_{k-1|k-1}\| + \bar{h}(\bar{k}^2 + \bar{\zeta}) \\ &\quad + (\bar{b}^2 + \bar{\lambda}^2\bar{c}^2\bar{\zeta})\left(n\sum_{s=1}^m \kappa_{s1}\kappa_s\bar{\tau} + n\bar{q}\right)\end{aligned}$$

where the second inequality comes from substituting (23). Since $\bar{a}^2(\bar{b}^2 + \bar{\lambda}^2\bar{c}^2\bar{\zeta}) < 1$, the sequence $\|M_{k|k}\|$ converges eventually. Using the fact that $M_{k|k}$ always is the upper bound of the real estimation error covariance $P_{k|k}$, we conclude that the filtering error dynamics is mean-square stable, which ends the proof. ■

Remark 2: According to (14b), it is clear that the gain variations do have a great impact on the covariance $M_{k|k}$. Moreover, it can be seem from condition (22) in the above theorem that, since $\lambda_{\max}(\bar{\Gamma}(k)\mathcal{C}(k)M_{k|k-1}\mathcal{C}'(k)\bar{\Gamma}(k)+\bar{\Upsilon}(k))\Lambda$ is a multiplicative term in the proliferation of $M_{k|k}$, the sequence $\{M_{k|k}\}$ will diverge quickly if δ_{ij} is too large. As such, it is observed that a smaller gain variation δ_{ij} is more beneficial for the mean-square boundedness.

VI. A Numerical Example

In this section, a numerical example is employed to demonstrate the effectiveness of the proposed distributed resilient filter scheme. A target tracking scenario is used to justify its potential applicability.

Consider a wireless sensor network with $n = 4$ sensor nodes. The network topology is represented by a directed graph $G = (V, E, \mathcal{H})$ with the set of nodes $V = \{1, 2, 3, 4\}$, the set of edges $E = \{(1,1), (1,2), (2,2), (2,3), (3,1), (3,2), (3,3), (4,3)\}$, and the adjacency matrix

$$\mathcal{H} = \begin{bmatrix} 1 & 0.3 & 0 & 0.1 \\ 0 & 1 & 0.2 & 0 \\ 0.5 & 1 & 1 & 0 \\ 0 & 0 & 0.8 & 1 \end{bmatrix}.$$

The discretized target system (1) with stochastic nonlinearities is described by

$$x(k+1) = \begin{bmatrix} 0.89 & 0.1+0.1\cos(0.12k) \\ 0 & 0.88 \end{bmatrix} x(k) + w(k) + f(k, x(k), \xi(k)).$$

The initial values of the state $x(0)$ and the process noise $w(k)$ follow the zero-mean Gaussian distribution with the respective covariances $\Sigma_0 = \text{diag}\{2, 2\}$ and $Q(k) = \text{diag}\{0.1, 0.15\}$. The parameters of the measurement models of the sensors (2) are described as follows:

$$C_1(k) = [0.92 + 0.05\cos(0.12k) \quad 0.82]$$
$$C_2(k) = [0.25 \quad 0.1 + 0.05\sin(0.1k)]$$
$$C_3(k) = [0.84 + 0.05\cos(0.1k) \quad 0.75 + 0.05\sin(0.1k)]$$
$$C_4(k) = [0.75 \quad 0.435]$$

