Variance-Constrained H_∞ Control for A Class of Nonlinear Stochastic Discrete Time-Varying Systems: The Event-Triggered Design ⋆

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Abstract

In this paper, a general event-triggered framework is set up to deal with the variance-constrained H_∞ control problem for a class of discrete time-varying systems with randomly occurring saturations, stochastic nonlinearities and state-multiplicative noises. Based on the relative error with respect to the measurement signal, an event indicator variable is introduced and the corresponding event-triggered scheme is proposed in order to determine whether the measurement output is transmitted to the controller or not. The stochastic nonlinearities under consideration are characterized by statistical means which can cover several classes of well-studied nonlinearities. A set of unrelated random variables is exploited to govern the phenomena of randomly occurring saturations, stochastic nonlinearities and state-dependent noises. The purpose of the addressed multiobjective control problem is to design a set of time-varying output feedback controller such that, over a finite horizon, the closed-loop system achieves both the prescribed H_∞ noise attenuation level and the state covariance constraints. A recursive matrix inequality approach is developed to derive the sufficient conditions for the existence of the desired finite-horizon controllers, and the analytical characterization of such controllers is also given. Simulation studies are conducted to demonstrate the effectiveness of the developed controller design scheme.

Key words: Event-triggered mechanism; Stochastic control; Time-varying systems; Variance constraints; Randomly occurring saturations.

1 Introduction

In the past few decades, there has been a surge of research interest in the stochastic control problem since stochastic modeling has been successfully applied in many fields. A large body of literature has been devoted to the stochastic control or filtering problem for different systems such as polynomial stochastic systems [1, 2, 4], Markovian jumping systems [20], switched stochastic systems [13], discrete-time stochastic systems with state-dependent noises [17], nonlinear stochastic systems [8, 19] and stochastic sampled-data control system [21]. Among various stochastic control schemes, the covariance control (CC) theory has gained particular research attention due primarily to the fact that the performance requirements of many engineering control systems are naturally expressed as the upper bounds on the steady-state variances [11]. It has been shown that the CC approach is ideally suited to handle the multi-objective design problems where the multiple objectives include, but are not limited to, variance constraints, H_2-norm specification, H_∞ performance index and pole placement [6]. The CC theory was originally developed for linear systems and has been recently extended to nonlinear stochastic systems [11]. It is worth pointing out that most results concerning the CC theory have focused on the steady-state behaviors for time-invariant systems over an infinite horizon. However, virtually almost all real-time control processes are time-varying especially when the noise inputs are nonstationary [10,12]. In such cases, it would make more sense to consider the covariance control problems for time-varying systems over a finite-horizon in order to provide a better transient performance.

Due to physical and safety constraints, the sensor saturation is probably one of the most commonly encountered phenomena in practical control systems that can severely degrade the system performance or even lead to unstable behaviors. So far, considerable research attention has been paid to the filtering and control problems for systems with sensor saturation, see [14, 16] and the references therein, where the saturation has been implicitly assumed to occur definitely, i.e., the sensor always undergoes saturation. Such an assumption, however, is

⋆ This work was supported in part by the National Natural Science Foundation of China under Grants 61329301, 61422301, 61374127, 61473076 and 61573246, the Outstanding Youth Science Foundation of Heilongjiang Province under Grant JC2015016, the Program for Professor of Special Appointment (Eastern Scholar) at Shanghai Institutions of Higher Learning, the Shanghai Rising-Star Programme of China under Grant 16QA1403000 and the Alexander von Humboldt Foundation of Germany.

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not always true. For example, in a networked environment, the sensor saturations may occur in a probabilistic way where the saturation amplitude/intensity may be randomly changeable. Such kind of randomly occurring sensor saturation (ROSS) may result from networked-induced intermittent sensor failures, sensor aging or sudden environment changes. On the other hand, stochastic nonlinearities are often found in networked control systems where the nonlinearities are induced by randomly fluctuated network loads due mainly to the communication limitations. Up to now, some initial efforts have been made on the filtering and control problems for systems with stochastic nonlinearities [10], while most available results have been concerned with additive noises only. Note that many plants may be modeled by systems with state-dependent noises and some characteristics of nonlinear systems can be closely approximated by models with state-multiplicative noises rather than by linearized models. It is, therefore, one of the motivations of the present research to investigate how the phenomena of ROSS, stochastic nonlinearities and state-multiplicative noises influence the state variance and H_∞ performance of a class of time-varying control systems.

In networked control systems, an important issue is how to transmit signals more effectively by utilizing the available but limited network bandwidth. To alleviate the unnecessary waste of communication/computation resources that often occurs in conventional time-triggered signal transmissions, a recently popular communication schedule called event-triggered strategy has been proposed in [9, 15, 18, 22, 23]. The triggering mechanism refers to the situation where the measurement output is transmitted to a remote controller/filter only when certain conditions are satisfied. In other words, a constant measurement signal is maintained until a specified event condition is violated in an event generator. In comparison with the conventional time-triggered communication, a notable advantage of the event triggering scheme is its capability of reducing redundant transmissions while preserving the guaranteed system performance. In recent years, increasing attention has been drawn on the event-triggered techniques for stochastic systems and many important results have been reported in the literature, see [5, 7]. However, it should be pointed out that, many established results referring to event-triggering schemes are in the framework of continuous-time systems and, when it comes to the discrete-time systems, the corresponding results have been scattered. The representative one [7] has addressed the problem of designing optimal event-triggered controllers under costly observations, in which the optimal event-trigger depends on the difference between the state estimates at controller and event-trigger. Although the importance of relative error-based event-triggering criterion has been widely recognized, the corresponding results for discrete-time systems have been very few especially when the variance-constrained H_∞ control problem becomes a research focus. It is also noticed that, despite its engineering significance, the event-triggered control problem for *time-varying* stochastic systems with *both variance and H_∞ performance constraints* over a *finite horizon* has not received adequate research attention yet, not to mention the case when *ROSS, stochastic nonlinearities and state-multiplicative noises* are simultaneously present. Therefore, the main motivation of this paper is to shorten such a gap by launching a systematic investigation.

In this paper, we make the first of the few attempts to consider the event-triggered multiobjective control problems for time-varying systems with ROSS, stochastic nonlinearities and state-multiplicative noise. The multiple objectives include the state variance constraints and the H_∞ disturbance rejection attenuation level. *The main contributions of this paper are highlighted as follows. 1) A variance-constrained H_∞ controller is proposed for discrete nonlinear time-varying systems in the framework of event-based communication protocol. 2) A new event indicator variable is constructed to reflect the event-triggered information in the controller analysis so as to decrease the frequency of data transmission and also reduce the conservatism in the controller design as compared to existing literature. 3) The developed recursive algorithm merits online applications. 4) The system model addressed is new, which is quite comprehensive to cover time-varying parameters, stochastic nonlinearities, multiplicative noise as well as ROSS, hence reflecting the reality more closely.*

Notation. The notation used here is standard except where otherwise stated. tr(A) represents the trace of a matrix A. The symbol $\otimes$ denotes the Kronecker product. Matrices, if they are not explicitly specified, are assumed to have compatible dimensions.

2 Problem Formulation

Consider the following class of discrete time-varying stochastic systems

$$\begin{cases} x(k+1)=\big(A(k)+\sum_{i=1}^{r}A_i(k)w_i(k)\big)x(k) \\ \qquad\qquad + g(k,x(k))+D(k)v(k)+B_1(k)u(k) \\ z(k)=L(k)x(k)+B_2(k)u(k) \end{cases} \tag{1}$$

and m sensor measurements with randomly occurring saturations

$$\begin{aligned} y_i(k) =&\alpha_i(k)\sigma(C_i(k)x(k))+(1-\alpha_i(k))C_i(k)x(k) \\ &+E_i(k)\varpi_i(k) \quad (i=1,2,\dots,m) \end{aligned} \tag{2}$$

where $x(k)\in\mathbb{R}^{n_x}$ represents the state vector; $y_i(k)\in\mathbb{R}$ is the measurement output measured by sensor i from the plant; $z(k)\in\mathbb{R}^{n_z}$ is the controlled output vector; $u(k)\in\mathbb{R}^{n_u}$ is the control input vector; $w_i(k)\in\mathbb{R}$ $(i=1,2,\dots,r)$, $v(k)\in\mathbb{R}^{n_v}$ and $\varpi_i(k)\in\mathbb{R}^{n_w}$ $(i=1,2,\dots,m)$ are, respectively, the multiplicative noise, the process noise and the measurement noise for sensor i. The noise sequences are mutually uncorrelated zero-mean Gaussian sequences with $\mathbb{E}\{w_i(k)w_i^T(k)\}=1$, $\mathbb{E}\{v(k)v^T(k)\}=V(k)$ and $\mathbb{E}\{\varpi_i(k)\varpi_i^T(k)\}=\bar{W}_i(k)$. $A(k)$, $A_i(k)$, $B_1(k)$, $B_2(k)$, $D(k)$, $L(k)$, $C_i(k)$ and $E_i(k)$ are known, real, time-varying matrices with appropriate dimensions.

