A method to solve optimal stopping problems for Lévy processes in infinite horizon

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Abstract

We present a method to solve optimal stopping problems in infinite horizon for a Lévy process when the reward function g can be non-monotone.

To solve the problem we introduce two new objects. Firstly, we define a random variable $\eta(x)$ which corresponds to the argmax of the reward function. Secondly, we propose a certain integral transform which can be built on any suitable random variable. It turns out that this integral transform constructed from $\eta(x)$ and applied to the reward function produces an easy and straightforward description of the optimal stopping rule. We illustrate our results with several examples.

The method we propose allows to avoid complicated differential or integro-differential equations which arise if the standard methodology is used.

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1 Introduction

In recent papers [14, 5, 6, 7, 8, 4], the solutions to optimal stopping problems for Lévy processes and random walks with monotone reward functions were found in terms of the maximum/minimum of the process. In [2], the two-sided optimal stopping problem for a strong Markov process was considered. In the present paper we use the method similar to [14], [7], and develop it further to the case of non-monotone reward functions.

In [14], even though some of the construction was defined for a wide class of reward functions, the actual stopping problem was solved only for monotone reward functions. Paper [2] obtains the necessary conditions for a function to solve the two-sided optimal stopping problem. In this paper, we present a constructive method to solve optimal stopping problems for a fairly general reward function g. That is, we show how to find the optimal stopping boundary. The key ingredient here is to construct the integral transform $\mathcal{A}^{\eta(x)}$ (see Definition 2.1) on the random variable $\eta(x) = \mathrm{argmaxg}_{0\le t\le e_q}(x + X_s) - x$ (see Definition 3.1).

Suppose $X = (X_s)_{s\ge 0}$ is a real-valued Lévy process. Let $\mathbf{P}$ and $\mathbf{E}$ denote the probability and the expectation, respectively, associated with the process X when started from 0. The natural filtration, generated by X, is denoted by $\mathcal{F} = (\mathcal{F}_t)_{t\ge 0}$, and $\mathcal{M}$ is the set of all stopping times with respect to $\mathcal{F}$. We aim to find the "value" function $V^* = V^*(\cdot)$ and the optimal stopping time τ^*, such that

$$V^*(x) = \sup_{\tau\in\mathcal{M}} \mathbf{E}\left(e^{-q\tau} g(x + X_\tau)\right) = \mathbf{E}\left(e^{-q\tau^*} g(x + X_{\tau^*})\right), \tag{1.1}$$

where $q > 0$, and g is a measurable function with respect to $\mathcal{F}$ under some conditions to be specified later.

From the general theory of optimal stopping [13] we know that optimal stopping problems can be solved through the so-called Markovian method, (see [9], chapter 1). Taking this path, the original problem can be reduced to the corresponding free-boundary problem. The free moving boundary divides the space into two subspaces. We are looking for the optimal stopping boundary among all possible moving boundaries. The optimal boundary divides the space into two subspaces, namely the "continuation region" (where it is optimal not to stop, but to continue observations), and the "stopping region" (where it is optimal to stop the process).

In [8, 7] Novikov and Shiryaev considered the optimal stopping problem (1.1) with the reward function $g(x) = (x^+)^\nu$, $\nu > 0$. They have discovered that the optimal stopping boundary can be found for $g(x) = (x^+)^n$ as a unique root of an Appell polynomial, see [8]. Or, more generally, the optimal stopping boundary can be found for $g(x) = (x^+)^\nu$ as a unique root of an Appell function, see [7]. The Appell function should be non-negative and increasing in the stopping region, and negative in the continuation region.

We follow a similar approach and generalize the result of Novikov and Shiryaev. We show that the optimal stopping boundaries can be found as zeros of some suitable integral transform $\mathcal{A}^{\eta(x)}\{g\}$ of the reward function g. The function $\mathcal{A}^{\eta(x)}\{g\}$ should be non-negative and co-monotone with g in the stopping regions and negative in the continuation regions.

Our algorithm to find the solution to the optimal stopping problem is the following

- We introduce an auxiliary random variable $\eta(x)$ pathwise tracking the value of X_t that achieved the running maximum of $g(x+X)$.
- We use $\eta(x)$ to define the transform $\mathcal{A}^{\eta(x)}$ mapping the reward function $g = g(\cdot)$ into the function $\mathcal{A}^{\eta(x)}\{g\}(\cdot)$ for each x.
- We define the region S as those arguments (x, y) at which $\mathcal{A}^{\eta(x)}\{g\}(\mathrm{y})$ is non-negative, i.e. $S = \{(x, y) \mid \mathcal{A}^{\eta(x)}\{g\}(y) \geq 0\}$.
- We define the candidate value function as $V(x) = \mathbf{E}\left(\mathcal{A}^{\eta(x)}\{g\}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right)$, and the candidate optimal stopping time as $\tau^+ = \inf\{s \geq 0 : \mathcal{A}^{\eta(x)}\{g\}(x+X_s) \geq 0\}$.
- We show that the obtained solution is the optimal solution indeed (see Theorem 4.1), i.e the candidate value function V and the candidate optimal stopping time τ^+ coincide with the value function V^* and the optimal stopping time τ^* from (1.1).

Finally, we illustrate our method by several examples.

2 $\mathcal{A}^\nu$-transform.

2.1 Definition

Suppose we are given a real function $g = g(\cdot)$ and a random variable ν , with $\mathbf{E}e^{\lambda|\nu|} < \infty$ for some $\lambda > 0$. Suppose function $g = g(\cdot)$ has an inverse bilateral Laplace transform $\mathcal{L}^{-1}\{g\}$. For the existence of the inverse bilateral Laplace transform we can assume, that g is vanishing at infinity and continuous. Alternatively, we can look at the function g as a formal power series, and take the inverse bilateral Laplace transform formally (for the motivation see Proposition 2.2).

Definition 2.1. *Let ν be a random variable, such that $\mathbf{E}e^{\lambda|\nu|} < \infty$ for some $\lambda > 0$. The transform $\mathcal{A}^\nu$ of function $g = g(\cdot)$ is the function $Q_g^\nu = Q_g^\nu(\cdot)$, defined by*

$$Q_g^\nu(y) = \mathcal{A}^\nu\{g\}(y) = \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u) \frac{e^{uy}}{\mathbf{E}e^{u\nu}} du. \tag{2.1}$$

The function $Q_g^\nu(y)$ is an integral over the product of the inverse bilateral Laplace transform of the function g and the Esscher transform $\frac{e^{uy}}{\mathbf{E}e^{u\nu}}$ of the random variable ν.

As it will be shown through examples below, the transform $\mathcal{A}^\nu$ was designed to convert a reward function g into a function of an “Appell type”, i.e. into a function with properties similar to the Appell function from [7] and the well-known Appell polynomials. We chose the notation $Q_g^\nu = Q_g^\nu(y)$ for the image of $\mathcal{A}^\nu$ -transform of the function g in order to be consistent with the existing notation for the Appell function from [7], and the Appell polynomials. However, as the term “Appell function” is already widely used for an extension of the hypergeometric function to two variables, and the term “Appell transform” is used in connection to heat conduction, we decided not to proceed with the term “Appell”, but to emphasize the “Appellness” by denoting the transform by the letter “$\mathcal{A}$”.

One can note, that instead of the bilateral Laplace transform we could have used any exponential transform with the same success. Our choice of the bilateral Laplace is motivated by the desire to have the Esscher transform in the definition.

