H_∞ Filtering for Nonlinear Systems with Stochastic Sensor Saturations and Markov Time-Delays: The Asymptotic Stability in Probability

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Abstract: This paper is concerned with the filtering problem for a class of nonlinear systems with stochastic sensor saturations and Markovian measurement transmission delays, where the asymptotic stability in probability is considered. The sensors are subject to random saturations characterized by a Bernoulli distributed sequence. The transmission time-delays are governed by a discrete-time Markov chain with finite states. In the presence of the nonlinearities, stochastic sensor saturations and Markovian time-delays, sufficient conditions are established to guarantee that the filtering process is asymptotically stable in probability without disturbances and also satisfies the H_∞ criterion with respect to nonzero exogenous disturbances under the zero-initial condition. Moreover, it is illustrated that the results can be specialized to linear filters. Two simulation examples are presented to show the effectiveness of the proposed algorithms.

1. Introduction

In the past decades, the filtering problem has been attracting considerable research attention due to its significance in signal processing, communication, navigation and tracking, finance, etc. There have been many different filtering methods reported in the literature. Traditional Kalman filter and extended Kalman filter can solve the estimation problem, respectively, for linear and nonlinear systems in the least mean square sense [1, 2]. The H_∞ filtering has been extensively studied to guarantee that the L_2 gain from the disturbance to the estimation error is less than a predefined positive level [3, 4]. For stochastic systems, the concept of stability in probability is important [5, 6] because it can describe the system dynamics in a probabilistic way. So far, the filtering problem in the sense of stability in probability has stirred some initial research interests [7, 8, 9].

In practical engineering, sensors cannot generate measured outputs with unbounded amplitudes due to physical and technological limitations such as nonlinear transition shift sensors [10], position sensors [11], displacement sensors [12], pressure sensors [13], and so on. This phenomenon, often referred to as sensor saturation, would bring in extra challenges to the filtering problem. So far, the filtering problem with sensor saturations has drawn much research attention [14, 15, 16, 17, 18]. In most of the reported literature, the sensor saturations have been assumed to occur definitely and described by the so-called sector-bounded conditions. Nevertheless, sensors in practical systems might be subject to some transient phenomena especially in unattended environments such as power grids [19, 20, 21]. In these cases, the saturations may occur in a random way owing to various reasons such as random sensor failures and abrupt environmental changes [22]. The filtering problem with stochastic sensor saturations has not received adequate research attention yet, not to mention the case where the stability in probability is also taken into account to quantify the performance. As such, it would be interesting to examine how the saturation levels and the statistical characteristics of the sensor saturations would influence the stability in probability for the filter design problem.

It is well known that time delays are frequently encountered in many practical systems and may deteriorate the system performances if they are not appropriately coped with in the design procedure [23, 24, 25, 26]. The filtering problem with time-delays has been investigated for a variety of systems such as two-dimensional systems [27] and neural networks [28]. A widely adopted way to formulate the stochastic time-delays is the finite state Markov chain method, which can reflect the relationship between the delays at different time steps [29, 30]. An H_∞ filter has been proposed in [31] for nonlinear systems with model uncertainties and Markov delays. Also, least-squares estimators for systems with Markov delays have been designed recursively in [32, 33]. Nevertheless, when the Markov time-delay issue is coupled with stochastic sensor saturations, the filtering problem for discrete nonlinear systems with guaranteed stability in probability still remains as an ongoing research issue. In fact, it is non-trivial to establish a unified framework to accommodate nonlinearities, stochastic sensor saturations as well as Markov time-delays simultaneously. The main purpose of this paper is to shorten such a gap.

In this paper, we aim to address filtering problem, in the sense of asymptotic stability in probability, for a class of nonlinear systems with stochastic sensor saturations and Markov time-delays. A Bernoulli-distributed sequence is employed to regulate the stochastic sensor saturations. Time delays in the measurement transmissions are governed by a discrete Markov chain with finite states. Sufficient conditions are established to guarantee the desired stability in probability and determine the filter parameters. The linear filter is further investigated as a special case. Two simulation examples are presented to show the effectiveness of the proposed method. The main novelty of the paper lies in the following aspects: 1) a comprehensive model is established which covers nonlinearities, stochastic sensor saturations, and Markov time delays; 2) sufficient conditions are achieved under which the designed filter is asymptotically stable in probability in the disturbance-free case and also robust to exogenous disturbances under the zero-initial condition; and 3) the conditions are specialized to some linear filters such that the simplified results are more applicable in practice.

The rest of paper is organized as follows. In Section 2, the formulation of the addressed nonlinear system with stochastic sensor saturations and Markov time delays is provided. In Section 3, the desired filter is designed in the sense of stability in probability, and a linear filter is established as a special case. Two simulation examples are presented in Section 4 and the paper is concluded in Section 5.

Notations. The notation used in the paper is fairly standard except where otherwise stated. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). I is the identity matrix with compatible dimension. $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x. $\|x\|$ refers to the Euclidean norm of vector x. $\circ$ stands for the Hadamard product with this product being defined as $[A \circ B]_{ij} = A_{ij}B_{ij}$. $\otimes$ is the

Kronecker product defined as $A \otimes B = \begin{bmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \cdots & a_{m,n}B \end{bmatrix}$. $\mathbf{1}$ represents a matrix whose entries are all one. $*$ in a symmetric block matrix represents a term that can be determined by symmetry.

2. Problem Formulation

Consider the following class of stochastic discrete-time nonlinear systems:

$$\begin{cases} x_{k+1} = f(x_k) + g(x_k)v_k + (h(x_k) + s(x_k)v_k)w_k, \\ z_k = m(x_k), \\ \tilde{y}_k = \lambda_k \sigma(l(x_k)) + (1-\lambda_k)l(x_k) + k(x_k)v_k, \\ y_k = \tilde{y}_{k-d_k}, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state; $z_k \in \mathbb{R}^{n_z}$ is the signal to be estimated; w_k is a one-dimensional and zero-mean Gaussian white noise sequence with $\mathbb{E}\{w_k^2\} = r^2$ (r is a known positive scalar); $v_k \in \mathbb{R}^{n_v}$ is the exogenous disturbance satisfying $\{v_k\}_{k\in\mathbb{N}} \in l_2([0,+\infty), \mathbb{R}^{n_v})$; $\tilde{y}_k \in \mathbb{R}^{n_y}$ is the measurement before transmission; $y_k \in \mathbb{R}^{n_y}$ is the received signal impaired by communication delays; d_k is the homogeneous discrete-time Markov chain defined on $N \triangleq \{0, 1, \cdots, d-1\}$ with the one-step transition matrix $(\pi_{ij})_{d\times d}$ and the initial distribution π_0, where $d > 0$ is a fixed integer.

The nonlinear functions $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, $g : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x \times n_v}$, $h : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, $s : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x \times n_v}$, $m : \mathbb{R}^{n_x} \to \mathbb{R}^{n_z}$, $l : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ and $k : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y \times n_v}$ are assumed to be smooth, matrix-valued functions with $f(0) = 0$, $h(0) = 0$, $m(0) = 0$ and $l(0) = 0$. $\lambda_k \in \mathbb{R}$ is a Bernoulli distributed white sequence taking values on 0 or 1 with

$$\begin{cases} \text{Prob}\{\lambda_k = 1\} = \bar{\lambda}, \\ \text{Prob}\{\lambda_k = 0\} = 1 - \bar{\lambda}, \end{cases} \tag{2}$$

where $\bar{\lambda} \in [0, 1]$ is a known scalar.

For a vector $q = \left[q_1, \ldots, q_{n_y}\right]^T$, the saturation function $\sigma : \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ is defined as:

$$\sigma(q) = \left[\sigma_1(q_1), \cdots, \sigma_{n_y}(q_{n_y})\right]^T, \tag{3}$$

with $\sigma_i(q_i) = \text{sign}(q_i)\text{min}(q_{i,\max}, |q_i|)$, where the notation of "sign" denotes the signum function and $q_{i,\max} > 0$ denotes the ith saturation level.

For notational brevity, we set

$$\begin{aligned} &\bar{x}_k = \left[x_k^T, \cdots, x_{k-(d-1)}^T\right]^T, \bar{v}_k = \left[v_k^T, \cdots, v_{k-(d-1)}^T\right]^T, \bar{f}(\bar{x}_k) = \left[f^T(x_k), x_k^T, \cdots, x_{k-(d-2)}^T\right]^T, \\ &\bar{g}(\bar{x}_k) = \text{diag}\left\{g(x_k), 0, \cdots, 0\right\}, \bar{h}(\bar{x}_k) = \left[h^T(x_k), 0, \cdots, 0\right]^T, \bar{s}(\bar{x}_k) = \text{diag}\left\{s(x_k), 0, \cdots, 0\right\}, \\ &\bar{y}_k = \left[\tilde{y}_k^T, \cdots, \tilde{y}_{k-(d-1)}^T\right]^T, \Lambda_k = \text{diag}\left\{\lambda_k, \cdots, \lambda_{k-(d-1)}\right\} \otimes I_{n_y}, \bar{m}(\bar{x}_k) = m(x_k), \\ &\bar{l}(\bar{x}_k) = \left[l^T(x_k), \cdots, l^T(x_{k-(d-1)})\right]^T, \bar{\sigma}(\bar{l}(\bar{x}_k)) = \left[\sigma^T(l(x_k)), \cdots, \sigma^T(l(x_{k-(d-1)}))\right]^T, \\ &\bar{k}(\bar{x}_k) = \text{diag}\left\{k(x_k), \cdots, k(x_{k-(d-1)})\right\}, \end{aligned}$$

and $C_{d_k} = [0, \cdots, 0, I, 0, \cdots, 0]$ with the $(d_k + 1)$th block being an identity.