The nonlinear function $g(k,x(k))$ with $g(k,0)=0$ is a stochastic nonlinear function having the following statistical characteristics:

$$\mathbb{E}\left\{g(k,x(k))\Big|x(k)\right\}=0,$$

$$\mathbb{E}\left\{g(k,x(k))g^T(j,x(k))\big|x(k)\right\}=0,\quad k\neq j$$

and

$$\mathbb{E}\left\{g(k,x(k))g^T(k,x(k))\big|x(k)\right\}$$

$$\begin{aligned} &:= \sum_{i=1}^{q} \pi_i \pi_i^T \mathbb{E}\left\{x^T(k)\Gamma_i(k)x(k)\right\} \\ &= \sum_{i=1}^{q} \Theta_i(k)\mathbb{E}\left\{x^T(k)\Gamma_i(k)x(k)\right\} \end{aligned} \quad (3)$$

where q is a known nonnegative integer, $\Theta_i(k)$ and $\Gamma_i(k)$ $(i = 1, 2, \ldots, q)$ are known matrices with appropriate dimensions.

The saturation function $\sigma(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ is defined as

$$\sigma(\vartheta) = sign(\vartheta)\min\{1, |\vartheta|\}. \quad (4)$$

Here, $sign(\cdot)$ denotes the signum function. Without loss of generality, the saturation level is taken as unity.

The variable $\alpha_i(k)$ $(i = 1, 2, \ldots, m)$ in (2), which governs the ROSS phenomenon, are Bernoulli distributed white sequences taking values on 0 or 1 with

$$\text{Prob}\{\alpha_i(k) = 1\} = \bar{\alpha}, \quad \text{Prob}\{\alpha_i(k) = 0\} = 1 - \bar{\alpha}, \quad (5)$$

where $\bar{\alpha} \in [0, 1]$ is a known constant. Throughout the paper, the stochastic variables $\alpha_i(k)$ $(i = 1, \ldots, m)$, $v(k)$, $\varpi_i(k)$ $(i = 1, 2, \ldots, m)$ and $w_i(k)$ $(i = 1, \ldots, r)$ are mutually uncorrelated.

For notational brevity, we set

$$\begin{aligned} & y(k) = \left[y_1^T(k)\ y_2^T(k)\ \cdots\ y_m^T(k)\right]^T, \ \bar{\Lambda}_\alpha = I_m \otimes \bar{\alpha}, \\ & \Lambda_\alpha(k) = \text{diag}\{\alpha_1(k), \alpha_2(k), \ldots, \alpha_m(k)\}, \\ & E(k) = \text{diag}\{E_1(k), E_2(k), \ldots, E_m(k)\}, \\ & C(k) = \left[C_1^T(k)\ C_2^T(k)\ \cdots\ C_m^T(k)\right]^T, \\ & \varpi(k) = \left[\varpi_1^T(k)\ \varpi_2^T(k)\ \cdots\ \varpi_m^T(k)\right]^T. \end{aligned} \quad (6)$$

Then, the sensor model (2) can be expressed in the following compact form:

$$\begin{aligned} y(k) =& \Lambda_\alpha(k)\sigma(C(k)x(k)) + (I - \Lambda_\alpha(k))C(k)x(k) \\ &+ E(k)\varpi(k) \end{aligned} \quad (7)$$

where $\sigma(C(k)x(k)) := \Big[\sigma^T(C_1(k)x(k))\ \sigma^T(C_2(k)x(k))\ \cdots\ \sigma^T(C_m(k)x(k))\Big]^T$. In this paper, the notation σ has been slightly abused to denote both the vector-valued and the scalar-valued saturation functions.

As in [14], for diagonal matrices K_1 and K_2 satisfying $0 \le K_1 < I \le K_2$, the saturation function in (7) can be decomposed into a linear and a nonlinear part as

$$\sigma(C(k)x(k)) = K_1 C(k)x(k) + \Psi(C(k)x(k)) \quad (8)$$

where $\Psi(C(k)x(k))$ is a nonlinear vector-valued function satisfying the following sector condition:

$$\Psi^T(C(k)x(k))\left(\Psi(C(k)x(k)) - KC(k)x(k)\right) \le 0 \quad (9)$$

with $K = K_2 - K_1$.

In this paper, we define the event generator function $f(\cdot, \cdot)$ as follows:

$$f(\varphi(k), \delta) = \varphi^T(k)\varphi(k) - \delta y^T(k)y(k) \quad (10)$$

where $\varphi(k) := y(k_i) - y(k)$, $y(k_i)$ is the measurement at latest event time, $y(k)$ is the current measurement and $\delta \in [0,\ 1)$.

The execution is triggered as long as the condition

$$f(\varphi(k), \delta) > 0 \quad (11)$$

is satisfied. Therefore, the sequence of event-triggered instants $0 \le k_0 \le k_1 \le \cdots \le k_i \le \cdots$ is determined iteratively by

$$k_{i+1} = \inf\{k \in \mathbb{N} | k > k_i, f(\varphi(k), \delta) > 0\}. \quad (12)$$

Accordingly, any measurement data satisfying the event condition (11) will be transmitted to the controller.

Remark 1 *The event triggering mechanism is adopted here in order to effectively decrease the data communication frequency and network bandwidth usages.*

For system (1), the following controller structure is adopted:

$$\begin{cases} x_c(k+1) = A_c(k)x_c(k) + B_c(k)y(k_i) \\ u(k) = C_c(k)x_c(k) \end{cases} \quad (13)$$

where $x_c(k) \in \mathbb{R}^{n_c}$ is the controller state, $A_c(k)$, $B_c(k)$ and $C_c(k)$ are the controller parameters to be designed.

Under the output feedback controller (13), the closed-loop system becomes

$$\begin{cases} \eta(k+1) = \mathcal{Y}_l(k) + H_0 g(k, x(k)) + \sum_{i=1}^{r} w_i(k)\bar{A}_i(k)\eta(k) \\ \qquad + \tilde{B}_c(k)\bar{C}(k)\eta(k) + \tilde{B}_c(k)\Psi(\hat{C}(k)\eta(k)) \\ \qquad + \bar{D}(k)\xi(k) \\ z(k) = \bar{M}(k)\eta(k) \end{cases} \quad (14)$$

where

$$\begin{aligned} & \eta(k) = \left[x^T(k)\ x_c^T(k)\right]^T, \ \tilde{\Lambda}_\alpha(k) = \Lambda_\alpha(k) - \bar{\Lambda}_\alpha, \\ & \xi(k) = \left[v^T(k)\ \varpi^T(k)\right]^T, \ \bar{M}(k) = \left[L(k)\ B_2(k)C_c(k)\right], \\ & \mathcal{Y}_l(k) = \big(\bar{A}(k) + \bar{B}_c(k)\bar{\Lambda}_\alpha K_1 \hat{C}(k)\big)\eta(k) \\ & \qquad + \bar{B}_c(k)\big(\varphi(k) + \bar{\Lambda}_\alpha \Psi(\hat{C}(k)\eta(k))\big), \\ & \bar{A}(k) = \begin{bmatrix} A(k) & B_1(k)C_c(k) \\ B_c(k)(I - \bar{\Lambda}_\alpha)C(k) & A_c(k) \end{bmatrix}, \\ & \bar{B}_c(k) = \left[0\ B_c^T(k)\right]^T, \ \bar{A}_i(k) = \text{diag}\{A_i(k), 0\}, \\ & \bar{C}(k) = \left[(K_1 - I)C(k)\ 0\right], \ H_0 = \left[I\ 0\right]^T, \\ & \tilde{B}_c(k) = \left[0\ (B_c(k)\tilde{\Lambda}_\alpha(k))^T\right]^T, \ \hat{C}(k) = \left[C(k)\ 0\right], \\ & \bar{D}(k) = \text{diag}\{D(k), B_c(k)E(k)\}. \end{aligned}$$