2.2 Properties

2.2.1 The averaging property

Lemma 2.1. *Let ν be a random variable with $\mathbf{E}e^{\lambda|\nu|} < \infty$ for some $\lambda > 0$, $\mathcal{A}^\nu\{g\}(y)$ be an $\mathcal{A}^\nu$-transform of function g given by Definition 2.1. Then the $\mathcal{A}^\nu$-transform satisfies the averaging property $\mathbf{E}(\mathcal{A}^\nu\{g\}(y+\nu)) = g(y)$.*

Proof. Indeed,

$$\mathbf{E}(\mathcal{A}^\nu\{g\}(y+\nu)) = \mathbf{E}Q_g^\nu(y+\nu) = \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u)\frac{\mathbf{E}e^{u(y+\nu)}}{\mathbf{E}e^{u\nu}}du = \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u)e^{uy}du = g(y).$$

$\triangle$

2.2.2 The martingale property of $\mathcal{A}^{X_t}\{g\}(X_t)$ for a Lévy process X.

Lemma 2.2. *Let X_t be a Lévy process such that for any t there is some $\lambda > 0$ such that $\mathbf{E}e^{\lambda|X_t|} < \infty$. Then $\mathcal{A}^{X_t}\{g\}(X_t)$ is a martingale.*

Proof. Indeed,

$$\begin{aligned}
\mathbf{E}\left(\mathcal{A}^{X_t}\{g\}(X_t)\middle|\,\mathcal{F}_s\right) &= \mathbf{E}\left(\int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u)\frac{e^{uX_t}}{\mathbf{E}e^{uX_t}}du\middle|\,\mathcal{F}_s\right) \\
&= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u)\mathbf{E}\left(\frac{e^{uX_t}}{\mathbf{E}e^{uX_t}}\middle|\,\mathcal{F}_s\right)du \\
&= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u)\frac{e^{uX_s}}{\mathbf{E}e^{uX_s}}du \\
&= \mathcal{A}^{X_s}\{g\}(X_s)
\end{aligned}$$

$\triangle$

2.2.3 The linearity

The linearity of $\mathcal{A}^\nu$-transform follows from linearity of the inverse bilateral Laplace transform.

Lemma 2.3. *Let $\mathcal{A}^\nu$-transform exist for the real functions f and g. Then*

$$\mathcal{A}^\nu\{c_1f + c_2g\}(y) = c_1\mathcal{A}^\nu\{f\}(y) + c_2\mathcal{A}^\nu\{g\}(y), \tag{2.2}$$

where c_1 and c_2 are some constants.

Proof.

$$\begin{aligned}
\mathcal{A}^\nu\{c_1f + c_2g\}(y) &= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{c_1f + c_2g\}(u)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du \\
&= \int_{-\infty}^{\infty} \left(c_1\mathcal{L}^{-1}\{f\}(u) + c_2\mathcal{L}^{-1}\{g\}(u)\right)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du \\
&= c_1\int_{-\infty}^{\infty} \mathcal{L}^{-1}\{f\}(u)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du + c_2\int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du \\
&= c_1\mathcal{A}^\nu\{f\}(y) + c_2\mathcal{A}^\nu\{g\}(y).
\end{aligned}$$

$\triangle$

2.2.4 The differential property

Lemma 2.4. *Let f be a differentiable function on $\mathbb{R}$. Suppose $\mathcal{A}^\nu$-transform of f is differentiable. Then it satisfies the following differential property*

$$\frac{d}{dy}\left(\mathcal{A}^\nu\{f\}(y)\right) = \mathcal{A}^\nu\left\{\frac{df}{dx}\right\}(y). \tag{2.3}$$

PROOF. It is well known that if $\mathcal{L}^{-1}(f)$ is the inverse bilateral Laplace transform of $f(y)$, then $u\mathcal{L}^{-1}(f)$ is the inverse bilateral Laplace transform of $\frac{d}{dy}f(y)$. Therefore, we have

$$\begin{aligned}
\frac{d}{dy}\left(\mathcal{A}^\nu\{f\}(y)\right) = \frac{d}{dy}Q^\nu_f(y) &= \int_{-\infty}^{\infty}\mathcal{L}^{-1}(f)(u)\frac{d}{dy}\left(\frac{e^{uy}}{\mathbf{E}e^{u\nu}}\right)du \\
&= \int_{-\infty}^{\infty}\left(u\mathcal{L}^{-1}(f)(u)\right)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du \\
&= \int_{-\infty}^{\infty}\mathcal{L}^{-1}\left(\frac{df}{dx}\right)(u)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du \\
&= Q^\nu_{\frac{df}{dx}}(y) = \mathcal{A}^\nu\left\{\frac{df}{dx}\right\}(y).
\end{aligned}$$

△

The differential property of $\mathcal{A}^\nu$-transform allows us to say, that if $\mathcal{A}^\nu$-transform exists for some derivative of the function f, then taking into account $\frac{d}{dy}Q^\nu_f(y) = Q^\nu_{\frac{df}{dx}}(y)$ we can extend the definition of $\mathcal{A}^\nu$-transform to the function f such that the averaging property is satisfied.

To show how it works let the real function f be differentiable on $\mathbb{R}$. Suppose the $\mathcal{A}^\nu$-transform exists for $\frac{df}{dx}$, i.e. there exists function $Q^\nu_{\frac{df}{dx}} = Q^\nu_{\frac{df}{dx}}(y)$. Then we can build a function $Q^\nu_f(y)$ as

$$Q^\nu_f(y) = Q^\nu_f(0) + \int_0^y Q^\nu_{\frac{df}{dx}}(z)dz, \tag{2.4}$$

with $Q^\nu_f(0) = -\mathbf{E}\int_0^\nu Q^\nu_{\frac{df}{dx}}(z)dz + f(0)$, assuming the mathematical expectation exists.

Now the function defined by (2.4) satisfies the averaging property as in Lemma 2.1. Indeed,

$$\begin{aligned}
\mathbf{E}(Q^\nu_f(y+\nu)) &= Q^\nu_f(0) + \mathbf{E}\int_0^{y+\nu} Q^\nu_{\frac{df}{dx}}(z)dz \\
&= Q^\nu_f(0) + \mathbf{E}\int_0^{\nu} Q^\nu_{\frac{df}{dx}}(z)dz + \mathbf{E}\int_\nu^{y+\nu} Q^\nu_{\frac{df}{dx}}(z)dz \\
&= -\mathbf{E}\int_0^{\nu} Q^\nu_{\frac{df}{dx}}(z)dz + f(0) + \mathbf{E}\int_0^{\nu} Q^\nu_{\frac{df}{dx}}(z)dz + \int_\nu^{y+\nu} \mathbf{E}Q^\nu_{\frac{df}{dx}}(z)dz \\
&= f(0) + \int_\nu^{y+\nu} \mathbf{E}Q^\nu_{\frac{df}{dx}}(z)dz \\
&= f(0) + \int_0^{y} \mathbf{E}Q^\nu_{\frac{df}{dx}}(z+\nu)dz \\
&= f(0) + \int_0^{y} \frac{df}{dz}dz \\
&= f(y).
\end{aligned}$$

2.3 Examples

2.3.1 Monomials.

Appell polynomials $Q_n^\nu(y)$ are traditionally defined as

$$Q_n^\nu(y) = \frac{d^n}{du^n}\left(\frac{e^{uy}}{\mathbf{E}\left(e^{u\nu}\right)}\right)\Bigg|_{u=0} \tag{2.5}$$

in other words, $\frac{e^{uy}}{\mathbf{E}(e^{u\nu})}$ is the generating function for Appell polynomials

$$\frac{e^{uy}}{\mathbf{E}(e^{u\nu})} = \sum_{n=0}^{\infty} \frac{u^n}{n!} Q_n^\nu(y). \tag{2.6}$$

Proposition 2.1. *The $\mathcal{A}^\nu$-transform of the monomial y^n is the corresponding Appell polynomial $Q_n^\nu(y)$.*