Suppose that the stochastic variables $v_i(k)$ are independent zero-mean Gaussian white noise sequences with the covariances $R_i(k) = 0.25, \ i = 1, 2, 3, 4$. The stochastic sensor gain degradation of individual sensors has the following probability density function

$$p_k^i(s) = \begin{cases} 0.05, & s = 0 \\ 0.10, & s = 0.5 \\ 0.85, & s = 1 \end{cases}$$

for $i = 1, 2, \cdots, 4$. As such, the expectation and variance can be easily calculated as $\bar{\gamma}_i(k) = 0.9$ and $\sigma^2_{\gamma_i}(k) = 0.065$, respectively. The stochastic nonlinearities $f(k, x(k), \xi(k))$ and $g_i(k, x(k), \zeta_i(k))$ are selected as follows:

$$f(k, x(k), \xi(k)) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} (0.3\text{sign}(x^{(1)}(k))x^{(1)}(k)\xi^{(1)}(k) + 0.4\text{sign}(x^{(2)}(k))x^{(2)}(k)\xi^{(2)}(k))$$
$$g_i(k, x(k), \zeta_i(k)) = 0.3(0.3\text{sign}(x^{(1)}(k))x^{(1)}(k)\zeta_i^{(1)}(k) + 0.4\text{sign}(x^{(2)}(k))x^{(2)}(k)\zeta_i^{(2)}(k))$$

where $x^{(j)}(k)$, $\xi^{(j)}(k)$ and $\zeta_i^{(j)}(k)$ ($j = 1, 2$) denote the jth elements of the system state $x(k)$, and the stochastic variables $\xi(k)$ and $\zeta_i(k)$, respectively. Obviously, the expectations and the covariances of the above stochastic nonlinearities meet the form in (3) and (4) with the integer $m = 1$, parameter matrices $\Pi_{s1} = [0.1 \ \ 0.2]'[0.1 \ \ 0.2]$, $\Pi_{s2}^{(i)} = 0.09$ and $\Gamma_s(k) = \text{diag}\{0.09, \ 0.16\}$. The initial parameters of the filters are chosen as $\hat{x}_i(0|0) = 0$ and $M_{0|0} = \mathbf{1}_4 \otimes \Sigma_0$. Additionally, assume that $\delta_{ij} = 0.1$, for $i, j = 1, \cdots, n$. We can compute the filter gain parameters according to $(14a)$, $(14b)$, and (18), and then exploit the algorithm given by (5a)-(5b) to estimate the state vector in a distributed manner .

The simulation results are presented in Figs. 1-3. Among them, Figs. 1-2 depict the trajectories of the true states $x^{(j)}(k)$ and the corresponding estimates $\hat{x}_i^{(j)}(k|k)$. To quantify the estimation accuracy, the mean square estimation error is defined as follows

$$\mathbf{MSE}(k) = \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{n} e_i'(k|k)e_i(k|k)$$

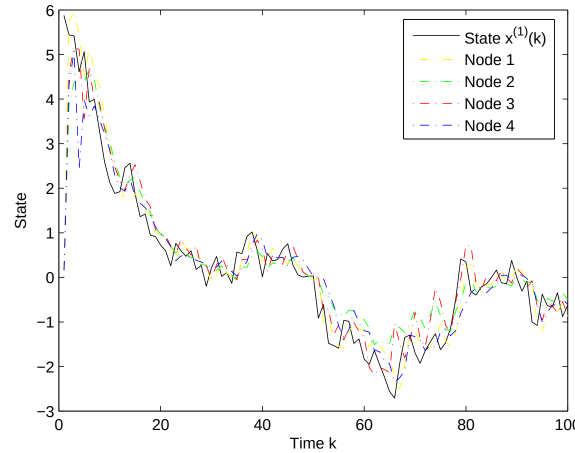

Fig. 1. The true state $x^{(1)}(k)$ and its estimates $\hat{x}_i^{(1)}(k|k)$.

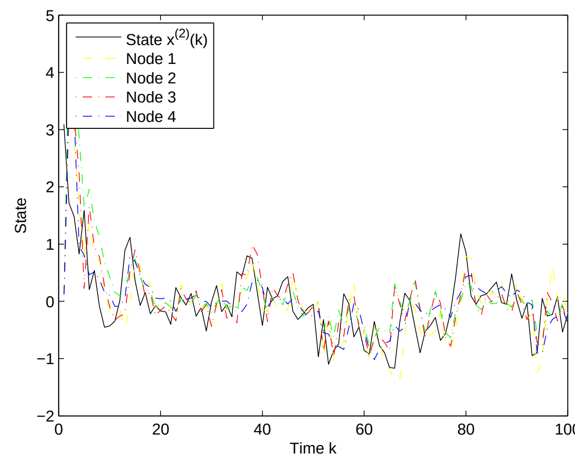

Fig. 2. The true state $x^{(2)}(k)$ and its estimates $\hat{x}_i^{(2)}(k|k)$.