The state covariance matrix of the dynamical system (14) can be defined as

$$\mathbb{X}(k) := \mathbb{E}\left\{\eta(k)\eta^T(k)\right\} = \mathbb{E}\left\{\begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}\begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}^T\right\}. \quad (15)$$

Our objective of this paper is to design a dynamic output feedback controller of the form (13) over the finite horizon $[0, N]$ such that the following two requirements are satisfied simultaneously:

- (Q1): For the given disturbance attenuation level $\gamma > 0$, the positive definite matrices U_1, U_2, S and the initial state $x(0)$, the controlled output $z(k)$ satisfies the following performance constraint:

$$J_1 := \mathbb{E}\left\{\sum_{k=0}^{N-1}\left(\|z(k)\|^2 - \gamma^2\|\xi(k)\|_U^2\right)\right\}$$
$$-\gamma^2\mathbb{E}\left\{x^T(0)Sx(0)\right\} < 0 \quad (\forall\{\xi(k)\} \neq 0) \quad (16)$$

where $\|\xi(k)\|_U^2 = \xi^T(k)U\xi(k)$, $U = \mathrm{diag}\{U_1, U_2\}$.

- (Q2): The state covariances satisfy the following constraints:

$$J_2 := \mathbb{E}\left\{x(k)x^T(k)\right\} \leq \Upsilon(k) \quad (17)$$

where $\Upsilon(k)$ $(0 \leq k < N)$ is a sequence of given matrices specifying the acceptable covariance upper bounds obtained from the engineering requirements.

Remark 2 *In the desired performance requirement (Q2), the estimation error variance at each sampling time point is required to be not more than an individual upper bound.*

3 Main results

Before deriving the main results, we define the event indicator variable $\mu(k)$ as follows:

$$\mu(k) := \begin{cases} 1 & \text{if the event generator condition is satisfied at the current instant } k \\ 0 & \text{if no event is triggered} \end{cases} \quad (18)$$

3.1 Event-triggered H_∞ Performance

For presentation clearly, we denote

$$\begin{aligned}
&\Xi_{11}(k)\\
&= 2\big(\bar{A}(k) + \bar{B}_c(k)\bar{\Lambda}_\alpha K_1\hat{C}(k)\big)^T P(k+1) \times \\
&\quad \big(\bar{A}(k) + \bar{B}_c(k)\bar{\Lambda}_\alpha K_1\hat{C}(k)\big) + \bar{M}^T(k)\bar{M}(k) \\
&\quad + \sum_{i=1}^{q}\bar{\Gamma}_i(k)\cdot \mathrm{tr}\Big[H_0^T P(k+1)H_0\Theta_i(k)\Big] \\
&\quad + \sum_{i=1}^{r}\bar{A}_i^T(k)P(k+1)\bar{A}_i(k) - P(k) \\
&\quad + \bar{\alpha}(1-\bar{\alpha})\bar{C}^T(k)\hat{B}_{cp}^T(k)P(k+1)\hat{B}_{cp}(k)\bar{C}(k) \\
&\quad + \delta((\bar{\Lambda}_\alpha K_1 + I - \bar{\Lambda}_\alpha)\hat{C}(k))^T(\bar{\Lambda}_\alpha K_1 + I - \bar{\Lambda}_\alpha) \\
&\quad \times\hat{C}(k) + \delta\bar{\alpha}(1-\bar{\alpha})((I-K_1)\hat{C}(k))^T(I-K_1)\hat{C}(k), \\
&\Xi_{22}(k) = 2\bar{B}_c^T(k)P(k+1)\bar{B}_c(k) - I, \\
&\Xi_{31}(k) \\
&= 2(\bar{B}_c(k)\bar{\Lambda}_\alpha)^T P(k+1)\big(\bar{A}(k) + \bar{B}_c(k)\bar{\Lambda}_\alpha K_1\hat{C}(k)\big) \\
&\quad + \bar{\alpha}(1-\bar{\alpha})\hat{B}_{cp}^T(k)P(k+1)\hat{B}_{cp}(k)\bar{C}(k) \\
&\quad + \lambda_1(k)K\hat{C}(k) + \delta\bar{\Lambda}_\alpha^T(\bar{\Lambda}_\alpha K_1 + I - \bar{\Lambda}_\alpha)\hat{C}(k) \\
&\quad + \delta\bar{\alpha}(1-\bar{\alpha})(I-K_1)\hat{C}(k), \\
&\Xi_{33}(k) \\
&= 2(\bar{B}_c(k)\bar{\Lambda}_\alpha)^T P(k+1)\bar{B}_c(k)\bar{\Lambda}_\alpha - \lambda_1(k)I \\
&\quad + \bar{\alpha}(1-\bar{\alpha})\hat{B}_{cp}^T(k)P(k+1)\hat{B}_{cp}(k) \\
&\quad + \delta\bar{\Lambda}_\alpha^T\bar{\Lambda}_\alpha + \delta\bar{\alpha}(1-\bar{\alpha})I, \\
&\Xi_{41}(k) = \delta\bar{E}^T(k)(\bar{\Lambda}_\alpha K_1 + I - \bar{\Lambda}_\alpha)\hat{C}(k), \\
&\Xi_{44}(k) \\
&= 2\bar{D}^T(k)P(k+1)\bar{D}(k) + \delta\bar{E}^T(k)\bar{E}(k) - \gamma^2 U \\
&\quad + \mu^2(k)\bar{E}^T(k)\bar{B}_c^T(k)P(k+1)\bar{B}_c(k)\bar{E}(k), \\
&\bar{\Gamma}_i(k) = \mathrm{diag}\{\Gamma_i(k), 0\},\ \bar{E}(k) = [\,0\ \ E(k)\,], \\
&P(k+1) = \mathrm{diag}\{M(k+1), N(k+1)\}, \\
&\hat{B}_{cp}(k) = \Big[\check{B}_c^T(k)\ \hat{B}_c^T(k)\Big]^T,\ \bar{S} = \mathrm{diag}\{S, 0\}, \\
&\check{B}_c(k) = \mathrm{diag}\{\check{B}_{c_1}(k), \check{B}_{c_2}(k), \ldots, \check{B}_{c_m}(k)\}, \\
&\hat{B}_c(k) = \mathrm{diag}\{\hat{B}_{c_1}(k), \hat{B}_{c_2}(k), \ldots, \hat{B}_{c_m}(k)\}, \\
&\check{B}_{c_i}(k) = \Big[\bar{B}_{c_{1,i}}^T(k)\ \bar{B}_{c_{2,i}}^T(k)\ \cdots\ \bar{B}_{c_{n_x,i}}^T(k)\Big]^T, \\
&\hat{B}_{c_i}(k) = \Big[\bar{B}_{c_{n_x+1,i}}^T(k)\ \bar{B}_{c_{n_x+2,i}}^T(k)\ \cdots\ \bar{B}_{c_{n_x+n_c,i}}^T(k)\Big]^T \\
&(i = 1, 2, \ldots, m).
\end{aligned} \quad (19)$$

Theorem 1 *Consider the discrete time-varying nonlinear stochastic system described by (1)–(2). Let the disturbance attenuation level $\gamma > 0$, the positive definite weighted matrices $U_1 > 0$, $U_2 > 0$ and $S > 0$, the scalar $\delta \in [0,\ 1)$ and the controller parameters $A_c(k)$, $B_c(k)$ and $C_c(k)$ in (13) be given. The performance criterion defined in (16) is guaranteed for all nonzero $\xi(k)$ if, with the initial condition $P(0) = \mathrm{diag}\{M(0), N(0)\} \leq \gamma^2\bar{S}$, there exist families of positive scalars $\{\lambda_1(k)\}_{k\in[0,\ N-1]}$ and a sequence of positive definite matrices $P(k) = \mathrm{diag}\{M(k), N(k)\}_{k\in[0,\ N]} > 0$ satisfying the following recursive matrix inequalities:*

$$\Xi(k) = \begin{bmatrix} \Xi_{11}(k) & * & * & * \\ 0 & \Xi_{22}(k) & * & * \\ \Xi_{31}(k) & 0 & \Xi_{33}(k) & * \\ \Xi_{41}(k) & 0 & \delta\bar{E}^T(k)\bar{\Lambda}_\alpha & \Xi_{44}(k) \end{bmatrix} < 0 \quad (20)$$

Proof: See Appendix I.