PROOF. Before we proceed any further, let us introduce the necessary notation. By $\mathcal{L}^{-1}\{g\}$ we denote the inverse bilateral Laplace transform for some function $g = g(\cdot)$. By $\delta^{(n)}(u)$ we denote the n'th derivative of the delta function (see [3], ch.I, §2). More precisely,

$$\int_{-\infty}^{\infty} \delta^{(n)}(u)\phi(u)du = (-1)^n \phi^{(n)}(0) \tag{2.7}$$

Note, that the inverse bilateral Laplace transform of y^n is the n'th derivative of the delta function,

$$\mathcal{L}^{-1}\{y^n\}(u) = (-1)^n \delta^{(n)}(u).$$

Indeed,

$$\int_{-\infty}^{\infty} \mathcal{L}^{-1}\{y^n\}(u)e^{uy}du = \int_{-\infty}^{\infty} (-1)^n \delta^{(n)}(u)e^{uy}du = (-1)^{2n}\frac{d^n}{du^n}\left(e^{uy}\right)\Big|_{u=0} = y^n.$$

Therefore,

$$\begin{aligned} Q_{y^n}^\nu(y) &= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{y^n\}(u)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du \\ &= \int_{-\infty}^{\infty} (-1)^n\delta^{(n)}(u)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du = \\ &= \frac{d^n}{du^n}\left(\frac{e^{uy}}{\mathbf{E}\left(e^{u\nu}\right)}\right)\Bigg|_{u=0} \\ &= Q_n^\nu(y) \end{aligned}$$

△

Thus with a slight abuse of notation we write for simplicity $Q_n^\nu(y)$ instead of $Q_{y^n}^\nu(y)$.

2.3.2 Polynomials (analytic functions/formal power series).

Assume that the function $g = g(\cdot)$ is a polynomial, (analytic or a formal power series in the style of umbral calculus [11]), then we can show, that the $\mathcal{A}^\nu$-transform of g can be represented as a linear combination (power series) of Appell polynomials.

Proposition 2.2. *Let*

$$g(y) = \sum_{k=0}^{n} c_k y^k. \tag{2.8}$$

Then the $\mathcal{A}^\nu$-transform of g is a linear combination in Appell polynomials with the same coefficients as $g(\cdot)$, i.e.

$$\mathcal{A}^\nu\{g\}(y) = Q_g^\nu(y) = \sum_{k=0}^n c_k Q_k^\nu(y),$$

where $Q_k^\nu(y)$ are Appell polynomials of order k generated by the random variable ν.

Proof.

$$\begin{aligned}\mathcal{A}^\nu\{g\}(y) = Q_g^\nu(y) &= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du = \int_{-\infty}^{\infty}\left(\sum_{k=0}^n c_k \mathcal{L}^{-1}\{y^k\}(u)\right)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du \\ &= \sum_{k=0}^n c_k \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{y^k\}(u)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du = \sum_{k=0}^n c_k \int_{-\infty}^{\infty}(-1)^k\delta^{(k)}(u)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du \\ &= \sum_{k=0}^n c_k \frac{d^k}{du^k}\left(\frac{e^{uy}}{\mathbf{E}e^{u\nu}}\right)\Bigg|_{u=0} = \sum_{k=0}^n c_k Q_k^\nu(y)\end{aligned}$$

2.3.3 Linear combination of exponentials.

Let the reward function g be given by linear combination of exponentials

$$g(y) = \sum_{k=0}^n c_k e^{r_k y}.$$

One can notice that the inverse bilateral Laplace transform of $e^{r_k y}$ is the delta function at r_k,

$$\mathcal{L}^{-1}\{e^{r_k y}\}(u) = \delta(u - r_k).$$

Indeed,

$$\int_{-\infty}^{\infty} \mathcal{L}^{-1}\{e^{r_k y}\}(u)e^{uy}du = \int_{-\infty}^{\infty}\delta(u-r_k)e^{uy}du = e^{uy}\big|_{u=r_k} = e^{r_k y}.$$

Proposition 2.3. *Let*

$$g(y) = \sum_{k=0}^n c_k e^{r_k y}. \tag{2.9}$$

Then the $\mathcal{A}^\nu$-transform of g is a sum of the corresponding Esscher transforms, i.e.

$$\mathcal{A}^\nu\{g\}(y) = Q_g^\nu(y) = \sum_{k=0}^n c_k \frac{e^{r_k y}}{\mathbf{E}e^{r_k \nu}}.$$

Proof.

$$\begin{aligned}\mathcal{A}^\nu\{g\}(y) = Q_g^\nu(y) &= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du = \int_{-\infty}^{\infty}\left(\sum_{k=0}^n c_k \mathcal{L}^{-1}\{e^{r_k y}\}(u)\right)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du \\ &= \sum_{k=0}^n c_k \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{e^{r_k y}\}(u)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du = \sum_{k=0}^n c_k \int_{-\infty}^{\infty}\delta(u-r_k)\frac{e^{uy}}{\mathbf{E}e^{u\nu}}du \\ &= \sum_{k=0}^n c_k \frac{e^{uy}}{\mathbf{E}e^{u\nu}}\Bigg|_{u=r_k} = \sum_{k=0}^n c_k \frac{e^{r_k y}}{\mathbf{E}e^{r_k\nu}}.\end{aligned}$$

△

2.3.4 Linear combinations of exponential polynomials.

Let the reward function g be given by an exponential polynomial

$$g(y) = \sum_{k=0}^{n} c_k y^k e^{r_k y}.$$

Note that the inverse bilateral Laplace transform of $y^k e^{r_k y}$ is the k'th derivative of the delta function at r_k,

$$\mathcal{L}^{-1}\{y^k e^{r_k y}\}(u) = (-1)^k \delta^{(k)}(u - r_k).$$

Indeed,

$$\int_{-\infty}^{\infty} \mathcal{L}^{-1}\{y^k e^{r_k y}\}(u) e^{uy} du = \int_{-\infty}^{\infty} (-1)^k \delta^{(k)}(u - r_k) e^{uy} du = (-1)^{2k} \frac{d^k}{du^k} (e^{uy}) \Big|_{u=r_k} = y^k e^{r_k y}.$$

Denote the k-th derivative in u of $\frac{e^{uy}}{\mathbf{E}e^{u\nu}}$ at $u = a$ by $Q_k^\nu(y; a)$:

$$Q_k^\nu(y; a) := \frac{d^k}{du^k} \left(\frac{e^{uy}}{\mathbf{E}e^{u\nu}} \right) \Big|_{u=a} \tag{2.10}$$

Proposition 2.4. *Let*

$$g(y) = \sum_{k=0}^{n} c_k y^k e^{r_k y}. \tag{2.11}$$

Then the $\mathcal{A}^\nu$-transform of g is a sum of products of corresponding Esscher transforms and Appell polynomials, i.e.

$$\mathcal{A}^\nu\{g\}(y) = Q_g^\nu(y) = \sum_{k=0}^{n} c_k Q_k^\nu(y; r_k).$$

Proof.

$$\begin{aligned}
\mathcal{A}^\nu\{g\}(y) = Q_g^\nu(y) &= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u) \frac{e^{uy}}{\mathbf{E}e^{u\nu}} du = \int_{-\infty}^{\infty} \left(\sum_{k=0}^{n} c_k \mathcal{L}^{-1}\{y^k e^{r_k y}\}(u) \right) \frac{e^{uy}}{\mathbf{E}e^{u\nu}} du \\
&= \sum_{k=0}^{n} c_k \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{y^k e^{r_k y}\}(u) \frac{e^{uy}}{\mathbf{E}e^{u\nu}} du = \sum_{k=0}^{n} c_k \int_{-\infty}^{\infty} (-1)^k \delta^{(k)}(u - r_k) \frac{e^{uy}}{\mathbf{E}e^{u\nu}} du \\
&= \sum_{k=0}^{n} c_k \frac{d^k}{du^k} \left(\frac{e^{uy}}{\mathbf{E}e^{u\nu}} \right) \Big|_{u=r_k} \\
&= \sum_{k=0}^{n} c_k Q_k^\nu(y; r_k).
\end{aligned}$$