With the defined notations, then the original model (1) can be written in the following form:

$$\begin{cases} \bar{x}_{k+1} = \bar{f}(\bar{x}_k) + \bar{g}(\bar{x}_k)\bar{v}_k + (\bar{h}(\bar{x}_k) + \bar{s}(\bar{x}_k)\bar{v}_k)w_k, \\ z_k = \bar{m}(\bar{x}_k), \\ \bar{y}_k = \Lambda_k \bar{\sigma}(\bar{l}(\bar{x}_k)) + (I - \Lambda_k)\bar{l}(\bar{x}_k) + \bar{k}(\bar{x}_k)\bar{v}_k, \\ y_k = C_{d_k}\bar{y}_k. \end{cases} \tag{4}$$

Remark 1. The measurement equations in (1) are employed to address the Markov transmission delays and the stochastic sensor saturations. The random variables Λ_k and d_k account for the sensor saturations and time delays, respectively. The statistics of Λ_k and d_k would be utilized in the filter design procedure.

In this paper, a full-order filter of the following structure is adopted:

$$\begin{cases} \hat{x}_{k+1} = \hat{f}(\hat{x}_k) + \hat{g}(\hat{x}_k, d_k)y_k, \\ \hat{z}_k = \hat{m}(\hat{x}_k), \end{cases} \tag{5}$$

where $\hat{x}_k \in \mathbb{R}^{d \times n_x}$ is the state estimate; $\hat{z}_k \in \mathbb{R}^{n_z}$ is the estimate of z_k, and $\hat{f}$, $\hat{g}$ and $\hat{m}$ are filter parameters of appropriate dimensions that are to be determined with $f(0) = 0$, $m(0) = 0$ and $\hat{x}_0 = 0$.

By introducing a new vector $\eta_k = \left[\bar{x}_k^T, \hat{x}_k^T\right]^T$ and letting the filtering error be $\tilde{z}_k = z_k - \hat{z}_k$, an augmented system is obtained as follows:

$$\begin{cases} \eta_{k+1} = \tilde{f}(\eta_k, d_k) + \tilde{g}(\eta_k, d_k)v_k + (\tilde{h}(\eta_k) + \tilde{s}(\eta_k)\bar{v}_k)w_k, \\ \tilde{z}_k = \bar{m}(\bar{x}_k) - \hat{m}(\hat{x}_k), \end{cases} \tag{6}$$

where

$$\tilde{f}(\eta_k, d_k) = \begin{bmatrix} \bar{f}(\bar{x}_k) \\ \hat{f}(\hat{x}_k) + \hat{g}(\hat{x}_k, d_k)C_{d_k}\Lambda_k\bar{\sigma}(\bar{l}(\bar{x}_k)) + \hat{g}(\hat{x}_k, d_k)C_{d_k}(I - \Lambda_k)\bar{l}(\bar{x}_k) \end{bmatrix},$$
$$\tilde{g}(\eta_k, d_k) = \begin{bmatrix} \bar{g}(\bar{x}_k) \\ \hat{g}(\hat{x}_k, d_k)C_{d_k}\bar{k}(\bar{x}_k) \end{bmatrix}, \tilde{h}(\eta_k) = \begin{bmatrix} \bar{h}(\bar{x}_k) \\ 0 \end{bmatrix}, \tilde{s}(\eta_k) = \begin{bmatrix} \bar{s}(\bar{x}_k) \\ 0 \end{bmatrix}.$$

Before proceeding, let us first introduce the following definition, which is a discrete version of that in [34].

Definition 1. The solution $\eta_k = 0$ of (6) is said to be

1. stable in probability if for every pair of $\epsilon \in (0, 1)$ and $\alpha > 0$, there exists a $\delta = \delta(\epsilon, \alpha) > 0$ such that
$$\text{Prob}\left\{\|\eta_k\| < \alpha, \forall k \in \mathbb{N}\right\} \geq 1 - \epsilon, \quad \text{whenever } \|\eta_0\| < \delta;$$
2. asymptotically stable in probability if the origin of (1) is stable in probability, and
$$\lim_{k \to \infty} \text{Prob}\left\{\|\eta_k\| = 0\right\} = 1, \ \forall \eta_0 \in \mathbb{R}^{2n_x}.$$

In this paper, we aim to design the filter parameters $\hat{f}(\hat{x}_k)$, $\hat{g}(\hat{x}_k, d_k)$ and $\hat{m}(\hat{x}_k)$ in (5) such that the following requirements are simultaneously satisfied:

1. The zero-solution of the augmented system (6) with $\bar{v}_k = 0$ is asymptotically stable in probability;
2. Under the zero-initial condition, the filtering error $\tilde{z}_k$ satisfies

$$\sum_{k=0}^{\infty} \mathbb{E}\left\{\|\tilde{z}_k\|^2\right\} < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\left\{\|\bar{v}_k\|^2\right\} \tag{7}$$

   for all nonzero $\bar{v}_k$, where $\gamma > 0$ is a given disturbance attenuation level.

The filtering problem for the addressed nonlinear system will be solved in the next section, and the results will be applied to some special cases for practical convenience.

3. Main Results

We start with the following definitions and lemma which will be used in the development of the main results.

Definition 2. [35] A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$.

Definition 3. [36] A function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a $\mathcal{K}$ class function if it is continuous, strictly increasing and $\kappa(0) = 0$. A $\mathcal{K}$ class function $\kappa(\cdot)$ is said to be a $\mathcal{K}_\infty$ class function if $\kappa(r) \to +\infty$ as $r \to +\infty$.

Lemma 1. If there exists a positive definite function $V : \mathbb{R}^{2d\times n_x} \to \mathbb{R}_+$, two $\mathcal{K}_\infty$ class functions κ_1, κ_2, and a $\mathcal{K}$ class function κ_3 such that for all $k \in \mathbb{N}$,

$$\kappa_1(\|\eta\|) \le V(\eta) \le \kappa_2(\|\eta\|), \forall \eta \in \mathbb{R}^{2d\times n_x}, \tag{8}$$

$$\mathbb{E}\left\{V(\eta_{k+1})|\eta_k\right\} - V(\eta_k) \le -\kappa_3(\|\eta_k\|), \tag{9}$$

then the origin of (6) with $\bar{v}_k = 0$ is asymptotically stable in probability.

Proof. Based on (9) and the definitions of $\mathcal{K}$ class functions and positive definite functions, we have

$$0 \le \mathbb{E}\left\{V(\eta_{k+1})\right\} \le \mathbb{E}\left\{V(\eta_k)\right\} \le \cdots V(\eta_0). \tag{10}$$

Then, for any $\alpha > 0$ and $k \in \mathbb{N}$, we have

$$\text{Prob}\left\{\|\eta_k\| \ge \alpha\right\} = \text{Prob}\left\{\kappa_1\left(\|\eta_k\|\right) \ge \kappa_1(\alpha)\right\} \le \text{Prob}\left\{V\left(\eta_k\right) \ge \kappa_1(\alpha)\right\}. \tag{11}$$

Since $V(\eta_k)$ is nonnegative, we have

$$\mathbb{E}\left\{V(\eta_k)\right\} \ge \kappa_1(\alpha)\text{Prob}\{V(\eta_k) \ge \kappa_1(\alpha)\}. \tag{12}$$

With (11) and (12), it can be obtained that

$$\text{Prob}\{V(\eta_k) \ge \kappa_1(\alpha))\} \le \frac{\mathbb{E}\left\{V(\eta_k)\right\}}{\kappa_1(\alpha))} \le \frac{V\left(\eta_0\right)}{\kappa_1(\alpha))} \le \frac{\kappa_2\left(\|\eta_0\|\right)}{\kappa_1(\alpha)}. \tag{13}$$

Therefore, for any $\epsilon > 0$, we can choose $\delta = \kappa_2^{-1}(\epsilon\kappa_1(\alpha)) > 0$ such that

$$\text{Prob}\{\|\eta_k\| < \alpha, \forall k \in \mathbb{N}\} \geq 1 - \epsilon, \text{whenever } \|\eta_0\| < \delta.$$

With (10), we can also find a $V_\infty \geq 0$ such that

$$\lim_{k\to\infty} \mathbb{E}\{V(\eta_k)\} = V_\infty. \tag{14}$$

Substituting (14) into (9) yields

$$\lim_{k\to\infty} \mathbb{E}\{\kappa_3(\|\eta_k\|)\} = 0 \tag{15}$$

which implies that

$$\lim_{k\to\infty} \text{Prob}\{\|\eta_k\| = 0\} = 1.$$

The proof is now complete. □

With Lemma 1, sufficient conditions are going to be established in the following theorem to facilitate the filter design.