3.2 Event-triggered Variance Analysis

Theorem 2 *Consider the discrete time-varying nonlinear stochastic system described by (1)–(2). Let the scalar $\delta \in [0,\ 1)$ and the controller parameters $A_c(k)$, $B_c(k)$ and $C_c(k)$ in (13) be given. We have $Q(k) \geqslant \mathbb{X}(k)$ $(\forall k \in \{1, 2, \cdots, N+1\})$ if, with initial condition $Q(0) = \mathbb{X}(0)$, there exists a sequence of positive definite matrices $\{Q(k)\}_{1\leq k\leq N+1}$ satisfying the following matrix inequality:*

$$Q(k+1) \geqslant \Phi\left(Q(k)\right) \quad (21)$$

where

$$\begin{aligned}
&\Phi(Q(k)) \\
&:= 3\bar{A}(k)Q(k)\bar{A}^T(k) + 3\bar{B}_c(k)\Bigg(\delta\cdot\mathrm{tr}\Big[\bar{E}^T(k)\bar{E}(k)\bar{V}(k)\Big]I \\
&\quad + 2\delta\bigg(\mathrm{tr}\Big[m\big(\bar{\Lambda}_\alpha^T\bar{\Lambda}_\alpha + \bar{\alpha}(1-\bar{\alpha})I\big)\Big] \\
&\quad + \mathrm{tr}\Big[\hat{C}^T(k)\big((I-\bar{\Lambda}_\alpha)^T(I-\bar{\Lambda}_\alpha) \\
&\quad + \bar{\alpha}(1-\bar{\alpha})I\big)\hat{C}(k)Q(k)\Big]\bigg)I\Bigg)\bar{B}_c^T(k) \\
&\quad + 3m\bar{B}_c(k)\bar{\Lambda}_\alpha\bar{\Lambda}_\alpha^T\bar{B}_c^T(k) + \sum_{i=1}^{r}\bar{A}_i(k)Q(k)\bar{A}_i^T(k) \\
&\quad + \sum_{i=1}^{q}H_0\Theta_i(k)H_0^T\cdot\mathrm{tr}\left[\bar{\Gamma}_i(k)Q(k)\right]
\end{aligned} \quad (22)$$

$$
\begin{aligned}
&+ 2\bar{\alpha}(1-\bar{\alpha})\bar{B}_c(k)\hat{\mathcal{C}}(k)(I_m \otimes Q(k))\hat{\mathcal{C}}^T(k)\bar{B}_c^T(k) \\
&+ 2\bar{\alpha}(1-\bar{\alpha})m\bar{B}_c(k)\bar{B}_c^T(k) + 2\bar{D}(k)\bar{V}(k)\bar{D}^T(k) \\
&+ \mu^2(k)\bar{B}_c(k)\bar{E}(k)\bar{V}(k)\bar{E}^T(k)\bar{B}_c^T(k), \\
&\hat{C}_i(k) = \left[\hat{C}_{i,1}(k)\ \hat{C}_{i,2}(k)\ \cdots\ \hat{C}_{i,n_x+n_c}(k)\right], \\
&(i = 1, 2, \ldots, m),\ \bar{V}(k) = \mathrm{diag}\{V(k), \bar{W}(k)\}, \\
&\hat{\mathcal{C}}(k) = \mathrm{diag}\{\hat{C}_1(k), \hat{C}_2(k), \ldots, \hat{C}_m(k)\}, \\
&\bar{W}(k) = \mathrm{diag}\{\bar{W}_1(k), \bar{W}_2(k), \ldots, \bar{W}_m(k)\}.
\end{aligned}
$$

Proof: See Appendix II.

Corollary 1 *The inequality holds*

$$
\begin{aligned}
J_2 :=& \mathbb{E}\left\{x(k)x^T(k)\right\} = \left[I\ 0\right]\mathbb{X}(k)\left[I\ 0\right]^T = H_0^T\mathbb{X}(k)H_0 \\
&\le H_0^T Q(k)H_0,\quad k \in [0, N-1].
\end{aligned}
$$

For presentation simplicity, we denote

$$
\begin{aligned}
\bar{\Xi}_{11}(k) =& \mathrm{diag}\left\{\sum_{i=1}^{q}\bar{\Gamma}_i(k)R_i(k) - P(k), -I\right\}, \\
\hat{A}_r(k) =& \left[\bar{A}_1^T(k)\ \bar{A}_2^T(k)\ \cdots\ \bar{A}_r^T(k)\right]^T, \\
\bar{\Xi}_{21}(k) =& \mathrm{diag}\left\{\lambda_1(k)K\hat{C}(k), 0\right\}, \\
\bar{\Xi}_{22}(k) =& \mathrm{diag}\left\{-\lambda_1(k)I, -\gamma^2 U\right\}, \\
\bar{\Xi}_{31}(k) =& \left[\bar{\Xi}_{311}(k)\ 0\right],\ \bar{\Xi}_{412}(k) = \left[\sqrt{2}\bar{B}_c^T(k)\ 0\ 0\right]^T, \\
\bar{\Xi}_{311}(k) =& \begin{bmatrix} \sqrt{2}\big(\bar{A}(k) + \bar{B}_c(k)\bar{\Lambda}_\alpha K_1\hat{C}(k)\big) \\ \hat{A}_r(k) \\ \sqrt{\bar{\alpha}(1-\bar{\alpha})}\hat{B}_{cp}(k)\bar{C}(k) \\ \sqrt{\delta}(\bar{\Lambda}_\alpha K_1 + I - \bar{\Lambda}_\alpha)\hat{C}(k) \\ \sqrt{\delta}\sqrt{\bar{\alpha}(1-\bar{\alpha})}(I - K_1)\hat{C}(k) \\ \bar{M}(k) \end{bmatrix}, \\
\bar{\Xi}_{33}(k) =& \mathrm{diag}\Big\{-P^{-1}(k+1), -I_r \otimes P^{-1}(k+1), \\
&- I_m \otimes P^{-1}(k+1), -I, -I, -I\Big\}, \\
\bar{\Xi}_{44}(k) =& -I_3 \otimes P^{-1}(k+1), \\
\bar{\Xi}_{32}(k) =& \begin{bmatrix} \sqrt{2}\bar{B}_c(k)\bar{\Lambda}_\alpha & 0 \\ 0 & 0 \\ \sqrt{\bar{\alpha}(1-\bar{\alpha})}\hat{B}_{cp}(k) & 0 \\ \sqrt{\delta}\bar{\Lambda}_\alpha & \sqrt{\delta}\bar{E}(k) \\ \sqrt{\delta}\sqrt{\bar{\alpha}(1-\bar{\alpha})}I & 0 \\ 0 & 0 \end{bmatrix}, \\
\bar{\Xi}_{41}(k) =& \left[0\ \bar{\Xi}_{412}(k)\right],\ \bar{\Xi}_{42}(k) = \left[0\ \bar{\Xi}_{422}(k)\right], \\
\bar{\Xi}_{422}(k) =& \left[0\ \sqrt{2}\bar{D}^T(k)\ \mu(k)\bar{E}^T(k)\bar{B}_c^T(k)\right]^T, \\
\hat{\Xi}_{21}(k) =& \left[\sqrt{3}\bar{A}(k)\ \tau(k)\bar{B}_c(k)\ \sqrt{3m}\bar{B}_c(k)\bar{\Lambda}_\alpha\right]^T, \\
\hat{\Xi}_{22}(k) =& \mathrm{diag}\left\{-Q^{-1}(k), -I, -I\right\}, \\
\hat{\Xi}_{41}(k) =& \Big[\check{A}_r(k)\ \sqrt{2\bar{\alpha}(1-\bar{\alpha})}\bar{B}_c(k)\hat{\mathcal{C}}(k)\ \sqrt{2}\bar{D}(k) \\
&\mu(k)\bar{B}_c(k)\bar{E}(k)\Big], \\
\bar{m}_1 =& \mathrm{tr}\Big[m\big(\bar{\Lambda}_\alpha^T\bar{\Lambda}_\alpha + \bar{\alpha}(1-\bar{\alpha})I\big)\Big], \\
\hat{\Xi}_{44}(k) =& \mathrm{diag}\Big\{-I_r \otimes Q^{-1}(k), \\
&- I_m \otimes Q^{-1}(k), -\bar{V}^{-1}(k), -\bar{V}^{-1}(k)\Big\}, \\
\check{A}_r(k) =& \left[\bar{A}_1(k)\ \bar{A}_2(k)\ \cdots\ \bar{A}_r(k)\right], \\
\tau(k) =& \sqrt{\tau_1(k) + 2\bar{\alpha}(1-\bar{\alpha})m}, \\
\tau_1(k) =& 6\delta(\bar{m}_1 + \rho(k)) + 3\delta\,\mathrm{tr}\Big[\bar{E}^T(k)\bar{E}(k)\bar{V}(k)\Big], \\
\Theta_{51} =& \left[H_0\pi_1\ H_0\pi_2\ \cdots\ H_0\pi_q\right]^T, \\
\Theta_{55} =& \mathrm{diag}\left\{\chi_1 I, \chi_2 I, \ldots, \chi_q I\right\}, \\
\chi_i =& (\mathrm{tr}\left[\bar{\Gamma}_i(k)Q(k)\right])^{-1}\quad (i = 1, 2, \ldots, q), \\
\rho(k) =& \mathrm{tr}\Big[\hat{C}^T(k)\big((I-\bar{\Lambda}_\alpha)^T(I-\bar{\Lambda}_\alpha) + \bar{\alpha}(1-\bar{\alpha})I\big)\hat{C}(k)Q(k)\Big].
\end{aligned}
$$