$\triangle$

3 The random variable $\eta(x)$

3.1 Construction of $\eta(x)$

Definition 3.1. *Denote $\varsigma_s = \inf\{r, s \le r \le t \mid g(X_r) \ge g(X_u)$ for any $u, s \le u \le t\}$. The running* argmaxg *starting at time s and running up to time t is defined by*

$$X_{\varsigma_s} = \underset{s \le u \le t}{\mathrm{argmaxg}}(x + X_u). \tag{3.1}$$

Our aim is to deliver a pathwise construction for the running argmaxg of the process X. Consider a trajectory of X starting at time 0, $X_0 = 0$, and running up to time t. Then $g(\text{argmaxg}_{0\le u\le t}(x+X_u))$ is the maximum of the path

$$[0,t] \ni u \to g(x+X_u) \in \mathbb{R}$$

Note that if g is a nondecreasing function, then the running argmaxg of the process X coincides with the running max of the process X, i.e.

$$\underset{0\le s\le t}{\text{argmaxg}}(x+X_u) = \max_{0\le u\le t}(x+X_u). \tag{3.2}$$

Similarly, if g is a nonincreasing function, then the running argmaxg coincides with the running min of the process

$$\underset{0\le u\le t}{\text{argmaxg}}(x+X_u) = \min_{0\le u\le t}(x+X_u). \tag{3.3}$$

Now we are ready to define the random variable $\eta(x)$ that will be used to define the A^ν-transform.

Definition 3.2. *Let e_q be an exponentially distributed random variable with mean $1/q$ and independent of the process X. We define the random variable $\eta(x)$ as*

$$\eta(x) = \underset{0\le s\le e_q}{\text{argmaxg}}(x+X_s) - x. \tag{3.4}$$

In the same way as above, if g is a nondecreasing function, then

$$\eta(x) = \max_{0\le s\le e_q}(x+X_s) - x = \max_{0\le s\le e_q}(X_s),$$

and if g is a nonincreasing function, then

$$\eta(x) = \min_{0\le s\le e_q} X_s.$$

It is useful to note that if g is a monotone function, then $\eta(x)$ does not depend on the starting position x.

3.2 Properties of $\eta(x)$.

Recall $\varsigma_s = \inf\{r, s\le r\le t \mid g(x+X_r)\ge g(x+X_u)$ for any $s\le u\le t\}$. Let τ_a be the first moment at which $g(x+X_s)$ reaches the level a, i.e. $\tau_a = \inf\{s\ge 0 : g(x+X_s)\ge a\}$, (see the Figure 1).

In this section we present several propositions showing the connection between $\eta(x+X_{\tau_a})$ and $\eta(x)$.

Proposition 3.1. *Let $\tau_a = \inf\{s\ge 0 : g(x+X_s)\ge a\}$. Then, conditionally on $\mathcal{F}_{\tau_a}$ and $\{e_q > \tau_a\}$, we have $x+\eta(x) \stackrel{d}{=} x + X_{\tau_a} + \eta(x+X_{\tau_a})$.*

Proof. Indeed, as τ_a is the first moment at which $g(x+X_s)$ reaches a, then, conditionally on $\mathcal{F}_{\tau_a}$ and $\{e_q>\tau_a\}$, pathwise we have

$$\begin{aligned}
\underset{0\le s\le e_q}{\text{argmaxg}}(x+X_s) &\stackrel{d}{=} \underset{\tau_a\le s\le e_q}{\text{argmaxg}}(x+X_s)\\
&\stackrel{d}{=} \underset{\tau_a\le s\le e_q}{\text{argmaxg}}(x+X_{\tau_a}-X_{\tau_a}+X_s)\\
&\stackrel{d}{=} \underset{\tau_a\le s\le e_q}{\text{argmaxg}}(x+X_{\tau_a}+X_{s-\tau_a})\\
&\stackrel{d}{=} \underset{0\le t\le e_q}{\text{argmaxg}}(x+X_{\tau_a}+X_t)
\end{aligned}$$

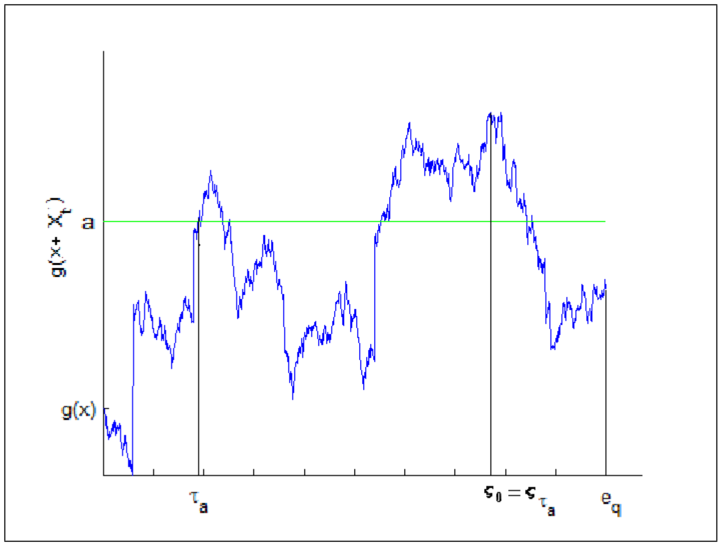

Figure 1: The trajectory $g(x + X_t)$ starting from $g(x)$, τ_a is the first moment at which $g(x + X_s)$ reaches the level a. Here ς_0 coincides with ς_{τ_a}.

Therefore, conditionally on $\mathcal{F}_{\tau_a}$ and $\{e_q > \tau_a\}$, we get

$$\begin{aligned}
x + \eta(x) &= \underset{0\le s\le e_q}{\operatorname{argmaxg}}(x + X_s) \\
&\overset{d}{=} \underset{0\le t\le e_q}{\operatorname{argmaxg}}(x + X_{\tau_a} + X_t) \\
&\overset{d}{=} x + X_{\tau_a} + \underset{0\le t\le e_q}{\operatorname{argmaxg}}(x + X_{\tau_a} + X_t) - x - X_{\tau_a} \\
&\overset{d}{=} x + X_{\tau_a} + \eta(x + X_{\tau_a}).
\end{aligned}$$

△

Proposition 3.2. *Denote $\tau_a = \inf\{s \ge 0 : g(x+X_s) \ge a\}$. Conditionally on $\mathcal{F}_{\tau_a}$ and $\{e_q > \tau_a\}$, we have $g(x + \eta(x + X_{\tau_a})) \le g(x + X_{\tau_a} + \eta(x + X_{\tau_a}))$ almost surely.*

Proof. Indeed, by definition of $\eta(x)$ and argmaxg we have

$$\begin{aligned}
g(x + X_{\tau_a} + \eta(x + X_{\tau_a})) &= g(\underset{0\le s\le e_q}{\operatorname{argmaxg}}(x + X_{\tau_a} + X_s)) \\
&\ge g(\underset{0\le s\le e_q}{\operatorname{argmaxg}}(x + X_{\tau_a} + X_s) - X_{\tau_a}) \\
&= g(x + \eta(x + X_{\tau_a})).
\end{aligned}$$