Theorem 1. Given a disturbance attenuation level $\gamma > 0$. If there exist two positive definite matrices $P = P^T > 0$ and $Q = Q^T > 0$, and a $\mathcal{K}$ class function κ_3 satisfying the inequalities

$$\begin{cases} \mathbb{H}(\bar{x}, \hat{x}, \tau) = B(\bar{x}, \hat{x}, \tau)A^{-1}(\bar{x}, \hat{x}, \tau)B^T(\bar{x}, \hat{x}, \tau) + r^2\bar{h}^T(\bar{x})P\bar{h}(\bar{x}) + D(\bar{x}, \hat{x}, \tau) \\ \qquad + \|\tilde{z}\|^2 + \kappa_3(\|\eta\|) < 0, \text{ for any } \bar{x}, \hat{x} \in \mathbb{R}^{d\times n_x}, \tau \in N, \\ A(\bar{x}, \hat{x}, \tau) > 0, \text{ for any } \bar{x}, \hat{x} \in \mathbb{R}^{d\times n_x}, \tau \in N, \end{cases} \tag{16}$$

where

$$\bar{\Lambda} = \bar{\lambda} I_{d\times n_y}, \quad \bar{\Lambda}_2 = \begin{bmatrix} 1-\bar{\lambda} & (1-\bar{\lambda})^2 & \cdots & (1-\bar{\lambda})^2 \\ * & 1-\bar{\lambda} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ * & \cdots & \cdots & 1-\bar{\lambda} \end{bmatrix} \otimes \mathbf{1}_{n_y\times n_y},$$

$$\bar{\Lambda}_1 = \begin{bmatrix} \bar{\lambda} & \bar{\lambda}^2 & \cdots & \bar{\lambda}^2 \\ * & \bar{\lambda} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ * & \cdots & \cdots & \bar{\lambda} \end{bmatrix} \otimes \mathbf{1}_{n_y\times n_y}, \quad \bar{\Lambda}_3 = \begin{bmatrix} 0 & \bar{\lambda}(1-\bar{\lambda}) & \cdots & \bar{\lambda}(1-\bar{\lambda}) \\ * & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ * & \cdots & \cdots & 0 \end{bmatrix} \otimes \mathbf{1}_{n_y\times n_y},$$

$$A(\bar{x}, \hat{x}, \tau) = \gamma^2 I - r^2\bar{s}^T(\bar{x})P\bar{s}(\bar{x}) - \bar{g}^T(\bar{x})P\bar{g}(\bar{x}) - \bar{k}^T(\bar{x})S_1(\bar{x}, \hat{x}, \tau)\bar{k}(\bar{x}), \tag{17}$$

$$\begin{aligned} B(\bar{x}, \hat{x}, \tau) = & r^2\bar{h}^T(\bar{x})P\bar{s}(\bar{x}) + \bar{f}^T(\bar{x})P\bar{g}(\bar{x}) + \hat{f}^T(\hat{x})Q\hat{g}(\hat{x}, \tau)C_\tau\bar{k}(\bar{x}) \\ & + S_2^T(\bar{x}, \hat{x}, \tau)Q\hat{g}(\hat{x}, \tau)C_\tau\bar{k}(\bar{x}), \end{aligned} \tag{18}$$

$$\begin{aligned} D(\bar{x}, \hat{x}, \tau) = & (\bar{f}(\bar{x}) + \bar{x})^T P(\bar{f}(\bar{x}) - \bar{x}) + (\hat{f}(\hat{x}) + \hat{x})^T Q(\hat{f}(\hat{x}) - \hat{x}) + 2\hat{f}^T(\hat{x})QS_2(\bar{x}, \hat{x}, \tau) \\ & + \bar{\sigma}^T(\bar{l}(\bar{x}))\big(\bar{\Lambda}_1 \circ S_1(\hat{x}, \tau)\big)\bar{\sigma}(\bar{l}(\bar{x})) + 2\bar{\sigma}^T(\bar{l}(\bar{x}))\big(\bar{\Lambda}_3 \circ S_1(\hat{x}, \tau)\big)\bar{l}(\bar{x}) \\ & + \bar{l}^T(\bar{x})\big(\bar{\Lambda}_2 \circ S_1(\hat{x}, \tau)\big)\bar{l}(\bar{x}), \end{aligned} \tag{19}$$

$$S_1(\hat{x}, \tau) = C_\tau^T\hat{g}^T(\hat{x}, \tau)Q\hat{g}(\hat{x}, \tau)C_\tau, \tag{20}$$

$$S_2(\bar{x}, \hat{x}, \tau) = \hat{g}(\hat{x}, \tau)C_\tau\bar{\Lambda}\bar{\sigma}(\bar{l}(\bar{x})) + \hat{g}(\hat{x}, \tau)C_\tau(I - \bar{\Lambda})\bar{l}(\bar{x}), \tag{21}$$

then the filtering problem for system (1) is solved by (5) in the sense of asymptotic stability in probability.

Proof. Set $V^{(1)}(\bar{x}) = \bar{x}^T P \bar{x}$ and $V^{(2)}(\hat{x}) = \hat{x}^T Q \hat{x}$. Define

$$\bar{\rho} = \max\{\lambda_{\max}(P), \lambda_{\max}(Q)\}, \quad \underline{\rho} = \min\{\lambda_{\min}(P), \lambda_{\min}(Q)\},$$

where $\lambda_{\max}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the maximum and the minimum eigenvalue of the square matrix, respectively. Then, two $\mathcal{K}_\infty$ functions can be defined as $\kappa_1(\|\eta\|) = \underline{\rho}\|\eta\|^2$ and $\kappa_2(\|\eta\|) = \bar{\rho}\|\eta\|^2$, and it follows that $\kappa_1(\|\eta\|) \leq V(\eta) \leq \kappa_2(\|\eta\|)$.

With $\mathbb{E}\{w_k\} = 0$, $\mathbb{E}\{w_k^2\} = r^2$ and (6), it can be obtained that

$$\begin{aligned}
&\mathbb{E}\{V(\eta_{k+1})|\eta_k\} - V(\eta_k) + \mathbb{E}\{\|\tilde{z}_k\|^2\} - \gamma^2\mathbb{E}\{\|\bar{v}_k\|^2\} + \kappa_3(\|\eta_k\|)\\
=&\mathbb{E}\Big\{\bar{f}^T(\bar{x}_k)P\bar{f}(\bar{x}_k) + \big(\hat{f}(\hat{x}_k) + \hat{g}(\hat{x}_k, d_k)C_{d_k}\bar{\Lambda}\bar{\sigma}(\bar{l}(\bar{x}_k)) + \hat{g}(\hat{x}_k, d_k)C_{d_k}(I-\bar{\Lambda})\bar{l}(\bar{x}_k)\big)^T Q\big(\hat{f}(\hat{x}_k)\\
&+ \hat{g}(\hat{x}_k, d_k)C_{d_k}\bar{\Lambda}\bar{\sigma}(\bar{l}(\bar{x}_k)) + \hat{g}(\hat{x}_k, d_k)C_{d_k}(I-\bar{\Lambda})\bar{l}(\bar{x}_k)\big) + r^2\Big(\bar{v}_k^T s^T(\bar{x}_k)Ps(\bar{x}_k)\bar{v}_k + \bar{h}^T(\bar{x}_k)\\
&\times P\bar{h}^T(\bar{x}_k) + 2\bar{h}^T(\bar{x}_k)Ps(\bar{x}_k)\bar{v}_k\Big) + \bar{v}_k^T\big(\bar{g}^T(\bar{x})P\bar{g}(\bar{x}) + \bar{k}^T(\bar{x})C_\tau^T\hat{g}^T(\hat{x},\tau)Q\hat{g}(\hat{x},\tau)C_\tau\bar{k}(\bar{x})\big)\bar{v}_k\\
&+ 2\bar{f}^T(\bar{x}_k)P\bar{g}(\bar{x}_k)\bar{v}_k + 2\big(\hat{f}(\hat{x}_k) + \hat{g}(\hat{x}_k, d_k)C_{d_k}\bar{\Lambda}\bar{\sigma}(\bar{l}(\bar{x}_k)) + \hat{g}(\hat{x}_k, d_k)C_{d_k}(I-\bar{\Lambda})\bar{l}(\bar{x}_k)\big)^T Q\\
&\times \hat{g}(\hat{x},\tau)C_\tau\bar{k}(\bar{x})\bar{v}_k\Big\} - \bar{x}_k^T P\bar{x}_k - \hat{x}_k^T Q\hat{x}_k + \mathbb{E}\{\|\tilde{z}_k\|^2\} - \gamma^2\mathbb{E}\{\|\bar{v}_k\|^2\} + \kappa_3(\|\eta_k\|).
\end{aligned}$$