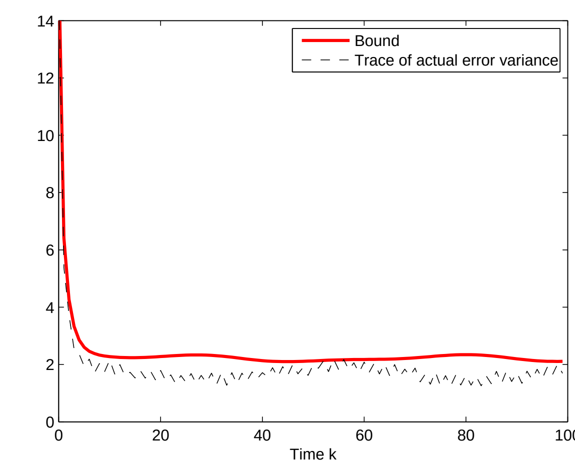

Fig. 3. The MSE and its upper bound $\text{tr}\{M_{k|k}\}$.

Fig. 3 presents the trace of the matrix $M_{k|k}$ calculated from Theorem 1 and the mean square error (MSE) obtained from $T = 1,000$ independent experiments. The result confirms that the solutions of the difference equation (14a)-(14b) are actually the upper bounds of the error variance. Moreover, we compare the MSE of our resilient distributed filter with that of the filter proposed in [15]. Form the simulation results in Fig. 4, it can be seen that our resilient distributed filter performs better, which is not surprising as we have made specific efforts to account for the stochastic gain variations, the nonlinearities and the sensor gain degradation.

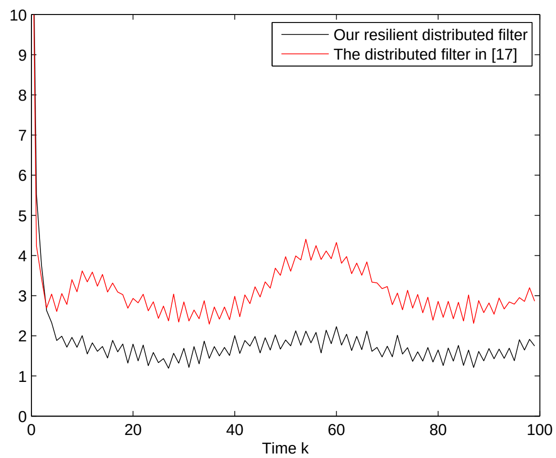

Fig. 4. MSE Comparison for the proposed resilient filter and the filter in [15].

VII. CONCLUSION

In this paper, we have investigated the distributed filtering problem for discrete time-varying systems subject to complicated stochastic phenomena including stochastic nonlinearities, sensor degradation and gain variations. In the presence of these stochastic phenomena, it is impossible to obtain the exact error covariance in an explicit form, let alone the design of the filter gains. To tackle this problem, a sub-optimal distributed resilient filter design scheme has been established. Specifically, we have derived a matrix difference equation whose solution is the upper bound of the actual error covariance. Filter gains have been designed through minimizing such an upper bound at each step iteratively. After that, a sufficient condition has been established to guarantee the mean-square stability of the distributed resilient filter. Finally, the effectiveness of the proposed filtering algorithm has been illustrated by a numerical example. One topic of future research would be the extension of our results to the systems with more complicated dynamical behaviors addressed in [3], [14], [17], [32].

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