△

Proposition 3.3. *Let τ be any stopping moment, $\tau \in \mathcal{M}$. Then, conditionally on $\mathcal{F}_\tau$ and $\{e_q > \tau\}$, we have $g(x + X_\tau + \eta(x)) \le g(x + X_\tau + \eta(x + X_\tau))$ almost surely.*

Proof. Indeed, by definition of $\eta(x)$ and argmaxg we have

$$\begin{aligned}
g(x + X_\tau + \eta(x + X_\tau)) &= g(\underset{0\le s\le e_q}{\operatorname{argmaxg}}(x + X_\tau + X_s)) \\
&\ge g(\underset{0\le s\le e_q}{\operatorname{argmaxg}}(x + X_s) + X_\tau) \\
&= g(x + X_\tau + \eta(x)).
\end{aligned}$$

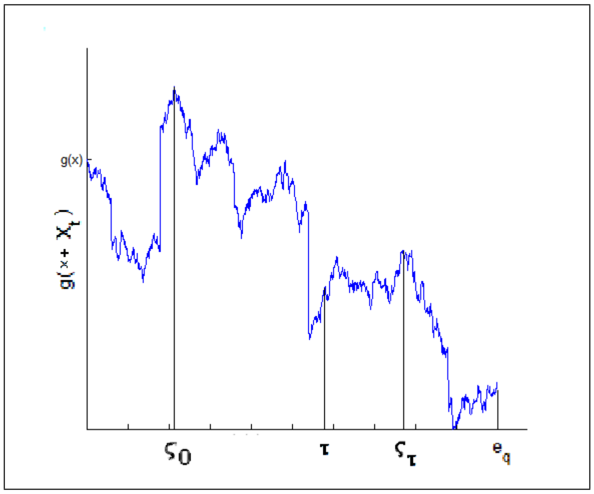

Figure 2: The trajectory $g(x+X_t)$ starting from $g(x)$, τ is any stopping moment. One can easily see $g(x+X_{\varsigma_0}) = g(\text{argmaxg}_{0\le u\le e_q}(x+X_u)) \ge g(\text{argmaxg}_{\tau\le u\le e_q}(x+X_u)) = g(x+X_{\varsigma_\tau})$.

$\triangle$

Proposition 3.4. *Let τ be any stopping moment, $\tau \in \mathcal{M}$. Then, conditionally on $\mathcal{F}_\tau$ and $\{e_q > \tau\}$, we have $g(x+X_\tau+\eta(x+X_\tau)) \le g(x+\eta(x))$ almost surely.*

Proof. Indeed, by definition of $\eta(x)$ and argmaxg we have

$$\begin{aligned} g(x+X_\tau+\eta(x+X_\tau)) &= g(\underset{0\le s\le e_q}{\text{argmaxg}}(x+X_\tau+X_s)) \\ &\le g(\underset{0\le s\le e_q}{\text{argmaxg}}(x+X_s)) \\ &= g(x+\eta(x)). \end{aligned}$$

$\triangle$

3.3 The property of $\mathcal{A}^{\eta(x)}$-transform

Let the function $Q_g^{\eta(x)}(\cdot)$ be given by the definition 3.1 as the image of the $\mathcal{A}^{\eta(x)}$-transform of the function g.

The following proposition shows how the functions $Q_g^{\eta(x)}(\cdot)$ and $Q_g^{\eta(x+X_{\tau_a})}(\cdot)$ are connected.

Proposition 3.5. *Let $\tau_a = \inf\{s \ge 0 : g(x+X_s) \ge a\}$. Then, conditionally on $\{X_{\tau_a} = z\}$ and $\{e_q > \tau_a\}$, we have $Q_g^{\eta(x)}(y) = Q_g^{\eta(x+z)}(y-z)$.*

Proof. The proof follows directly from the definition. Indeed, conditionally on $\{X_{\tau_a} = z\}$ and

$\{e_q > \tau_a\}$ we have

$$Q_g^{\eta(x)}(y) = \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u) \frac{e^{-uy}}{\mathbf{E} e^{-u\eta(x)}} du \tag{3.5}$$

$$\stackrel{proposition\, 3.1}{=} \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u) \frac{e^{-uy}}{\mathbf{E} e^{-u(\eta(x+z)+z)}} du \tag{3.6}$$

$$= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u) \frac{e^{-uy}}{e^{-uz}\mathbf{E} e^{-u\eta(x+z)}} du \tag{3.7}$$

$$= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{g\}(u) \frac{e^{-u(y-z)}}{\mathbf{E} e^{-u\eta(x+z)}} du \tag{3.8}$$

$$= Q_g^{\eta(x+z)}(y-z) \tag{3.9}$$

$\triangle$

4 Main results. Solution to the optimal stopping problem

4.1 The candidate value function

To solve the optimal stopping problem we have to find the "value" function $V^* = V^*(\cdot)$ and the optimal stopping time τ^*, such that

$$V^*(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}\left(e^{-q\tau} g(x + X_\tau)\right) = \mathbf{E}\left(e^{-q\tau^*} g(x + X_{\tau^*})\right), \tag{4.1}$$

Here we introduce *the candidate optimal stopping time* τ^+ and the *candidate value function* V. Let the candidate optimal stopping time τ^+ be defined as

$$\tau^+ := \inf\{s \geq 0 : Q_g^{\eta(x)}(x + X_s) \geq 0\}. \tag{4.2}$$

In order to define the candidate value function we introduce the set S. By S we denote the set of all pairs (x, y) such that the function $Q_g^{\eta(x)}(y)$ (i.e. $\mathcal{A}^{\eta(x)}$-transform of the reward function g) is non-negative

$$S := \{(x, y) \mid Q_g^{\eta(x)}(y) \geq 0\}. \tag{4.3}$$

Now we can say that τ^+ is the first moment at which $(x, x + X_s)$ reaches S.

Let the candidate value function V be defined as

$$V(x) := \mathbf{E}\left(Q_g^{\eta(x)}(x + \eta(x)) \mathbf{1}_{\{(x, x+\eta(x)) \in S\}}\right). \tag{4.4}$$

To show that our candidates are indeed the solution to the optimal stopping problem we have to show $V(x) = V^*(x)$ for any x; and τ^+ is the optimal stopping moment, i.e. $V(x) = \mathbf{E}\left(e^{-q\tau^+} g(x + X_{\tau^+})\right)$. To prove the optimality we will need two auxiliary lemmas which are presented below.

4.2 Auxiliary lemmas

To prove the optimality we will need the notion of co-monotonicity. Let us recall the definition of co-monotone functions.

Definition 4.1. *The two real functions f and g are co-monotone on $\Gamma \subset \mathbb{R}$, if for any $u, v \in \Gamma$ we have $(f(u) - f(v))(g(u) - g(v)) \geq 0$.*

Lemma 4.1. *Let τ be any stopping moment, $\tau \in \mathcal{M}$. Suppose for $(x, y) \in S$ the functions $g = g(y)$ and $Q_g^{\eta(x)}(y)$ are co-monotone in y on S for each fixed x. Then, conditionally on $\mathcal{F}_\tau$ and $\{e_q > \tau\}$, the following inequality holds*

$$\mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right) \geq \mathbf{E}\left(e^{-q\tau}g(x+X_\tau)\right).$$

Proof. Note that for any x and y

$$Q_g^{\eta(x)}(y)\mathbf{1}_{\{(x,y)\in S\}} \geq Q_g^{\eta(x)}(y) \tag{4.5}$$

by definition of the set S. Thus

$$\begin{aligned}
\mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right) &\overset{tower\ property}{=} \mathbf{E}\left(\mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}} \mid \mathcal{F}_\tau\right)\right) \\
&\underset{co-monot.}{\overset{prop.\,3.4}{\geq}} \mathbf{E}\left(\mathbf{E}\left(Q_g^{\eta(x)}(x+X_\tau+\eta(x+X_\tau))\mathbf{1}_{\{(x,x+X_\tau+\eta(x+X_\tau))\in S\}} \mid \mathcal{F}_\tau\right); e_q > \tau\right) \\
&\underset{co-monot.}{\overset{prop.\,3.3}{\geq}} \mathbf{E}\left(\mathbf{E}\left(Q_g^{\eta(x)}(x+X_\tau+\eta(x))\mathbf{1}_{\{(x,x+X_\tau+\eta(x))\in S\}}\right); e_q > \tau\right) \\
&\overset{(4.5)}{\geq} \mathbf{E}\left(\mathbf{E}\left(Q_g^{\eta(x)}(x+X_\tau+\eta(x))\right); e_q > \tau\right) \\
&\overset{lemma\,2.1}{=} \mathbf{E}\left(g(x+X_\tau); e_q > \tau\right) \\
&= \mathbf{E}\left(e^{-q\tau}g(x+X_\tau)\right)
\end{aligned}$$

△

Recall $\tau^+ = \inf\{s \geq 0 : Q_g^{\eta(x)}(x+X_s) \geq 0\}$.