Completing the squares with respect to $\bar{v}_k$ yields that

$$\begin{aligned}
&\mathbb{E}\{V(\eta_{k+1})|\eta_k\} - V(\eta_k) + \mathbb{E}\{\|\tilde{z}_k\|^2\} - \gamma^2\mathbb{E}\{\|\bar{v}_k\|^2\} + \kappa_3(\|\eta_k\|)\\
=&\mathbb{E}\Big\{-(\bar{v}_k - \bar{v}_k^*)^T A(\bar{x}_k, \hat{x}_k, d_k)(\bar{v}_k - \bar{v}_k^*) + B(\bar{x}_k, \hat{x}_k, d_k)A^{-1}(\bar{x}_k, \hat{x}_k, d_k)B^T(\bar{x}_k, \hat{x}_k, d_k)\\
&+ D(\bar{x}_k, \hat{x}_k, d_k) + r^2\bar{h}^T(\bar{x}_k)P\bar{h}(\bar{x}_k) + \|\tilde{z}_k\|^2 + \kappa_3(\|\eta_k\|)\Big\},
\end{aligned}$$

where $\bar{v}_k^* = A^{-1}(\eta_k, \eta_{\alpha_k}, d_k)B^T(\eta_k, \eta_{\alpha_k}, d_k)$. Based on (16), we have

$$\begin{aligned}
&\mathbb{E}\{V(\eta_{k+1})|\eta_k\} - V(\eta_k) + \mathbb{E}\{\|\tilde{z}_k\|^2\} - \gamma^2\mathbb{E}\{\|\bar{v}_k\|^2\} + \kappa_3(\|\eta_k\|)\\
\leq&B(\bar{x}_k, \hat{x}_k, d_k)A^{-1}(\bar{x}_k, \hat{x}_k, d_k)B^T(\bar{x}_k, \hat{x}_k, d_k) + D(\bar{x}_k, \hat{x}_k, d_k) + r^2\bar{h}^T(\bar{x}_k)P\bar{h}(\bar{x}_k) + \|\tilde{z}_k\|^2\\
&+ \kappa_3(\|\eta_k\|)\\
=&\mathbb{H}(\bar{x}_k, \hat{x}_k, d_k) < 0.
\end{aligned}$$

Now we have proved that

$$\mathbb{E}\{V(\eta_{k+1})|\eta_k\} - V(\eta_k) + \mathbb{E}\{\|\tilde{z}_k\|^2\} - \gamma^2\mathbb{E}\{\|\bar{v}_k\|^2\} + \kappa_3(\|\eta_k\|) < 0. \tag{22}$$

Noticing that $\kappa_3(\|\eta_k\|) > 0$ and summing up (22) from 0 to positive integer N with respect to k, we have

$$\sum_{k=0}^{N}\mathbb{E}\{\|\tilde{z}_k\|^2\} < \gamma^2\sum_{k=0}^{N}\mathbb{E}\{\|\bar{v}_k\|^2\} + \mathbb{E}\{V(0)\} - \mathbb{E}\{V(\eta_{N+1})\}. \tag{23}$$

Considering $V(\eta_{N+1}) > 0$, $V(0) = 0$ under the zero-initial condition and letting $N \to \infty$, we can get

$$\sum_{k=0}^{\infty}\mathbb{E}\{\|\tilde{z}_k\|^2\} < \gamma^2\sum_{k=0}^{\infty}\mathbb{E}\{\|\bar{v}_k\|^2\}, \tag{24}$$

which means that the desired H_∞ performance requirement is met.

When $\bar{v}_k = 0$, it follows from (22) and $\mathbb{E}\{\|\tilde{z}_k\|^2\} > 0$ that

$$\mathbb{E}\{V(\eta_{k+1})|\eta_k\} - V(\eta_k) \leq -\kappa_3(\|\eta_k\|). \tag{25}$$

It follows directly from (16), (25) and Lemma 1 that the augmented system (6) is asymptotically stable in probability, and this concludes the proof. □

Remark 2. In Theorem 1, a nonlinear filter has been designed to guarantee the asymptotic stability in probability. The nonlinear function $\bar{\sigma}$ and the matrices $\bar{\Lambda}$ and $\bar{\Lambda}_i (i = 1, 2, 3)$ represent the influences of stochastic sensor saturations, and the parameter τ quantifies the effects of Markov time delays. The scalar r represents the consideration of the noise w_k. It is noted that the conditions established in (1) are in a very general form that will be applied to some special cases later.

Lemma 2. [2, 37] Let $x, y \in \mathbb{R}^n$ and $\epsilon > 0$. Then we have

$$2x^T y \leq \epsilon x^T x + \epsilon^{-1} y^T y. \tag{26}$$

In the case that $\bar{k}(\bar{x}) \equiv I$, the conditions of Theorem 1 can be further simplified/decoupled.

Corollary 1. Given a disturbance attenuation level $\gamma > 0$ and $\bar{k}(\bar{x}) \equiv I$. If there exist two positive constants μ_1, μ_2, two positive definite matrices $P = P^T > 0$ and $Q = Q^T > 0$ and two $\mathcal{K}$ class functions κ_1 and κ_2 satisfying the following inequalities for all $\tau \in N$:

$$C_\tau^T \hat{g}^T(\hat{x}, \tau) Q \hat{g}(\hat{x}, \tau) C_\tau \leq \mu_1 I, \tag{27}$$

$$\gamma^2 I - r^2 \bar{s}^T(\bar{x}) P \bar{s}(\bar{x}) - \bar{g}^T(\bar{x}) P \bar{g}(\bar{x}) > (\mu_1 + \mu_2) I, \tag{28}$$

$$\begin{aligned}\mathbb{H}_1(\bar{x}, \tau) =& \frac{5}{\mu_2}\left(\|\bar{f}^T(\bar{x}) P \bar{g}(\bar{x})\|^2 + r^4 \|\bar{h}^T(\bar{x}) P \bar{s}(\bar{x})\|^2\right) + \left(\bar{f}(\bar{x}) + \bar{x}\right)^T P \left(\bar{f}(\bar{x}) - \bar{x}\right) \\ &+ r^2 \bar{h}^T(\bar{x}) P \bar{h}(\bar{x}) + \left(\frac{5\bar{\lambda}^2 \mu_1^2}{\mu_2} + \bar{\lambda} + 2\bar{\lambda}\mu_1\right) \|\bar{\sigma}(\bar{l}(\bar{x}))\|^2 + \left(\frac{5(1-\bar{\lambda})^2 \mu_1^2}{\mu_2}\right. \\ &\left. + (1-\bar{\lambda}) + 2(1-\bar{\lambda})\mu_1\right) \|\bar{l}(\bar{x})\|^2 + 2\|\bar{m}(\bar{x})\|^2 + \kappa_1(\|\bar{x}\|) < 0,\end{aligned} \tag{29}$$

$$\begin{aligned}\mathbb{H}_2(\hat{x}, \tau) =& (\hat{f}(\hat{x}) + \hat{x})^T Q (\hat{f}(\hat{x}) - \hat{x}) + \left(\frac{5}{\mu_2} + 1\right) \|\hat{f}^T(\hat{x}) Q \hat{g}(\hat{x}, \tau) C_\tau\|^2 + 2\|\hat{m}(\hat{x})\|^2 \\ &+ \kappa_2(\|\hat{x}\|) < 0,\end{aligned} \tag{30}$$

then the filtering problem for system (1) is solved by (5) in the sense of asymptotic stability in probability.

Proof. With the element inequality $\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$, we have

$$\|\tilde{z}\|^2 = \|\bar{m}(\bar{x}) - \hat{m}(\hat{x})\|^2 \leq 2\|\bar{m}(\bar{x})\|^2 + 2\|\hat{m}(\hat{x})\|^2. \tag{31}$$

Considering (17), (27) and (28), we have

$$\begin{aligned}A(\bar{x}, \hat{x}, \tau) =& \gamma^2 I - r^2 \bar{s}^T(\bar{x}) P \bar{s}(\bar{x}) - \bar{g}^T(\bar{x}) P \bar{g}(\bar{x}) - C_\tau^T \hat{g}^T(\hat{x}, \tau) Q \hat{g}(\hat{x}, \tau) C_\tau \\ \geq& (\mu_1 + \mu_2) I - \mu_1 I = \mu_2 I.\end{aligned} \tag{32}$$

Then it follows from (18), (27), (28) and (32) that

$$\begin{aligned}B(\bar{x},\hat{x},\tau)A^{-1}(\bar{x},\hat{x},\tau)B^T(\bar{x},\hat{x},\tau) \leq & \frac{1}{\mu_2}B(\bar{x},\hat{x},\tau)B^T(\bar{x},\hat{x},\tau) \\ \leq & \frac{5}{\mu_2}\Big(\|\bar{f}^T(\bar{x})P\bar{g}(\bar{x})\|^2 + r^4\|\bar{h}^T(\bar{x})P\bar{s}(\bar{x})\|^2 + \|\hat{f}^T(\hat{x})Q \\ & \times \hat{g}(\hat{x},\tau)C_\tau\|^2 + \bar{\lambda}^2\mu_1^2\|\bar{\sigma}(\bar{l}(\bar{x}))\|^2 + (1-\bar{\lambda})^2\mu_1^2\|\bar{l}(\bar{x})\|^2\Big).\end{aligned} \tag{33}$$

With Lemma 2, we can obtain

$$2\hat{f}^T(\hat{x})Q\hat{g}(\hat{x},\tau)C_\tau\bar{\Lambda}\bar{\sigma}(\bar{l}(\bar{x})) \leq \bar{\lambda}\left(\|\hat{f}^T(\hat{x})Q\hat{g}(\hat{x},\tau)C_\tau\|^2 + \|\bar{\sigma}(\bar{l}(\bar{x}))\|^2\right), \tag{34}$$

$$2\hat{f}^T(\hat{x})Q\hat{g}(\hat{x},\tau)C_\tau(I-\bar{\Lambda})\bar{\sigma}(\bar{l}(\bar{x})) \leq (1-\bar{\lambda})\left(\|\hat{f}^T(\hat{x})Q\hat{g}(\hat{x},\tau)C_\tau\|^2 + \|\bar{l}(\bar{x})\|^2\right). \tag{35}$$