Theorem 3 *Consider the discrete time-varying nonlinear stochastic system described by (1)–(2). Let the disturbance attenuation level $\gamma > 0$, the positive definite weighted matrices $U_1 > 0$, $U_2 > 0$ and $S > 0$, the scalar $\delta \in [0,\ 1)$ and the controller parameters $A_c(k)$, $B_c(k)$ and $C_c(k)$ in (13) be given. Then, for the closed-loop system (14), we have $J_1 < 0$ and $J_2 < 0$ ($\forall k \in \{0, 1, \ldots, N+1\}$) if there exist families of positive scalars $\{\lambda_1(k)\}_{k\in[0,\ N-1]}$, $\{R_i(k)\}_{0\le k\le N}$ ($i = 1, 2, \cdots, q$) and families of positive definite matrices $\{M(k)\}_{1\le k\le N+1}$, $\{N(k)\}_{1\le k\le N+1}$ and $\{Q(k)\}_{1\le k\le N+1}$ satisfying the following recursive matrix inequalities:*

$$
\begin{bmatrix} -R_i(k) & * \\ H_0\pi_i & -P^{-1}(k+1) \end{bmatrix} < 0,\ (i = 1, 2, \cdots, q) \tag{23}
$$

$$
\begin{bmatrix} \bar{\Xi}_{11}(k) & * & * & * \\ \bar{\Xi}_{21}(k) & \bar{\Xi}_{22}(k) & * & * \\ \bar{\Xi}_{31}(k) & \bar{\Xi}_{32}(k) & \bar{\Xi}_{33}(k) & * \\ \bar{\Xi}_{41}(k) & \bar{\Xi}_{42}(k) & 0 & \bar{\Xi}_{44}(k) \end{bmatrix} < 0, \tag{24}
$$

$$
\begin{bmatrix} -Q(k+1) & * & * & * \\ \hat{\Xi}_{21}(k) & \hat{\Xi}_{22}(k) & * & * \\ \Theta_{51} & 0 & -\Theta_{55} & * \\ \hat{\Xi}_{41}(k) & 0 & 0 & \hat{\Xi}_{44}(k) \end{bmatrix} \le 0, \tag{25}
$$

with the initial condition

$$
\begin{cases} P(0) \le \gamma^2\bar{S} \\ Q(0) = \mathbb{X}(0) \end{cases} \tag{26}
$$

where the system data are defined in (23).

Proof: Based on some straightforward algebraic manipulations and under the initial conditions (26), we can see that the inequalities (23) and (24) imply (20), and the inequality (25) is equivalent to (21). Therefore, according to Theorem 1, Theorem 2 and Corollary 1, the H_∞ index defined in (16) satisfies $J_1 < 0$ and, at the same time, the system state covariances achieves $\mathbb{E}\left\{x(k)x^T(k)\right\} \leqslant \begin{bmatrix} I & 0 \end{bmatrix} Q(k) \begin{bmatrix} I & 0 \end{bmatrix}^T$, $\forall k \in \{0, 1, \cdots, N+1\}$. The proof is complete.

4 Variance Constrained Controller Design

Theorem 4 *Let the disturbance attenuation level $\gamma > 0$, positive definite weighted matrices U_1, U_2 and $S > 0$, a scalar $\delta \in [0,\ 1)$ and a sequence of prespecified variance upper bounds $\{\Upsilon(k)\}_{0\leqslant k\leqslant N+1}$ be given. The addressed event-triggered variance constrained controller design problem is solvable for the stochastic nonlinear system (14) if there exist families of positive definite matrices $\{\mathcal{M}(k)\}_{1\leqslant k\leqslant N+1}$, $\{\mathcal{N}(k)\}_{1\leqslant k\leqslant N+1}$, $\{Q_1(k)\}_{1\leqslant k\leqslant N+1}$, $\{Q_2(k)\}_{1\leqslant k\leqslant N+1}$, positive scalars $\{\lambda_1(k)\}_{k\in[0,\ N-1]}$, $\{R_i(k)\}_{0\leq k\leq N}$ $(i = 1, 2, \cdots, q)$ and families of real-valued matrices $\{Q_3(k)\}_{1\leqslant k\leqslant N+1}$, $\{A_c(k)\}_{0\leqslant k\leqslant N}$, $\{B_c(k)\}_{0\leqslant k\leqslant N}$ and $\{C_c(k)\}_{0\leqslant k\leqslant N}$ satisfying the following recursive matrix inequalities:*

$$\begin{bmatrix} -R_i(k) & * & * \\ \pi_i & -\mathcal{M}(k+1) & * \\ 0 & 0 & -\mathcal{N}(k+1) \end{bmatrix} < 0, \ (i=1,2,\ldots,q) \quad (27)$$

$$\begin{bmatrix} \Phi_{11}(k) & * & * & * & * \\ \Phi_{21}(k) & \Phi_{22}(k) & * & * & * \\ \Phi_{31}(k) & \Phi_{32}(k) & \Phi_{33}(k) & * & * \\ \Phi_{41}(k) & 0 & 0 & \Phi_{44}(k) & * \\ 0 & \Phi_{52}(k) & 0 & 0 & \Phi_{55}(k) \end{bmatrix} < 0, \quad (28)$$

$$\begin{bmatrix} -Q(k+1) & * & * & * & * \\ \tau_1(k)\bar{\Phi}_{21}(k) & -Q(k) & * & * & * \\ \bar{\Phi}_{31}(k) & 0 & \bar{\Phi}_{33}(k) & * & * \\ \bar{\Phi}_{41}(k) & 0 & 0 & \bar{\Phi}_{44}(k) & * \\ \bar{\Phi}_{51}(k) & 0 & 0 & 0 & \bar{\Phi}_{55}(k) \end{bmatrix} < 0, \quad (29)$$

$$Q_1(k+1) - \Upsilon(k+1) \leqslant 0, \quad (30)$$

with the initial conditions

$$\begin{cases} \begin{bmatrix} M(0) - \gamma^2 S & 0 \\ 0 & N(0) \end{bmatrix} \leqslant 0 \\ \mathbb{E}\left\{x(0)x^T(0)\right\} = Q_1(0) \leq \Upsilon(0) \end{cases} \quad (31)$$