Lemma 4.2. *Suppose for $(x, y) \in S$ the functions $g = g(y)$ and $Q_g^{\eta(x)}(y)$ are co-monotone in y on S for each fixed x. Then, conditionally on $\mathcal{F}_{\tau^+}$ and $\{e_q > \tau^+\}$, the following identity holds*

$$\mathbf{E}\left(e^{-q\tau^+}g(x+X_{\tau^+})\right) \geq \mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right).$$

Proof. The functions $g = g(\cdot)$ and $Q_g^{\eta(x)}(\cdot)$ are co-monotone on S for any fixed x. Therefore, we have

$$\begin{aligned}
\{(x, x+\eta(x)) \in S\} &= \{(x, \underset{0\leq s\leq e_q}{\mathrm{argmaxg}}(x+X_s)) \in S\} \\
&\overset{d}{=} \{Q_g^{\eta(x)}(\underset{0\leq s\leq e_q}{\mathrm{argmaxg}}(x+X_s)) \geq 0\} \\
&\overset{d}{=} \{e_q > \tau^+\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\,\mathbf{1}_{\{(x,x+\eta(x))\in S_a\}}\right) &= \mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\,\mathbf{1}_{\{e_q>\tau^+\}}\right) \\
&\overset{tower\ property}{=} \mathbf{E}\left(\mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\,\mathbf{1}_{\{e_q>\tau^+\}}|\mathcal{F}_{\tau^+}\right)\right) \\
&= \mathbf{E}\left(\mathbf{1}_{\{e_q>\tau^+\}}\mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\,|\mathcal{F}_{\tau^+}\right)\right) \\
&\overset{proposition\,3.1}{=} \mathbf{E}\left(\mathbf{1}_{\{e_q>\tau^+\}}\mathbf{E}\left(Q_g^{\eta(x)}(x+X_{\tau^+}+\eta(x+X_{\tau^+}))\,|\mathcal{F}_{\tau^+}\right)\right) \\
&\overset{proposition\,3.5}{=} \mathbf{E}\left(\mathbf{1}_{\{e_q>\tau^+\}}\mathbf{E}\left(Q_g^{\eta(x+X_{\tau^+})}(x+\eta(x+X_{\tau^+}))\,|\mathcal{F}_{\tau^+}\right)\right) \\
&\underset{co-monot.}{\overset{proposition\,3.2}{\leq}} \mathbf{E}\left(\mathbf{1}_{\{e_q>\tau^+\}}\mathbf{E}\left(Q_g^{\eta(x+X_{\tau^+})}(x+X_{\tau^+}+\eta(x+X_{\tau^+}))\right)\right) \\
&\overset{lemma\,2.1}{=} \mathbf{E}\left(\mathbf{1}_{\{e_q>\tau^+\}}g(x+X_{\tau^+})\right) \\
&= \mathbf{E}\left(e^{-q\tau^+}g(x+X_{\tau^+})\right)
\end{aligned}$$

$\triangle$

Corollary 4.1. *The following fluctuation identity holds for τ^+ defined by (4.2)*

$$\mathbf{E}\left(e^{-q\tau^+}g(x+X_{\tau^+})\right) = \mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right).$$

Proof. By lemma 4.1 we have for any $\tau \in \mathcal{M}$

$$\mathbf{E}\left(e^{-q\tau}g(x+X_{\tau})\right) \leq \mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right).$$

If the inequality is true for any τ, then it is true for τ^+ too. Therefore, we have

$$\mathbf{E}\left(e^{-q\tau^+}g(x+X_{\tau^+})\right) \leq \mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right).$$

On the other hand, by lemma 4.2, we have

$$\mathbf{E}\left(e^{-q\tau^+}g(x+X_{\tau^+})\right) \geq \mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right).$$

Thus, it follows

$$\mathbf{E}\left(e^{-q\tau^+}g(x+X_{\tau^+})\right) = \mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right).$$

$\triangle$

4.3 The main theorem

Theorem 4.1. *Let S and τ^+ be given by (4.3) and (4.2) correspondingly. Suppose for $(x,y)\in S$ the functions $g=g(y)$ and $Q_g^{\eta(x)}(y)$ are co-monotone in y on S for each fixed x. Then the stopping time τ^+ is optimal for the problem (1.1), and the value function is given by*

$$V^*(x) = \mathbf{E}\left(e^{-q\tau^+}g(x+X_{\tau^+})\right) = \mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right).$$

Proof. Let τ be any stopping moment, and recall

$$V(x) \quad = \quad \mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right).$$

Let us show $V(x) \geq V^*(x)$. By lemma 4.1 we have for any $\tau \in \mathcal{M}$

$$\begin{aligned} V(x) &= \mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right) \\ &\overset{lemma\,4.1}{\geq} \left(e^{-q\tau}g(x+X_{\tau})\right). \end{aligned}$$

As we obtained $V(x) \geq \mathbf{E}\left(e^{-q\tau}g(x+X_{\tau})\right)$ for *any* stopping time τ, then it also holds for the optimal stopping time τ^*:

$$V(x) \geq \mathbf{E}\left(e^{-q\tau^*}g(x+X_{\tau^*})\right) = \sup_{\tau}\mathbf{E}\left(e^{-q\tau}g(x+X_{\tau})\right) = V^*(x).$$

On the other hand, by lemma 4.2 we have the following inequalities for the moment τ^+

$$\begin{aligned} V^*(x) &= \sup_{\tau}\mathbf{E}\left(e^{-q\tau}g(x+X_{\tau})\right) \\ &\geq \mathbf{E}\left(e^{-q\tau^+}g(x+X_{\tau^+})\right) \\ &\overset{lemma\,4.2}{\geq} \mathbf{E}\left(Q_g^{\eta(x)}(x+\eta(x))\mathbf{1}_{\{(x,x+\eta(x))\in S\}}\right) = V(x). \end{aligned}$$

Thus, we see $V(x) \geq V^*(x) \geq V(x)$ for any $x \in \mathbb{R}$, i.e. we have proved $V(x) = V^*(x)$ for any $x \in \mathbb{R}$. Furthermore, by corollary 4.1 we have $V(x) = \mathbf{E}\left(e^{-q\tau^+}g(x+X_{\tau^+})\right)$. Thus we have proved that τ^+ is the optimal stopping time. $\triangle$

5 Examples

5.1 The Novikov-Shiryaev optimal stopping problem with $g(x) = (x^+)^n$.

In [8] Novikov and Shiryaev solved the optimal stopping problem (1.1) with $g(x) = (x^+)^n$, $n = 1, 2, \ldots$ for random walks, and in [15] Kyprianou and Surya found the solution for Lévy processes.