Define the $\mathcal{K}$ class function as $\kappa(\|\eta\|) = \kappa_1(\|\bar{x}\|) + \kappa_2(\|\hat{x}\|)$, and it follows that

$$\begin{aligned}\mathbb{H}(\bar{x},\hat{x},\tau) \leq & \frac{5}{\mu_2}\Big(\|\bar{f}^T(\bar{x})P\bar{g}(\bar{x})\|^2 + r^4\|\bar{h}^T(\bar{x})P\bar{s}(\bar{x})\|^2 + \|\hat{f}^T(\hat{x})Q\hat{g}(\hat{x},\tau)C_\tau\|^2 + \bar{\lambda}^2\mu_1^2 \\ & \times\|\bar{\sigma}(\bar{l}(\bar{x}))\|^2 + (1-\bar{\lambda})^2\mu_1^2\|\bar{l}(\bar{x})\|^2\Big) + \left(\bar{f}(\bar{x})+\bar{x}\right)^T P\left(\bar{f}(\bar{x})-\bar{x}\right) + (\hat{f}(\hat{x})+\hat{x})^T \\ & \times Q(\hat{f}(\hat{x})-\hat{x}) + r^2\bar{h}^T(\bar{x})P\bar{h}(\bar{x}) + \bar{\lambda}\|\bar{\sigma}(\bar{l}(\bar{x}))\|^2 + \bar{\lambda}\|\hat{f}^T(\hat{x})Q\hat{g}(\hat{x},\tau)C_\tau\|^2 \\ & +(1-\bar{\lambda})\|\bar{l}(\bar{x})\|^2 + (1-\bar{\lambda})\|\hat{f}^T(\hat{x})Q\hat{g}(\hat{x},\tau)C_\tau\|^2 + 2\bar{\lambda}\mu_1\|\bar{\sigma}(\bar{l}(\bar{x}))\|^2 + 2(1-\bar{\lambda})\mu_1 \\ & \times\|\bar{l}(\bar{x})\|^2 + 2\|\bar{m}(\bar{x})\|^2 + 2\|\hat{m}(\hat{x})\|^2 + \kappa_1(\|\bar{x}\|) + \kappa_2(\|\hat{x}\|) \\ = & \mathbb{H}_1(\bar{x},\tau) + \mathbb{H}_2(\hat{x},\tau) < 0.\end{aligned} \tag{36}$$

The rest of the proof follows directly from that of Theorem 1. □

If the adopted positive definite function is dependent on not only the augmented states but also the values of time delays, then the sufficient conditions would be obtained subsequently.

Theorem 2. Given a disturbance attenuation level $\gamma > 0$. If there exist two sets of positive-definite matrices $P(\tau) = P^T(\tau) > 0$ and $Q(\tau) = Q^T(\tau) > 0$ for all $\tau \in N$, and a $\mathcal{K}$ class function κ satisfying the inequalities

$$\begin{cases}\mathbb{H}(\bar{x},\tilde{x},\tau) = & B(\bar{x},\tilde{x},\tau)A^{-1}(\bar{x},\tilde{x},\tau)B^T(\bar{x},\tilde{x},\tau) + r^2\bar{h}^T(\bar{x})\tilde{P}(\tau)\bar{h}(\bar{x}) + \bar{f}^T(\bar{x})\tilde{P}(\tau)\bar{f}(\bar{x}) \\ & +D(\bar{x},\tilde{x},\tau) - \bar{x}^TP(\tau)\bar{x} - \hat{x}^TQ(\tau)\hat{x} + \|\tilde{z}\|^2 + \kappa(\|\eta\|) < 0, \\ & \text{for any } \bar{x},\hat{x}\in\mathbb{R}^{d\times n_x}, \tau\in N, \\ A(\bar{x},\tilde{x},\tau) > 0, & \text{for any } \bar{x},\hat{x}\in\mathbb{R}^{d\times n_x}, \tau\in N,\end{cases} \tag{37}$$

where $\tilde{P}(\tau) = \sum_{i=0}^{d-1} \pi_{\tau i} P(i)$, $\tilde{Q}(\tau) = \sum_{i=0}^{d-1} \pi_{\tau i} Q(i)$ and

$$A(\bar{x}, \tilde{x}, \tau) =\gamma^2 I - r^2 \bar{s}^T(\bar{x}) \tilde{P}(\tau) \bar{s}(\bar{x}) - \bar{g}^T(\bar{x}) \tilde{P}(\tau) \bar{g}(\bar{x}) - \bar{k}^T(\bar{x}) S_1(\hat{x}, \tau) \bar{k}(\bar{x}), \tag{38}$$

$$B(\bar{x}, \tilde{x}, \tau) =\bar{f}(\bar{x}) \tilde{P}(\tau) \bar{g}(\bar{x}) + \hat{f}(\hat{x}) \tilde{Q}(\tau) \hat{g}(\hat{x}, \tau) C_\tau \bar{k}(\bar{x}) + r^2 \bar{h}^T(\bar{x}) \tilde{P}(\tau) \bar{s}(\bar{x}) + S_2^T(\bar{x}, \hat{x}, \tau) \tilde{Q}(\tau) \hat{g}(\hat{x}, \tau) C_\tau \bar{k}(\bar{x}), \tag{39}$$

$$D(\bar{x}, \tilde{x}, \tau) =\hat{f}^T(\hat{x}) \tilde{Q}(\tau) \hat{f}(\hat{x}) + 2\hat{f}^T(\hat{x}) \tilde{Q}(\tau) S_2(\bar{x}, \hat{x}, \tau) + \bar{\sigma}^T(\bar{l}(\bar{x})) \left(\bar{\Lambda}_1 \circ S_1(\hat{x}, \tau)\right) \bar{\sigma}(\bar{l}(\bar{x})) + 2\bar{\sigma}^T(\bar{l}(\bar{x}) \left(\bar{\Lambda}_3 \circ S_1(\hat{x}, \tau)\right) \bar{l}(\bar{x}) + \bar{l}^T(\bar{x}) \left(\bar{\Lambda}_2 \circ S_1(\hat{x}, \tau)\right) \bar{l}(\bar{x}), \tag{40}$$

$$S_1(\hat{x}, \tau) =C_\tau^T \hat{g}^T(\hat{x}, \tau) \tilde{Q}(\tau) \hat{g}(\hat{x}, \tau) C_\tau, \tag{41}$$

$$S_2(\bar{x}, \hat{x}, \tau) =\hat{g}(\hat{x}, \tau) C_\tau \bar{\Lambda} \bar{\sigma}(\bar{l}(\bar{x})) + \hat{g}(\hat{x}, \tau) C_\tau (I - \bar{\Lambda}) \bar{l}(\bar{x}), \tag{42}$$

then the filtering problem for system (1) is solved by (5) in the sense of asymptotic stability in probability.

Proof. The positive definite function is taken as:

$$V(\eta_k, d_k) = \bar{x}_k^T P(d_k) \bar{x}_k + \hat{x}_k^T Q(d_k) \hat{x}_k. \tag{43}$$

We take

$$\bar{\rho} = \max \left\{ \max_{\tau \in N} \lambda_{\max}(P(\tau)), \max_{\tau \in N} \lambda_{\max}(Q(\tau)) \right\}, \underline{\rho} = \min \left\{ \min_{\tau \in N} \lambda_{\min}(P(\tau)), \min_{\tau \in N} \lambda_{\min}(Q(\tau)) \right\},$$

then two $\mathcal{K}_\infty$ functions can be defined as $\kappa_1(\|\eta\|) = \underline{\rho}\|\eta\|^2$ and $\kappa_2(\|\eta\|) = \bar{\rho}\|\eta\|^2$, and it follows that $\kappa_1(\|\eta\|) \le V(\eta, \tau) \le \kappa_2(\|\eta\|)$ for all $\tau \in N$.

With notations above, one has

$$\begin{aligned}
&\mathbb{E}\left\{V(\eta_{k+1}, d_{k+1}) | \eta_k, d_k\right\} - V(\eta_k, d_k) + \mathbb{E}\left\{\|\tilde{z}_k\|^2\right\} - \gamma^2 \mathbb{E}\left\{\|\bar{v}_k\|^2\right\} + \kappa(\|\eta_k\|) \\
=&\mathbb{E}\Big\{ - (\bar{v}_k - \bar{v}_k^*)^T A(\bar{x}_k, \tilde{x}_k, d_k)(\bar{v}_k - \bar{v}_k^*) + B(\bar{x}_k, \tilde{x}_k, d_k) A^{-1}(\bar{x}_k, \tilde{x}_k, d_k) B^T(\bar{x}_k, \tilde{x}_k, d_k) \\
&+ D(\bar{x}_k, \tilde{x}_k, d_k) + r^2 \bar{h}^T(\bar{x}_k) \tilde{P}(d_k) \bar{h}(\bar{x}_k) + \bar{f}^T(\bar{x}_k) \tilde{P}(d_k) \bar{f}(\bar{x}_k) - \bar{x}_k^T P(d_k) \bar{x}_k - \hat{x}_k^T Q(d_k) \hat{x}_k \\
&+ \|\tilde{z}_k\|^2 + \kappa(\|\eta_k\|)\Big\} \\
\le&\mathbb{H}(\bar{x}_k, \hat{x}_k, d_k) < 0.
\end{aligned}$$

The rest of the proof follows directly from that of Theorem 1 and is therefore omitted. □

We also have the following corollary from Theorem 2 and Corollary 1.