and the parameters updated by

$$M(k+1) = \mathcal{M}^{-1}(k+1), \ \ N(k+1) = \mathcal{N}^{-1}(k+1) \quad (32)$$

where

$$\bar{\mathcal{N}}(k+1) = \mathrm{diag}\{\mathcal{M}(k+1), \mathcal{N}(k+1)\},$$

$$\phi_\alpha = \sqrt{\bar{\alpha}(1-\bar{\alpha})}, \ C_k(k) = (K_1 - I)C(k),$$

$$\Phi_{11}(k) = \mathrm{diag}\left\{ \sum_{i=1}^{q} \Gamma_i(k)R_i(k) - M(k), -N(k), -I \right\},$$

$$\Phi_{21}(k) = \mathrm{diag}\left\{\lambda_1(k)KC(k), 0, 0\right\},$$

$$Q_{13}(k) = \begin{bmatrix} Q_1^T(k) & Q_3^T(k) \end{bmatrix}^T, \ \Phi_{55}(k) = -I_2 \otimes \bar{\mathcal{N}}(k+1),$$

$$\Phi_{22}(k) = \mathrm{diag}\left\{ -\lambda_1(k)I, -\gamma^2 U_1, -\gamma^2 U_2 \right\},$$

$$\Phi_{31}(k) = \begin{bmatrix} \Omega_{311}(k) & \Omega_{312}(k) \\ \Omega_{321}(k) & 0 \end{bmatrix}, \ \Omega_{411}(k) = \begin{bmatrix} 0 \\ \sqrt{2}B_c(k)\bar{\Lambda}_\alpha \end{bmatrix},$$

$$\Omega_{311}(k) = \begin{bmatrix} \sqrt{2}A(k) \\ \sqrt{2}B_c(k)(C(k) + \bar{\Lambda}_\alpha C_k(k)) \end{bmatrix},$$

$$\Omega_{312}(k) = \begin{bmatrix} \sqrt{2}B_1(k)C_c(k) & 0 \\ \sqrt{2}A_c(k) & 0 \end{bmatrix}, \Phi_{32}(k) = \begin{bmatrix} \Omega_{411}(k) & 0 \\ \Omega_{421}(k) & \Omega_{422}(k) \end{bmatrix},$$

$$\Omega_{321}(k) = \Big[\, \tilde{A}_l^T(k) \ \ \phi_\alpha(\hat{B}_{cp}(k)C_k(k))^T \ \ \sqrt{\delta}\big(\bar{\Lambda}_\alpha C_k(k) + C(k)\big)^T \ \ -\phi_\alpha\sqrt{\delta}C_k^T(k) \,\Big]^T,$$

$$\Omega_{421}(k) = \begin{bmatrix} 0 & \phi_\alpha \hat{B}_{cp}^T(k) & \sqrt{\delta}\bar{\Lambda}_\alpha & \phi_\alpha\sqrt{\delta}I \end{bmatrix}^T,$$

$$\Omega_{422}(k) = \begin{bmatrix} 0 & \Omega_{4222}(k) \end{bmatrix}, \ \mathcal{D}(k) = \begin{bmatrix} 0 & \sqrt{2}D(k) \end{bmatrix},$$

$$\Omega_{4222}(k) = \begin{bmatrix} 0 & 0 & \sqrt{\delta}E^T(k) & 0 \end{bmatrix}^T, \ \Phi_{52}(k) = \mathrm{diag}\{\mathcal{D}(k), \mathcal{B}_c(k)\},$$

$$\Phi_{41}(k) = \mathrm{diag}\left\{ \begin{bmatrix} L(k) & B_2(k)C_c(k) \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{2}B_c^T(k) \end{bmatrix}^T \right\},$$

$$\Phi_{33}(k) = \mathrm{diag}\big\{ -\mathcal{M}(k+1), -\mathcal{N}(k+1), -I_r \otimes \bar{\mathcal{N}}(k+1), -I_m \otimes \bar{\mathcal{N}}(k+1), -I, -I \big\},$$

$$\Phi_{44}(k) = \mathrm{diag}\left\{ -I, -\mathcal{M}(k+1), -\mathcal{N}(k+1) \right\},$$

$$\mathcal{B}_c(k) = \begin{bmatrix} \sqrt{2}E^T(k)B_c^T(k) & 0 & \mu(k)E^T(k)B_c^T(k) \end{bmatrix}^T,$$

$$\tilde{A}_l(k) = \begin{bmatrix} A_{l1}^T(k) & A_{l2}^T(k) & \cdots & A_{lr}^T(k) \end{bmatrix}^T,$$

$$A_{li}(k) = \begin{bmatrix} A_i^T(k) & 0 \end{bmatrix}^T \ \ (i = 1, 2, \ldots, r),$$

$$\bar{\Phi}_{21}(k) = \begin{bmatrix} (A(k)Q_1(k) + B_1(k)C_c(k)Q_3(k))^T & (B_c(k)(I - \bar{\Lambda}_\alpha)C(k)Q_1(k) + A_c(k)Q_3(k))^T \\ (A(k)Q_3^T(k) + B_1(k)C_c(k)Q_2(k))^T & (B_c(k)(I - \bar{\Lambda}_\alpha)C(k)Q_3^T(k) + A_c(k)Q_2(k))^T \end{bmatrix},$$

$$\bar{\Omega}_{31}(k) = \begin{bmatrix} \hat{\tau}(k)B_c(k) & \sqrt{3m}B_c(k)\bar{\Lambda}_\alpha \end{bmatrix}^T,$$

$$\bar{\Phi}_{31}(k) = \begin{bmatrix} 0 & \bar{\Omega}_{31}(k) \end{bmatrix}, \ \bar{\Phi}_{33}(k) = \mathrm{diag}\{-I, -I\},$$

$$\bar{\Phi}_{41}(k) = \begin{bmatrix} \hat{\Theta}_{51}^T & \hat{A}_l^T(k) & 0 & \sqrt{2}D(k) \\ 0 & 0 & \sqrt{2}\phi_\alpha \hat{C}_l^T(k) & 0 \end{bmatrix}^T,$$

$$\hat{\chi}_i = \left(\mathrm{tr}\left[\breve{\Gamma}_i(k)\right] \right)^{-1} \ \ (i = 1, 2, \ldots, q),$$

$$\bar{\Phi}_{44}(k) = \mathrm{diag}\{-\hat{\Theta}_{55}, -I_r \otimes Q(k), -I_m \otimes Q(k), -V^{-1}(k)\},$$

$$\bar{\Phi}_{51}(k) = \begin{bmatrix} 0 & \bar{\Omega}_{51}(k) \end{bmatrix}, \ \hat{\Theta}_{51} = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_q \end{bmatrix}^T,$$

$$\bar{\Phi}_{55}(k) = \mathrm{diag}\left\{ -\bar{W}^{-1}(k), -V^{-1}(k), -\bar{W}^{-1}(k) \right\},$$

$$\bar{\Omega}_{51}(k) = \begin{bmatrix} \sqrt{2}B_c(k)E(k) & 0 & \mu(k)B_c(k)E(k) \end{bmatrix}^T,$$

Table 1
The Event-triggered Variance Constrained Controller Design (EVCCD) algorithm:

Step 1.	Given the disturbance attenuation level γ, the positive definite weighted matrices $U_1 > 0$, $U_2 > 0$, $S > 0$ and the scalar $\delta \in [0,\ 1)$. Select the initial value for matrix $M(0)$, $N(0)$ and $Q_1(0)$ $(i = 1, 2, \ldots, l)$ which satisfy the condition (31) and set $k = 0$.
Step 2.	Obtain the values of matrices $\{\mathcal{M}(k+1)$, $\mathcal{N}(k+1), Q_1(k+1), Q_2(k+1), Q_3(k+1)\}$ and the desired controller parameters $\{A_c(k), B_c(k), C_c(k)\}$ for the sampling instant k by solving the matrix inequalities (27)-(30).
Step 3.	Set $k = k + 1$ and obtain $\{M(k+1)\}$ and $\{N(k+1)\}$ by the parameter update formula (32).
Step 4.	If $k < N$, then go to Step 2, else go to Step 5.
Step 5.	Stop.