Here we repeat their results with our method.

Let X be a Lévy process with $X_0 = 0$, and the reward function $g(x) = (x^+)^n$. Then we have

$$\begin{aligned} \eta(x) &= \underset{0\le t\le e_q}{\mathrm{argmaxg}}(x+X_t)-x \\ &= \max_{0\le t\le e_q}(x+X_t)-x \\ &= \max_{0\le t\le e_q}(X_t) \\ &= \eta. \end{aligned}$$

The A^η-transform of x^n is an Appell polynomial of order n

$$\begin{aligned} Q^\eta_{x^n}(y) &= \int_{-\infty}^{\infty}\mathcal{L}^{-1}\{x^n\}(u)\frac{e^{uy}}{\mathbf{E}e^{u\eta}}du \\ &= \int_{-\infty}^{\infty}\delta(n,u)\frac{e^{uy}}{\mathbf{E}e^{u\eta}}du \\ &= Q^\eta_n(y) \end{aligned}$$

In such a way we repeat the results of Novikov and Shiryaev, and if $(Q^\eta_n(x))^+$ and $(x^n)^+$ are co-monotone, than the optimal stopping boundary is the positive root of the Appell polynomial $Q^\eta_n(x)$.

5.2 The Novikov-Shiryaev optimal stopping problem with $g(x) = (x^+)^\nu$.

In [7] Novikov and Shiryaev solved the optimal stopping problem (1.1) with $g(x) = (x^+)^\nu$ when the underlying process is a Lévy process.

Here we repeat their results with our method.

Let X be a Lévy process with $X_0 = 0$, and the reward function $g(x) = (x^+)^\nu$. Exactly as in the previous example we obtain

$$\begin{aligned} \eta(x) &= \underset{0\le t\le e_q}{\mathrm{argmaxg}}(x+X_t)-x \\ &= \max_{0\le t\le e_q}(x+X_t)-x \\ &= \max_{0\le t\le e_q}(X_t) \\ &= \eta. \end{aligned}$$

The inverse bilateral Laplace transform of x^ν with $\nu < 0$ is

$$\mathcal{L}^{-1}\{x^\nu\}(u) = \begin{cases} -\dfrac{(-u)^{-\nu-1}}{\Gamma(-\nu)}, & \text{if } u < 0 \\ 0, & \text{if } u \ge 0. \end{cases}$$

where Γ is a gamma function. Indeed,

$$\begin{aligned} \int_{-\infty}^{\infty}\mathcal{L}^{-1}\{x^\nu\}(u)e^{uy}du &= \int_{-\infty}^{0}-\frac{(-u)^{-\nu-1}}{\Gamma(-\nu)}e^{uy}du \\ &= \int_{0}^{\infty}\frac{u^{-\nu-1}}{\Gamma(-\nu)}e^{-uy}du \\ &= y^\nu \end{aligned}$$

Thus, for $\nu < 0$ the A^η-transform of x^ν is given by

$$
\begin{aligned}
Q^\eta_{x^\nu}(y) &= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{x^\nu\}(u)\frac{e^{uy}}{\mathbf{E}e^{u\eta}}du \\
&= \int_{-\infty}^{0} -\frac{(-u)^{-\nu-1}}{\Gamma(-\nu)}\frac{e^{uy}}{\mathbf{E}e^{u\eta}}du \\
&= \int_{0}^{\infty} \frac{u^{-\nu-1}}{\Gamma(-\nu)}\frac{e^{-uy}}{\mathbf{E}e^{-u\eta}}du.
\end{aligned}
$$

This coincides with the results obtained by Novikov and Shiryaev for $\nu < 0$. For $\nu > 0$, we use (2.4) to define $Q^\eta_{x^\nu}(y)$ in the same way as in [7].
Therefore, we repeat the results of [7] and state that if $(Q^\eta_{x^\nu}(x))^+$ and $(x^\nu)^+$ are co-monotone, then the optimal stopping boundary is the positive root of the A^ν -transform of x^ν, i.e. $Q^\eta_{x^\nu}(x)$.

5.3 Two-sided problem

Consider the optimal stopping problem (1.1) with the reward function

$$
g(x) = e^{ax} + e^{-bx} - 2, \tag{5.1}
$$

with $\sqrt{2q} > a > b > 0$ some constants, and q is an interest (killing) rate in our stopping problem. We assume that the underlying process is a Brownian motion B_t with $B_0 = 0$.

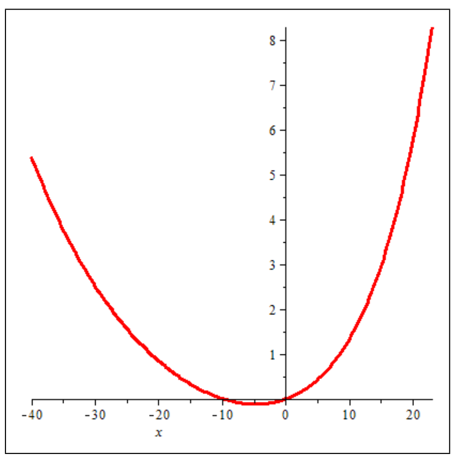

Figure 3: The reward function $g(x) = e^{x/10} + e^{-x/20} - 2$.

The function $g(x)$ is decreasing for $x \le \ln(b/a)/(a+b)$ and increasing for $x \ge \ln(b/a)/(a+b)$. Moreover, it is well known that $(\sup_{0\le t\le e_q} B_t)$ and $(-\inf_{0\le t\le e_q} B_t)$ are equal in distribution.
To find $\eta(x)$ we should compare

$$
g\left(x + \sup_{0\le t\le e_q} B_t\right) \text{ and } g\left(x + \inf_{0\le t\le e_q} B_t\right),
$$

or, equivalently, to compare

$$
g\left(x + \sup_{0\le t\le e_q} B_t\right) \text{ and } g\left(x - \sup_{0\le t\le e_q} B_t\right),
$$

or, equivalently, to compare

$$e^{(a+b)x} \quad \text{and} \quad \frac{\sinh\left(b \sup_{0\le t\le e_q} B_t\right)}{\sinh\left(a \sup_{0\le t\le e_q} B_t\right)}.$$

Define the function $f : [0,\infty) \to \mathbb{R}$ as

$$f(u) = \frac{\sinh(bu)}{\sinh(au)}. \tag{5.2}$$

The function f is decreasing on $[0,\infty)$ due to $a > b > 0$. We write f^{-1} for the inverse function of f, and denote by $c(x)$ the function $c(x) = f^{-1}\left(e^{(a+b)x}\right)$.