Corollary 2. Given a disturbance attenuation level $\gamma > 0$ and $\bar{k}(\bar{x}) \equiv I$. If there exist two positive constants μ_1, μ_2, two sets of positive definite matrices $P(\tau) = P^T(\tau) > 0$ and $Q(\tau) = Q^T(\tau) > 0$ for all $\tau \in N$, and two $\mathcal{K}$ class functions κ_1 and κ_2 satisfying the following inequalities for all

$\tau \in N$:

$$C_\tau^T \hat{g}^T(\hat{x},\tau)\tilde{Q}(\tau)\hat{g}(\hat{x},\tau)C_\tau \leq \mu_1 I, \tag{44}$$

$$\gamma^2 I - r^2 \bar{s}^T(\bar{x})\tilde{P}(\tau)\bar{s}(\bar{x}) - \bar{g}^T(\bar{x})\tilde{P}(\tau)\bar{g}(\bar{x}) > (\mu_1+\mu_2) I, \tag{45}$$

$$\begin{aligned}\mathbb{H}_1(\bar{x},\tau) =& \frac{5}{\mu_2}\left(\|\bar{f}^T(\bar{x})\tilde{P}(\tau)\bar{g}(\bar{x})\|^2 + r^4\|\bar{h}^T(\bar{x})\tilde{P}(\tau)\bar{s}(\bar{x})\|^2\right) + \bar{f}^T(\bar{x})\tilde{P}(\tau)\bar{f}(\bar{x}) \\ &+ r^2\bar{h}^T(\bar{x})\tilde{P}(\tau)\bar{h}(\bar{x}) - \bar{x}^T P(\tau)\bar{x} + \left(\frac{5\bar{\lambda}^2\mu_1^2}{\mu_2} + \bar{\lambda} + 2\bar{\lambda}\mu_1\right)\|\bar{\sigma}(\bar{l}(\bar{x}))\|^2 \\ &+ \left(\frac{5(1-\bar{\lambda})^2\mu_1^2}{\mu_2} + (1-\bar{\lambda}) + 2(1-\bar{\lambda})\mu_1\right)\|\bar{l}(\bar{x})\|^2 + 2\|\bar{m}(\bar{x})\|^2 + \kappa_1(\|\bar{x}\|) < 0,\end{aligned} \tag{46}$$

$$\begin{aligned}\mathbb{H}_2(\hat{x},\tau) =& \hat{f}^T(\hat{x})\tilde{Q}(\tau)\hat{f}(\hat{x}) - \hat{x}^T Q(\tau)\hat{x} + \left(\frac{5}{\mu_2}+1\right)\|\hat{f}^T(\hat{x})\tilde{Q}(\tau)\hat{g}(\hat{x},\tau)C_\tau\|^2 + 2\|\hat{m}(\hat{x})\|^2 \\ &+ \kappa_2(\|\hat{x}\|) < 0,\end{aligned} \tag{47}$$

then the filtering problem for system (1) is solved by (5) in the sense of asymptotic stability in probability.

Proof. The proof of the corollary is a straightforward consequence of that of Corollary 1 and is omitted here for conciseness. □

Remark 3. A set of nonlinear filters has been obtained via solving nonlinear matrix inequalities for some positive definite functions that could be either delay-dependent or delay-independent. Sufficient conditions have been achieved under which the estimation is asymptotically stable in probability in the disturbance-free case and robust to the exogenous disturbances under the zero-initial condition. In real-world applications, a linear filter is much easier to implement than a nonlinear one. As a result, a linear filter for the nonlinear system (1) would be investigated next.

Consider the filter of the following structure:

$$\begin{cases} \hat{x}_{k+1} = F\hat{x}_k + G(d_k)y_k, \\ \hat{z}_k = M\hat{x}_k, \end{cases} \tag{48}$$

where $\hat{x}_k \in \mathbb{R}^{d\times n_x}$ is the state estimate; $\hat{z}_k \in \mathbb{R}^{n_z}$ is the estimate of z_k. F, G and M are filter parameters of appropriate dimensions to be determined.

Similar to the nonlinear case, we can have the following augmented system.

$$\begin{cases} \eta_{k+1} = \tilde{f}(\eta_k,d_k) + \tilde{g}(\eta_k,d_k)v_k + (\tilde{h}(\eta_k) + \tilde{s}(\eta_k)\bar{v}_k)w_k, \\ \tilde{z}_k = \bar{m}(\bar{x}_k) - M\hat{x}_k, \end{cases} \tag{49}$$

where

$$\tilde{f}(\eta_k,d_k) = \begin{bmatrix} \bar{f}(\bar{x}_k) \\ F\hat{x}_k + G(d_k)C_{d_k}\Lambda_k\bar{\sigma}(\bar{l}(\bar{x}_k)) + G(d_k)C_{d_k}(I-\Lambda_k)\bar{l}(\bar{x}_k) \end{bmatrix},$$

$$\tilde{g}(\eta_k,d_k) = \begin{bmatrix} \bar{g}(\bar{x}_k) \\ G(d_k)C_{d_k}\bar{k}(\bar{x}_k) \end{bmatrix}, \tilde{h}(\eta_k) = \begin{bmatrix} \bar{h}(\bar{x}_k) \\ 0 \end{bmatrix}, \tilde{s}(\eta_k) = \begin{bmatrix} \bar{s}(\bar{x}_k) \\ 0 \end{bmatrix}.$$

In virtue of Theorem 1, the following results can be obtained.

Theorem 3. Given a disturbance attenuation level $\gamma > 0$. If there exist two positive definite matrices $P = P^T > 0$ and $Q = Q^T > 0$, and a $\mathcal{K}$ class function κ_3 satisfying the inequalities

$$\begin{cases} \mathbb{H}(\bar{x},\hat{x},\tau) = B(\bar{x},\hat{x},\tau)A^{-1}(\bar{x},\hat{x},\tau)B^T(\bar{x},\hat{x},\tau) + r^2\bar{h}^T(\bar{x})P\bar{h}(\bar{x}) + D(\bar{x},\hat{x},\tau) \\ \qquad +\|\tilde{z}\|^2 + \kappa_3(\|\eta\|) < 0, \text{ for any } \bar{x},\hat{x} \in \mathbb{R}^{d\times n_x}, \tau \in N \\ A(\bar{x},\tilde{x},\tau) > 0, \text{ for any } \bar{x},\tilde{x} \in \mathbb{R}^{d\times n_x}, \tau \in N, \end{cases} \tag{50}$$

where

$$A(\bar{x},\hat{x},\tau) = \gamma^2 I - r^2\bar{s}^T(\bar{x})P\bar{s}(\bar{x}) - \bar{g}^T(\bar{x})P\bar{g}(\bar{x}) - \bar{k}^T(\bar{x})S_1(\tau)\bar{k}(\bar{x}), \tag{51}$$

$$B(\bar{x},\hat{x},\tau) = r^2\bar{h}^T(\bar{x})P\bar{s}(\bar{x}) + \bar{f}^T(\bar{x})P\bar{g}(\bar{x}) + \hat{x}^T F^T QG(\tau)C_\tau\bar{k}(\bar{x}) + S_2^T(\bar{x},\tau)QG(\tau)C_\tau\bar{k}(\bar{x}), \tag{52}$$

$$\begin{aligned} D(\bar{x},\hat{x},\tau) =& (\bar{f}(\bar{x}) + \bar{x})^T P(\bar{f}(\bar{x}) - \bar{x}) + \hat{x}^T(F+I)^T Q(F-I)\hat{x} + 2\hat{x}^T F^T QS_2(\bar{x},\tau) \\ &+ \bar{\sigma}^T(\bar{l}(\bar{x}))\big(\bar{\Lambda}_1 \circ S_1(\tau)\big)\bar{\sigma}(\bar{l}(\bar{x})) + 2\bar{\sigma}^T(\bar{l}(\bar{x}))\big(\bar{\Lambda}_3 \circ S_1(\tau)\big)\bar{l}(\bar{x}) \\ &+ \bar{l}^T(\bar{x})\left(\bar{\Lambda}_2 \circ S_1(\tau)\right)\bar{l}(\bar{x}), \end{aligned} \tag{53}$$

$$S_1(\tau) = C_\tau^T G^T(\tau)QG(\tau)C_\tau, \tag{54}$$

$$S_2(\bar{x},\tau) = G(\tau)C_\tau\bar{\Lambda}\bar{\sigma}(\bar{l}(\bar{x})) + G(\tau)C_\tau(I-\bar{\Lambda})\bar{l}(\bar{x}), \tag{55}$$

then the filtering problem for system (1) is solved by (48) in the sense of asymptotic stability in probability.

Proof. This proof can be completed by following the similar line of Theorem 1 and is therefore omitted. □

Similar to Corollaries 1-2, we can have certain decoupled sufficient conditions in the following corollaries.