$$\hat{A}_l(k) = \Big[\Pi_1^T(k)\ \Pi_2^T(k)\ \cdots\ \Pi_r^T(k)\Big]^T,$$

$$\hat{\rho}(k) = \mathrm{tr}\Bigg(\Big[C(k)\ 0\Big]^T\big((I - \bar{\Lambda}_\alpha)^T(I - \bar{\Lambda}_\alpha) + \phi_a^2 I\big) \times \Big[C(k)Q_1(k)\ C(k)Q_3^T(k)\Big]\Bigg),$$

$$\hat{\Theta}_{55} = \mathrm{diag}\Big\{\hat{\chi}_1 I, \hat{\chi}_2 I, \ldots, \hat{\chi}_q I\Big\}, \hat{\tau}(k) = \sqrt{\hat{\tau}_1(k) + 2\bar{\alpha}(1-\bar{\alpha})m},$$

$$\hat{\tau}_1(k) = 6\delta(\bar{m}_1 + \hat{\rho}(k)) + 3\delta\mathrm{tr}\Big[\bar{E}^T(k)\bar{E}(k)\bar{V}(k)\Big],$$

$$\hat{C}_l(k) = \Big[\hat{\Pi}_1^T(k)\ \hat{\Pi}_2^T(k)\ \cdots\ \hat{\Pi}_m^T(k)\Big]^T,$$

$$\Pi_i(k) = Q_{13}(k)A_i^T(k)\ (i = 1, 2, \ldots, r),$$

$$\hat{\Pi}_i(k) = Q_{13}(k)C_i^T(k)\big[B_{c_{1,i}}^T(k)\ B_{c_{2,i}}^T(k)\ \cdots\ B_{c_{n_x,i}}^T(k)\big]\ (i = 1, 2, \ldots, m).$$

Proof: Decompose the variable $Q(k)$ as $Q(k) = \begin{bmatrix} Q_1(k) & * \\ Q_3(k) & Q_2(k) \end{bmatrix}$. Meanwhile, noticing that $P(k) = \mathrm{diag}\{M(k), N(k)\}$, we define $P^{-1}(k) = \mathrm{diag}\{\mathcal{M}(k), \mathcal{N}(k)\}$. It is easy to see that the inequalities (23)–(25) are equivalent to (27)–(29), respectively. Therefore, according to Theorem 3, we have $J_1 < 0$ and $\mathbb{E}\{x(k)x^T(k)\} \leqslant [\,I\ 0\,]Q(k)[\,I\ 0\,]^T$, $\forall k \in \{0, 1, \cdots, N+1\}$. From (30), it is obvious that

$$\mathbb{E}\{x(k)x^T(k)\} \leqslant \Big[\,I\ 0\,\Big]Q(k)\Big[\,I\ 0\,\Big]^T \leq \Upsilon_k,\ \forall k \in \{0, 1, \cdots, N\}.$$

It can now be concluded that the requirements ($Q1$) and ($Q2$) are simultaneously satisfied. The proof is complete.

By means of Theorem 4, the algorithm for designing the Event-triggered Variance Constrained Controller (EVCCD) can be outlined as Table 1.

5 An Illustrative Example

Following [3], we consider the networked control problem for an industrial continuous-stirred tank reactor system, where chemical species A react to form species B. Fig. 1 shows the cross-sectional diagram of continuous flow stirred-tank reactor and Fig. 2 illustrates the physical structure of the system, where C_{Ai}, C_A, T, T_C are, respectively, the input concentration of a key reactant A, the output concentration of chemical species A, the reaction temperature and the cooling medium temperature.

When modeling the industrial continuous-stirred tank reactor system, there exist modeling errors (state-multiplicative noises) and linearization errors (nonlinear disturbances). Moreover, since the system is in a network environment, the sensor saturations may occur in a probabilistic way and are randomly changeable in terms of their types and/or levels due to the random occurrence of networked-induced phenomena such as random sensor failures, sensor aging, or sudden environment changes.

By selecting the state and input variables as $x = \Big[C_A^T\ T^T\Big]^T$, $u = \Big[T_C^T\ C_{Ai}^T\Big]^T$. A discrete-space model is obtained as the form of (1)–(2), where system matrix $A(k)$ and the control matrix $B_1(k)$ are taken from the linearized model of an industrial continuous-stirred tank reactor system in [3]:

$$A(k) = \begin{bmatrix} 0.9719 & 0.0013 \\ 0.0340 & 0.8628 \end{bmatrix},\ B_1(k) = \begin{bmatrix} 0.0839 & 0.0232 \\ 0.0761 & 0.4144 \end{bmatrix}.$$

Our purpose is to design a time-varying controller in the form of (13) in order to control the cooling medium temperature T_C and the input concentration C_{Ai} of a key reactant A in a network environment. To this end, other parameters are set as follows:

$$A_1(k) = \begin{bmatrix} 0.05\cos(k) & -0.1 \\ 0.1 & 0.02\sin(k) \end{bmatrix}, E_1(k) = E_2(k) = E_3(k) = 0.1,$$

$$A_2(k) = \begin{bmatrix} 0.05\sin(k) & 0.1 \\ \sin(k) & 0.02 \end{bmatrix}, D(k) = \begin{bmatrix} 0.2 \\ -0.05 \end{bmatrix}, C_2(k) = \begin{bmatrix} -0.2 \\ 0.05 \end{bmatrix}^T,$$

$$L(k) = \Big[0.1\sin(k)\ -0.3\Big],\ B_2(k) = \Big[0.3\sin(k)\ 0.1\Big],$$

$$C_1(k) = \Big[-0.1\sin(k)\ 0.05\Big],\ C_3(k) = \Big[0.1\ -0.3\sin(k)\Big].$$

The disturbances $w_1(k)$, $w_2(k)$, $v(k)$ and $\varpi(k)$ are mutually independent Gaussian distributed sequences with unity variances. The probability is taken as $\bar{\alpha} = 0.8$. The saturation level is set to be 1, and other parameters are chosen as $K_1 = 0.4$, $K = 0.6$, $V(k) = 0.5$, $\bar{W}_1(k) = 0.8$, $\bar{W}_2(k) = 0.8$ and $\bar{W}_3(k) = 1$.

The nonlinear function $g(k, x(k))$ is chosen as follows:

$$g(k, x(k)) = [\,0.1\ 0.3\,]^T \times \big(0.2x_1(k)\xi_1(k) + 0.3x_2(k)\xi_2(k)\big)$$

where $x_i(k)$ $(i = 1, 2)$ is the ith element of $x(k)$, and $\xi_i(k)$ $(i = 1, 2)$ are zero mean, uncorrelated Gaussian white noise processes with unity variances that is also uncorrelated with $w_1(k)$, $w_2(k)$, $v(k)$ and $\varpi(k)$. The positive definite weighted matrices U_1, U_2, S and variance upper bounds $\{\Upsilon(k)\}_{0 \leq k \leq N+1}$ are chosen as $U_1 = U_2 = I$, $S = 2I$ and $\{\Upsilon(k)\}_{0 \leq k \leq N+1} = \mathrm{diag}\{1.5, 2.5\}$, respectively. The event-triggered transmission threshold is chosen as $\delta = 0.04$.