It is easy to see that

$$\begin{aligned} \eta(x) &= \underset{0\le t\le e_q}{\operatorname{argmax}} g(x+B_t) - x \\ &= \begin{cases} \sup\limits_{0\le t\le e_q} B_t & \text{if } x \ge \frac{\ln(b/a)}{(a+b)}, \\ \sup\limits_{0\le t\le e_q} B_t & \text{if } x < \frac{\ln(b/a)}{(a+b)} \text{ and } \sup\limits_{0\le s\le e_q} B_s \ge c(x), \\ \inf\limits_{0\le t\le e_q} B_t & \text{if } x < \frac{\ln(b/a)}{(a+b)} \text{ and } \inf\limits_{0\le s\le e_q} B_s > -c(x). \end{cases} \end{aligned} \tag{5.3}$$

As our reward function g is a linear combination of exponential functions plus some constant, then $A^{\eta(x)}$-transform of g is given by

$$A^{\eta(x)}\{g\}(y) = Q_g^{\eta(x)}(y) = \frac{e^{ay}}{\mathbf{E}(e^{a\,\eta(x)})} + \frac{e^{-by}}{\mathbf{E}(e^{-b\,\eta(x)})} - 2.$$

Let us calculate $\mathbf{E}(e^{u\eta(x)})$:

$$\mathbf{E}e^{u\,\eta(x)} = \begin{cases} \int_0^\infty e^{uy} p_{sup}(y)dy, & x \ge \frac{\ln(b/a)}{(a+b)}, \\ \int_{c(x)}^\infty e^{uy} p_{sup}(y)dy + \int_{-c(x)}^0 e^{uy} p_{inf}(y)dy, & x < \frac{\ln(b/a)}{(a+b)}, \end{cases} \tag{5.4}$$

where $p_{\inf}(y) = \sqrt{2q}\, e^{y\sqrt{2q}}$ and $p_{sup}(y) = \sqrt{2q}\, e^{-y\sqrt{2q}}$ are the probability density functions for $\inf_{0\le t\le e_q} B_t$ and $\sup_{0\le t\le e_q} B_t$ respectively.
Subsequently, for $u < \sqrt{2q}$ we have

$$\mathbf{E}e^{u\,\eta(x)} = \begin{cases} \frac{\sqrt{2q}}{\sqrt{2q}-u}, & x \ge \frac{\ln(b/a)}{(a+b)}, \\ \frac{\sqrt{2q}}{\sqrt{2q}-u} e^{-c(x)(\sqrt{2q}-u)} - \frac{\sqrt{2q}}{\sqrt{2q}+u} e^{-c(x)(\sqrt{2q}+u)} + \frac{\sqrt{2q}}{\sqrt{2q}+u}, & x < \frac{\ln(b/a)}{(a+b)}, \end{cases} \tag{5.5}$$

Therefore $\mathbf{E}e^{a\,\eta(x)}$ and $\mathbf{E}e^{-b\,\eta(x)}$ are given by

$$\mathbf{E}e^{a\,\eta(x)} = \begin{cases} \frac{\sqrt{2q}}{\sqrt{2q}-a}, & x \ge \frac{\ln(b/a)}{(a+b)}, \\ \frac{\sqrt{2q}}{\sqrt{2q}-a} e^{-c(x)(\sqrt{2q}-a)} - \frac{\sqrt{2q}}{\sqrt{2q}+a} e^{-c(x)(\sqrt{2q}+a)} + \frac{\sqrt{2q}}{\sqrt{2q}+a}, & x < \frac{\ln(b/a)}{(a+b)}, \end{cases} \tag{5.6}$$

and

$$\mathbf{E}e^{-b\,\eta(x)} = \begin{cases} \frac{\sqrt{2q}}{\sqrt{2q}+b}, & x \ge \frac{\ln(b/a)}{(a+b)}, \\ \frac{\sqrt{2q}}{\sqrt{2q}+b} e^{-c(x)(\sqrt{2q}+b)} - \frac{\sqrt{2q}}{\sqrt{2q}-b} e^{-c(x)(\sqrt{2q}-b)} + \frac{\sqrt{2q}}{\sqrt{2q}-b}, & x < \frac{\ln(b/a)}{(a+b)}, \end{cases} \tag{5.7}$$

One can easily check that for each fixed x the functions $g(y)$ and $\mathcal{A}^{\eta(x)}\{g\}(y)$ are co-monotone (in y) for those y where $\mathcal{A}^{\eta(x)}\{g\}(y)$ is nonnegative. Consequently, to find the optimal stopping boundaries we have to find the zeros of $\mathcal{A}^{\eta(x)}\{g\}(x)$.

In other words, there are two optimal stopping boundaries x_* and x^*, where $x^* > 0$ is the zero of the equation

$$\frac{\sqrt{2q}-a}{\sqrt{2q}}e^{a\,x}+\frac{\sqrt{2q}+b}{\sqrt{2q}}e^{-b\,x}-2=0, \tag{5.8}$$

and $x_* < 0$ is the zero of

$$\frac{e^{ax}}{\frac{\sqrt{2q}}{\sqrt{2q}-a}e^{-c(x)(\sqrt{2q}-a)}-\frac{\sqrt{2q}}{\sqrt{2q}+a}e^{-c(x)(\sqrt{2q}+a)}+\frac{\sqrt{2q}}{\sqrt{2q}+a}} + \frac{e^{-bx}}{\frac{\sqrt{2q}}{\sqrt{2q}+b}e^{-c(x)(\sqrt{2q}+b)}-\frac{\sqrt{2q}}{\sqrt{2q}-b}e^{-c(x)(\sqrt{2q}-b)}+\frac{\sqrt{2q}}{\sqrt{2q}-b}}-2=0, \tag{5.9}$$

where $c(x)=f^{-1}\left(e^{(a+b)x}\right)$.

The graph of function $\mathcal{A}^{\eta(x)}\{g\}(x)$ for $a=0.1$, $b=0.05$ and $q=0.02$ is shown in Fig.4

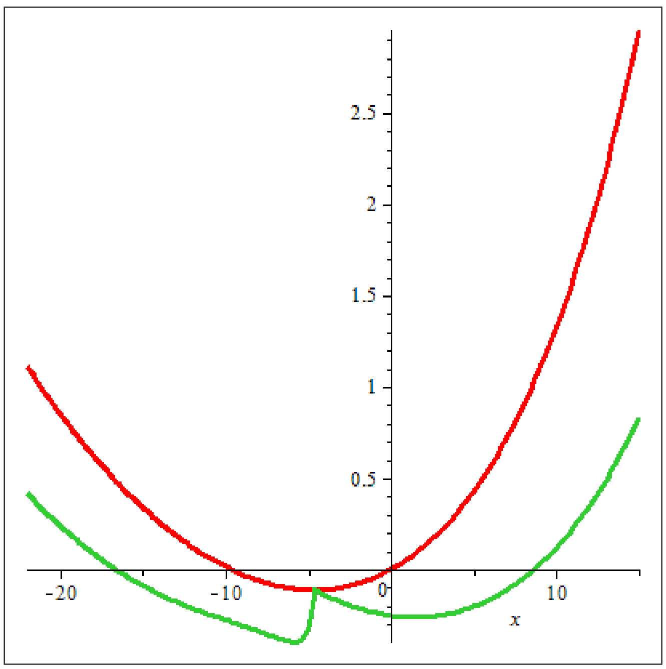

Figure 4: The graphs of the reward function $g=g(x)$ (red) and its $\mathcal{A}^{\eta(x)}$-transform as a function of x, i.e. $A^{\eta(x)}\{g\}(x)$ (green) for $a=0.1$, $b=0.05$ and $q=0.02$.

6 Conclusion and further development

In this paper, we presented a novel approach for solving optimal stopping problems by means of applying a specially designed integral transform to the reward function. The important feature of our method that it works for non-monotone reward functions. To construct the integral transform we need the reward function to have an inverse bilateral Laplace transform.

The newly defined random variable argmaxg plays the central role in the construction of the integral transform. Calculation of argmaxg for various Levy processes is the task to be explored.

We should mention the restriction of the proposed method, namely, the requirement for the reward function $g = g(y)$ and its transform $A^{\eta(x)}\{g\}(y)$ to be co-monotone in y for each fixed x on the set S defined by (4.3). Perhaps this requirement is too strong and one can find a weaker necessary condition.

Although our primary aim in this paper was to solve optimal stopping problems, it is worthwhile mentioning a by-product of our results. The integral transform we created produces a martingale if built on a Lévy process.

We showed that our method works particulary well when the reward function is a polynomial, an exponential or an exponential polynomial. This naturally leads us to explore the possibility of creating numerical methods for solving optimal stopping problems by approximating the reward functions by polynomials/exponential polynomials. The work on this topic has barely begun but looks very promising.

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