Corollary 3. Given a disturbance attenuation level $\gamma > 0$ and $\bar{k}(\bar{x}) \equiv I$. If there exist two positive constants μ_1, μ_2, two positive definite matrices $P = P^T > 0$ and $Q = Q^T > 0$, and two $\mathcal{K}$ class functions κ_1 and κ_2 satisfying the following inequalities for all $\tau \in N$:

$$C_\tau^T G^T QGC_\tau \le \mu_1 I, \tag{56}$$

$$\gamma^2 I - r^2\bar{s}^T(\bar{x})P\bar{s}(\bar{x}) - \bar{g}^T(\bar{x})P\bar{g}(\bar{x}) > (\mu_1+\mu_2)I, \tag{57}$$

$$\begin{aligned} \mathbb{H}_1(\bar{x},\tau) =& \frac{5}{\mu_2}\left(\|\bar{f}^T(\bar{x})P\bar{g}(\bar{x})\|^2 + r^4\|\bar{h}^T(\bar{x})P\bar{s}(\bar{x})\|^2\right) + \left(\bar{f}(\bar{x}) + \bar{x}\right)^T P\left(\bar{f}(\bar{x}) - \bar{x}\right) \\ &+ r^2\bar{h}^T(\bar{x})P\bar{h}(\bar{x}) + \left(\frac{5\bar{\lambda}^2\mu_1^2}{\mu_2} + \bar{\lambda} + 2\bar{\lambda}\mu_1\right)\|\bar{\sigma}(\bar{l}(\bar{x}))\|^2 \\ &+ \left(\frac{5(1-\bar{\lambda})^2\mu_1^2}{\mu_2} + (1-\bar{\lambda}) + 2(1-\bar{\lambda})\mu_1\right)\|\bar{l}(\bar{x})\|^2 + 2\|\bar{m}(\bar{x})\|^2 + \kappa_1(\|\bar{x}\|) < 0, \end{aligned} \tag{58}$$

$$\begin{aligned} \mathbb{H}_2(\hat{x},\tau) =& \hat{x}^T(F+I)^T Q(F-I)\hat{x} + \left(\frac{5}{\mu_2} + 1\right)\|\hat{x}^T F^T QGC_\tau\|^2 + 2\|M\hat{x}\|^2 \\ &+ \kappa_2(\|\hat{x}\|) < 0, \end{aligned} \tag{59}$$

then the filtering problem for system (1) is solved by (48) in the sense of asymptotic stability in probability where $G(\tau) = G$ for any $\tau \in N$.

Corollary 4. Given a disturbance attenuation level $\gamma > 0$ and $\bar{k}(\bar{x}) \equiv I$. If there exist two positive constants μ_1, μ_2, two sets of positive-definite matrices $P(\tau) = P^T(\tau) > 0$ and $Q(\tau) = Q^T(\tau) > 0$ for all $\tau \in N$, and two $\mathcal{K}$ class functions κ_1 and κ_2 satisfying the following inequalities for all $\tau \in N$:

$$C_\tau^T G^T(\tau)\tilde{Q}(\tau)G(\tau)C_\tau \le \mu_1 I, \tag{60}$$

$$\gamma^2 I - r^2\bar{s}^T(\bar{x})\tilde{P}(\tau)\bar{s}(\bar{x}) - \bar{g}^T(\bar{x})\tilde{P}(\tau)\bar{g}(\bar{x}) > (\mu_1 + \mu_2) I, \tag{61}$$

$$\begin{aligned}\mathbb{H}_1(\bar{x},\tau) =& \frac{5}{\mu_2}\left(\|\bar{f}^T(\bar{x})\tilde{P}(\tau)\bar{g}(\bar{x})\|^2 + r^4\|\bar{h}^T(\bar{x})\tilde{P}(\tau)\bar{s}(\bar{x})\|^2\right) + \bar{f}^T(\bar{x})\tilde{P}(\tau)\bar{f}(\bar{x}) \\ &+ r^2\bar{h}^T(\bar{x})\tilde{P}(\tau)\bar{h}(\bar{x}) - \bar{x}^T P(\tau)\bar{x} + \left(\frac{5\bar{\lambda}^2\mu_1^2}{\mu_2} + \bar{\lambda} + 2\bar{\lambda}\mu_1\right)\|\bar{\sigma}(\bar{l}(\bar{x}))\|^2 \\ &+ \left(\frac{5(1-\bar{\lambda})^2\mu_1^2}{\mu_2} + (1-\bar{\lambda}) + 2(1-\bar{\lambda})\mu_1\right)\|\bar{l}(\bar{x})\|^2 + 2\|\bar{m}(\bar{x})\|^2 + \kappa_1(\|\bar{x}\|) < 0,\end{aligned} \tag{62}$$

$$\begin{aligned}\mathbb{H}_2(\hat{x},\tau) =& \hat{x}^T F^T\tilde{Q}(\tau)F\hat{x} - \hat{x}^T Q(\tau)\hat{x} + \left(\frac{5}{\mu_2} + 1\right)\|\hat{x}^T F^T\tilde{Q}(\tau)\hat{g}(\hat{x},\tau)C_\tau\|^2 + 2\|M\hat{x}\|^2 \\ &+ \kappa_2(\|\hat{x}\|) < 0,\end{aligned} \tag{63}$$

then the filtering problem for system (1) is solved by (48) in the sense of asymptotic stability in probability.

The proofs of Corollaries 3-4 follow directly from those of Corollaries 1-2 and are therefore omitted.

Remark 4. The filtering problem for a class of nonlinear systems with stochastic sensor saturations and Markov time delays has been investigated in the sense of asymptotic stability in probability. Sufficient conditions have been established to guarantee the asymptotic stability in probability of the estimation process in the noise-free case and the robustness of the filtering error to the exogenous disturbances under the zero-initial condition. Specifically, the linear filters have been considered for the convenience of practical applications. Both time-dependent and time-independent linear filters have been obtained by choosing proper positive definite functions.

4. Simulation Examples

4.1. Example A

Consider the following nonlinear system

$$\begin{cases} x_{k+1} = 0.7x_k + 0.2\sin(x_k) + v_k + 0.4\cos(x_k)v_k w_k, \\ z_k = \frac{2}{3}x_k, \\ \tilde{y}_k = \lambda_k\sigma(l(x_k)) + (1-\lambda_k)l(x_k) + v_k, \\ y_k = \tilde{y}_{k-d_k}, \end{cases} \tag{64}$$

where d_k belongs to the set $\{0,1\}$ and its transition probability matrix is given by $\Pi = [\mathrm{col}\{0.7, 0.8\}, \mathrm{col}\{0.3, 0.2\}]$. The variance of w_k is 1 and $\mathrm{Prob}\{\lambda_k = 1\} = 0.9$. The saturation level is 0.1. The prescribed disturbance attenuation level is set to be $\gamma = 1.414$. Choose $\kappa_1(\|\bar{x}\|) = 0.01\|\bar{x}\|^2$, $\kappa_2(\|\hat{x}\|) = 0.01\|\hat{x}\|^2$, $\mu_1 = 0.5$ and $\mu_2 = 1$. Then we can get the following nonlinear filter with

Corollary 1:

$$\begin{cases} \hat{x}_{k+1} = 0.4\hat{x}_k \sin(\hat{x}_{k-1}) + 0.65y_k, \\ \hat{z}_k = 0.5\hat{x}_k + 0.3\hat{x}_{k-1}. \end{cases} \tag{65}$$

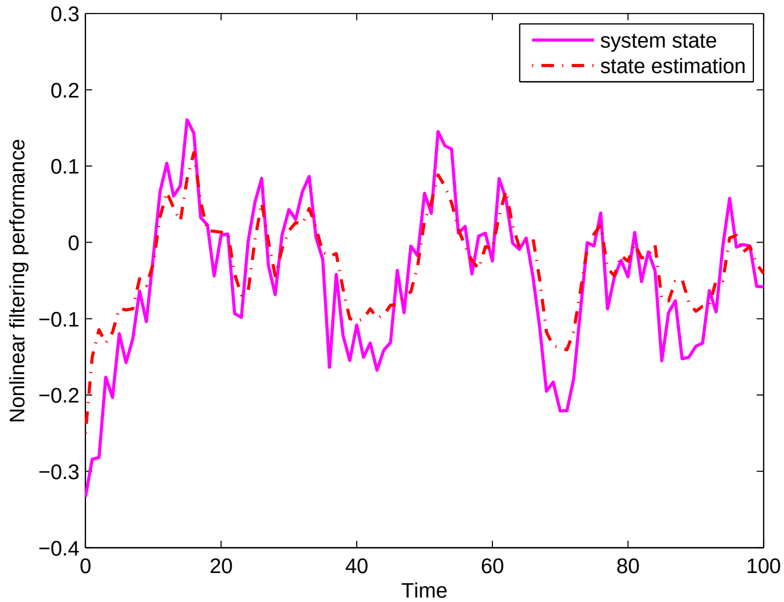

Fig. 1. *Nonlinear filtering performance*

As shown in Fig. 1, the established nonlinear filter can estimate z_k well.