In the simulation, the initial value of the state is $x(0) = [\,-0.7\ 0.6\,]^T$. The simulation results are shown

Table 2
The average event-triggered ratio with different threshold δ

The event-triggered threshold δ	$\delta = 0$	$\delta = 0.08$	$\delta = 0.2$	$\delta = 0.6$	$\delta = 0.8$
The event-triggered law	100%	68.2%	51.6%	46.7%	43.3%

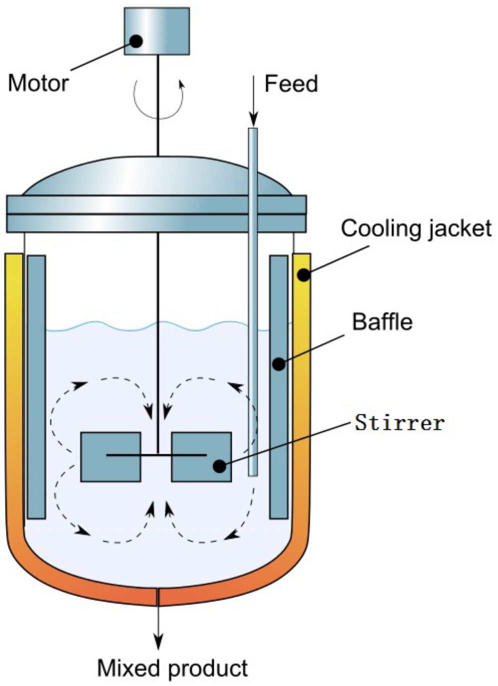

Fig. 1. Cross-sectional diagram of continuous flow stirred–tank reactor

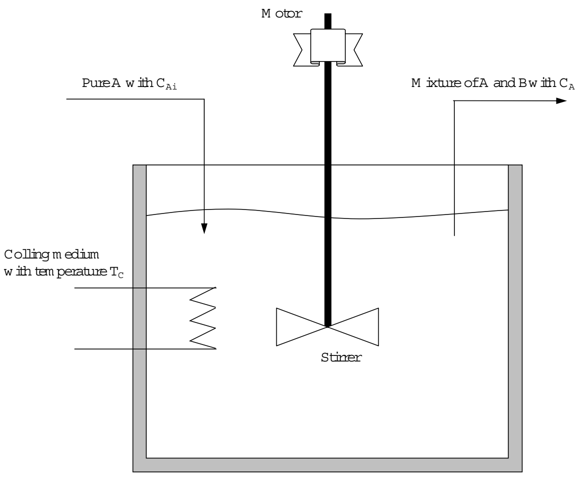

Fig. 2. A continuous-stirred tank reactor model

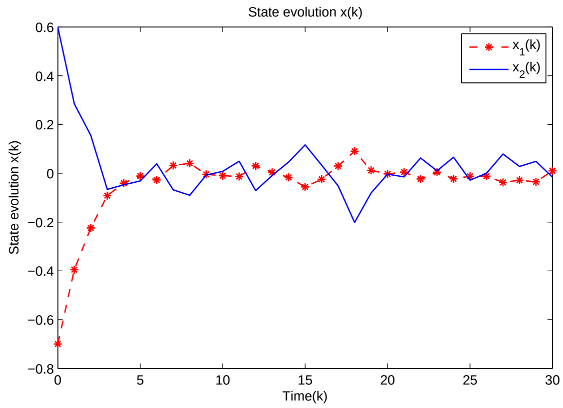

Fig. 3. The state evolution $x(k)$.

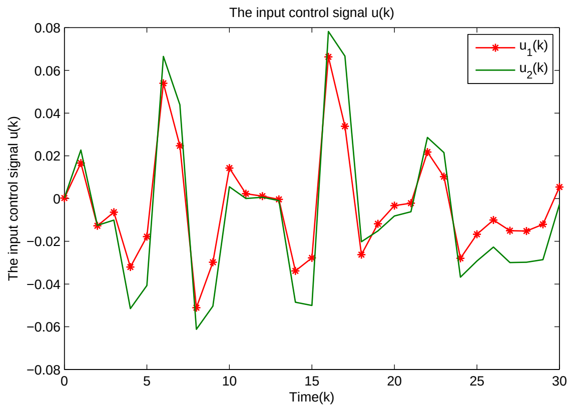

Fig. 4. The input control signal $u(k)$.

in Figs. 3–6, where Fig. 3 depicts the state evolution $x(k)$ and Fig. 4 plots the input control signal $u(k)$. Fig. 5 shows the measurement $y(k)$ and the measurement $y(k_i)$ for event-triggered instants, and the variance upper bound and actual variance are given in Fig. 6.

Furthermore, by conducting 200 independent simulation trials, the average event-triggered ratio (number of event-triggered updates over total number of time points) is displayed in Table 2, which shows that the number of event-triggered updates is quite small and the communication burden is effectively reduced. All the simulation results confirm that the approach addressed in this paper provides a desired finite-horizon performance and the proposed EVCCD algorithm is indeed effective.

6 Conclusion

In this paper, a novel event-triggered multiobjective controller has been designed for a class of discrete time-varying stochastic systems with randomly occurring saturations, stochastic nonlinearities and multiplicative

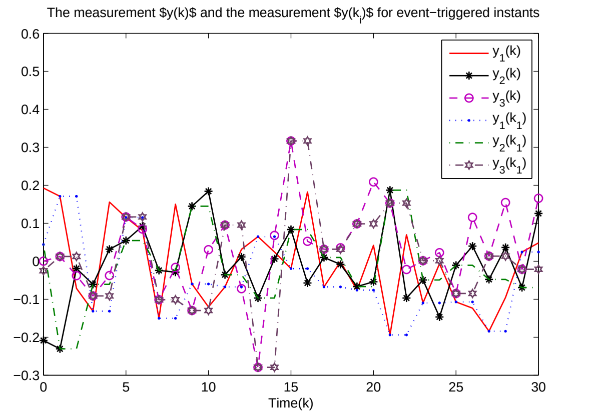

Fig. 5. The measurement $y(k)$ and the measurement $y(k_i)$ for event-triggered instants.

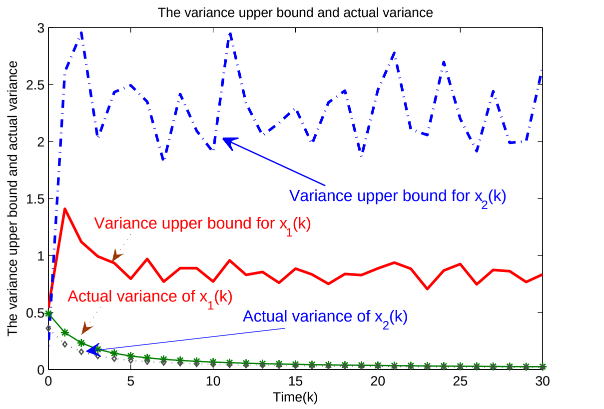

Fig. 6. The variance upper bound and actual variance.

noise. The aim of the addressed problem is to construct an output feedback controller to achieve the prescribed H_∞ noise attenuation level and system state covariance constraint simultaneously. An event indicator variable has been created and the corresponding event-triggered scheme has been proposed in order to reduce the sensor data transmission rate and the energy consumption. Sufficient conditions have been provided to ensure the solvability of the addressed controller design problem and the desired controller gain matrices have been derived in terms of solving recursive matrix inequalities. Finally, an illustrative example has highlighted the applicability and effectiveness of the event-triggered variance constrained control technology presented in this paper.

Appendix I

Proof of Theorem 1

Proof: Define $J(k) := \eta^T(k+1)P(k+1)\eta(k+1) - \eta^T(k)P(k)\eta(k)$.

$$\mathbb{E}\{J(k)\} \le \mathbb{E}\{\bar{\eta}^T(k)\Xi(k)\bar{\eta}(k)\} - \mathbb{E}\{\|z(k)\|^2 - \gamma^2\|\xi(k)\|_U^2\} \quad (33)$$

where $\bar{\eta}(k) := \begin{bmatrix}\eta^T(k) & \varphi^T(k) & \Psi^T(\hat{C}(k)\eta(k)) & \xi^T(k)\end{bmatrix}^T$.

Summing up (33) on both sides from 0 to $N-1$ with respect to k. Hence, the performance index defined in (16) is given by $J_1 < 0$ and the proof is now complete.

Appendix II

Proof of Theorem 2

Proof: We know that the Lyapunov-type equation governing the evolution of state covariance $\mathbb{X}(k)$ is given by

$$\mathbb{X}(k+1) \le \Phi(\mathbb{X}(k)). \quad (34)$$

We now complete the proof by induction. Obviously, $Q(0) \geqslant \mathbb{X}(0)$. Letting $Q(k) \geqslant \mathbb{X}(k)$, we arrive at

$$Q(k+1) \geqslant \Phi(Q(k)) \geqslant \Phi(\mathbb{X}(k)) \geqslant \mathbb{X}(k+1), \quad (35)$$

and therefore the proof is finished.

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