4.2. Example B

An inverted pendulum example is presented in this subsection to demonstrate the effectiveness of the proposed approach. An appropriate controller has been pre-designed to stabilize the system. The model of the inverted pendulum system is given by [31]

$$ml^2\ddot{\theta} - mgl\sin(\theta) + (\varsigma + \omega)\dot{\theta} + \kappa\theta = u + 2\nu_2, \tag{66}$$

where m is the mass, l is the length of the inverted pendulum, g is the gravitation coefficient, θ is the inclination angle, ς is the spring coefficient, κ is the damping parameter, ω is the white noise for the damping coefficient, ν_2 is the external disturbance, and u is the control input that has been pre-designed as $u = k_1\theta + k_2 ml^2\dot{\theta}$. Two output measurements without stochastic sensor saturation or transmission delays are $\tilde{y}_1 = \theta + \nu_1$ and $\tilde{y}_2 = ml^2\dot{\theta} + \dfrac{\sin(ml^2\dot{\theta})}{ml^2 g} + \nu_2$, respectively. The regulated output is described by $z = \dfrac{\theta + ml^2\dot{\theta}}{10}$. Since inertial sensors in reality usually undergo saturations [38], it is reasonable to consider sensor saturations in the inverted pendulum example (66) whose outputs are related to angle and angular acceleration.

Taking $x_1 = \theta$, $x_2 = ml^2\dot{\theta}$, and the sampling period as T, the system model can be discretized and realized by the state-space model as follows:

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{T}{ml^2} \\ -\kappa T + Tk_1 & 1 - \dfrac{T\varsigma}{ml^2} + Tk_2 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} 0 \\ Tmgl\sin(x_{1,k}) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 \\ 0 & 2T \end{bmatrix} \begin{bmatrix} \nu_{1,k} \\ \nu_{2,k} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{\sqrt{T}}{ml^2} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} \omega_k. \tag{67}$$

The output measurement equation with stochastic sensor saturation is discretized as

$$\begin{bmatrix} \tilde{y}_{1,k} \\ \tilde{y}_{2,k} \end{bmatrix} = \lambda_k \sigma\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{\sin(x_{2,k})}{ml^2 g} \end{bmatrix} \right) + (1-\lambda_k)\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} 0 \\ \dfrac{\sin(x_{2,k})}{ml^2 g} \end{bmatrix} \right) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \nu_{1,k} \\ \nu_{2,k} \end{bmatrix}. \tag{68}$$

Due to the delay, the received measurement is $y_k = \tilde{y}_{k-d_k}$, where d_k is the random time delay governed by a Markov chain. The state d_k belongs to the set $\{0, 1, 2\}$ and its transition probability matrix is given by $\Pi = [\mathrm{col}\{0.2, 0.1, 0.2\}, \mathrm{col}\{0.8, 0.4, 0.4\}, \mathrm{col}\{0, 0.5, 0.4\}]$. Furthermore, the regulated output can be discretized as $z_k = 0.1x_{1,k} + 0.1x_{2,k}$. The system parameters are $m = 0.5$kg, $l = 0.5$m, $\varsigma = 0.25$, $k_1 = -49.5$, $k_2 = -167.5$, sampling period $T = 0.01$s, and $\kappa = 0.5$N/m. The variance of ω is 0.01 and $\mathrm{Prob}\{\lambda_k = 1\} = 0.2$. The saturation level is 1. The prescribed disturbance attenuation level is set to be $\gamma = 0.707$.

Consider the linear filter in the form of (48) and let $\kappa_1(\|\bar{x}\|) = 0.5\|\bar{x}\|^2$, $\kappa_2(\|\hat{x}\|) = 0.5\|\hat{x}\|^2$. F and M have been selected to reflect the linear part of the system dynamics as follows:

$$F = \begin{bmatrix} F_0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \quad M = \begin{bmatrix} M_0 & 0 & 0 \end{bmatrix},$$

where

$$F_0 = \begin{bmatrix} 1 & \dfrac{T}{ml^2} \\ -\kappa T + Tk_1 & 1 - \dfrac{T\varsigma}{ml^2} + Tk_2 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}.$$

In such a case, G can be calculated using Matlab for both delay-dependent and delay-independent filters.

Using Corollaries 3 and 4 with $\mu_1 = \mu_2 = 0.15$, the feasible solutions for G can be obtained as

$$G = \begin{bmatrix} -0.0021 & -0.0031 & 0.0018 & -0.0034 & -0.0032 & 0.0009 \\ 0.0007 & -0.0119 & -0.0045 & -0.0125 & 0.0024 & 0.0003 \end{bmatrix}$$

for the delay-independent filter, and

$$G(0) = \begin{bmatrix} -0.0032 & -0.0013 & -0.0013 & -0.0014 & -0.0033 & 0.0003 \\ 0.0002 & -0.0077 & -0.0003 & -0.0083 & 0.0011 & -0.0048 \end{bmatrix},$$
$$G(1) = \begin{bmatrix} -0.0012 & -0.0028 & 0.0042 & -0.0028 & -0.0017 & 0.0017 \\ 0.0004 & -0.0079 & -0.0010 & -0.0097 & 0.0032 & 0.0003 \end{bmatrix},$$
$$G(2) = \begin{bmatrix} -0.0016 & -0.0024 & 0.0022 & -0.0021 & -0.0013 & 0.0007 \\ 0.0003 & -0.0076 & -0.0006 & -0.0094 & 0.0019 & -0.0012 \end{bmatrix}$$

for the delay-dependent filter.

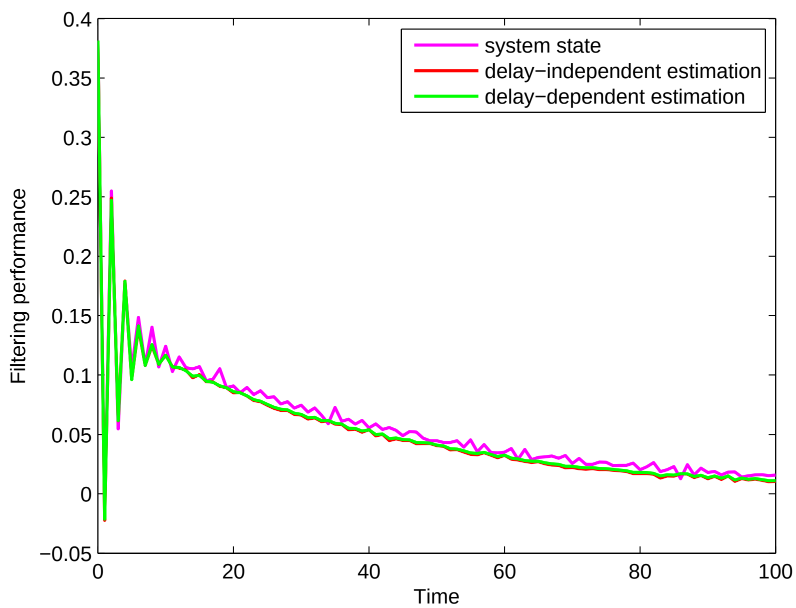

Fig. 2. *Delay-dependent and delay-independent filtering performances*

As illustrated in Fig. 2, both the delay-dependent and delay-independent filters achieve acceptable estimation performance. The filtering errors are depicted in Fig. 3. The average filtering error obtained with the delay-dependent filter is 5.258×10^{-3} in 100 Monte-Carlo simulations. With the delay-independent filter, the average filtering error is 5.789×10^{-3}. It can be easily seen that the delay-dependent filter can generate smaller filtering error compared to the delay-independent one due to the consideration of the information on time-delays.

5. Conclusion

In this paper, the filtering problem has been investigated, in the sense of asymptotic stability in probability, for a class of nonlinear systems with stochastic sensor saturations and Markovian measurement transmission delays. A Bernoulli distributed sequence and a discrete-time Markov chain with finite states have been introduced to govern the random sensor saturations and the transmission time delays, respectively. Sufficient conditions have been achieved to guarantee that the filtering process is asymptotically stable in probability in the disturbance-free case and satisfies the

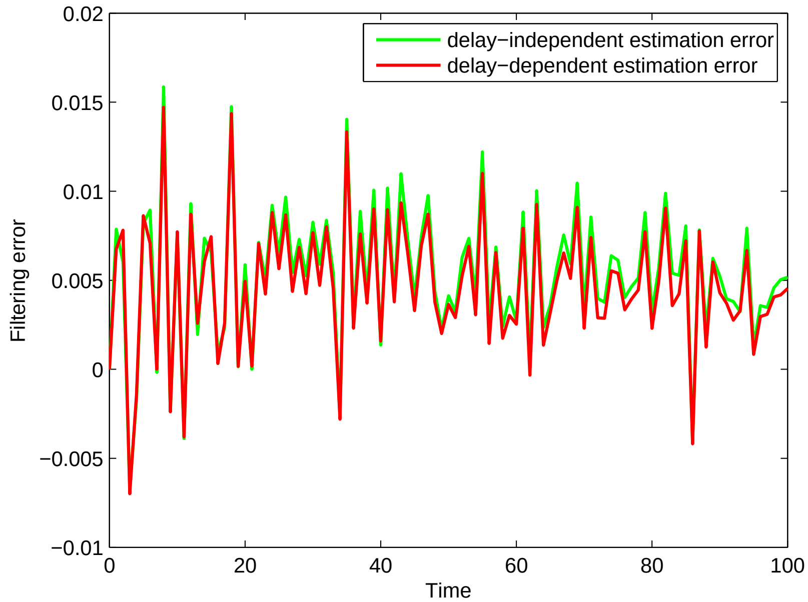

Fig. 3. *Delay-dependent and delay-independent filtering errors*

H_∞ criterion with respect to nonzero exogenous disturbances under the zero-initial condition. The results have been specialized to delay-independent and delay-dependent linear filters as well. Two simulation examples have been presented to show the effectiveness of the proposed algorithms.

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