Portfolio Optimisation Using Risky Assets with Options as Derivative Insurance*

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Abstract

We introduce options on FTSE100 index in portfolio optimisation with shares in which conditional value at risk (CVaR) is minimised. The option considered here is the one that follows FTSE100 Index Option standards. Price of options are calculated under the risk neutral valuation. The efficient portfolio composed under this addition of options shows that put option will be selected as part of the investment for every level of targeted returns. Main finding shows that the use of options does indeed decrease downside risk, and leads to better in-sample portfolio performance. Out-of-sample and back-testing also shows better performance of CVaR efficient portfolios in which index options are included. All models are coded using AMPL and the results are analysed using Microsoft Excel. Data used in this study are obtained from Datastream. We conclude that adding a put index option in addition to stocks, in order to actively create a portfolio, can substantially reduce the risk at a relatively low cost. Further research work will consider the case when short positions are considered, including writing call options.

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1 Introduction

The portfolio selection problem is about how to divide an investor's wealth amongst a set of available securities. One basic principle in finance is that, due to the lack of perfect information about future asset returns, financial decisions are made in the face of trade-offs. Markowitz[8] identified the trade-off faced by the investors as risk versus expected return and proposed variance as a measure of risk. He introduced the concepts of efficient portfolio and efficient frontier and proposed a computational method for finding efficient portfolios.

Following notations given by Roman et al. [15], we consider an example of portfolio selection with one investment period. A rational investor is interested in investing their capital in each of number of available assets so that at the end of the investment period the return is maximised. Consider a set of n assets, with asset $j \in \{1 \dots n\}$ having a return of R_j at the end of the investment period. Since the future price of the asset is unknown,

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R_j is a random variable. Let x_j be the percentage of capital invested in asset j and let $x = (x_1, \ldots, x_n)$ denote the portfolio choice. This portfolio's return is given as

$$R_x = x_1 R_1 + \ldots + x_n R_n$$

with distribution function $F(r) = P(R_x \leq r)$ depending on the choice of $x = (x_1, \ldots, x_n)$.

The weighs $(x_1, \ldots, x_n)$ belongs to a set of decision vectors given as

$$X = \{(x_1, \ldots, x_n) | \sum_{j=1}^{n} x_j = 1, x_j \geq 0, \forall j \in \{1, \ldots, n\}\} \quad (1)$$

This is the simplest way to represent a feasible set by the requirement that the weights must sum to 1 and no short selling is allowed.

In the mean-risk paradigm, a random variable R_x representing the return of a portfolio x is characterised using two statistics of its distribution: the expected value / mean (large value are desired) and a "risk" value (low values are desired).

An efficient portfolio is the one that has the lowest risk for a specified level of expected return. Varying the level of expected return, we obtain different efficient portfolios [1]. An efficient portfolio is found by solving an optimisation problem in which we minimise risk subject to a constraint on the expected return.

Markowitz [8] proposed variance as a measure of risk. Criticism of variance, mainly due to its symmetric nature that penalised upside potential as well as downside deviations, led to proposal of other risk measures.

The first objective of this paper is to present the results of an empirical study on the effect of different risk measures on portfolio choice. We examine portfolios with the same expected return, obtained by minimising various risk measures: variance together with downside risk measures (lower partial moment (LPM)) and quantile based risk measures (conditional Value-at-Risk (CVaR)).

The second objective of this paper is to analyse the effect of introducing index options, in addition to stocks, to the portfolio optimisation; this is done in the context of mean-CVaR optimisation. Naturally, options are used as insurance against unfavourable outcomes hence they reduce downside risk; this comes however at a cost that reduces overall the return of the portfolio.

The rest of this paper is organised as follows. Risk measures are presented in Section 2. The algebraic formulations of the corresponding mean-risk optimisation models is presented in Section 3. Section 4 describes the background for incorporating an index option in the portfolio optimisation. Computational results are presented in Section 5. Conclusions are drawn in Section 6.

2 Risk Measures

We give a review of mean risk models and risk measures presented by Roman and Mitra in [15] . Risk measures are classified following [1][14][9] into two categories. The first category measures the deviation from a target and concerned with the whole distribution of outcomes. The second category concerns only with the left tail of a distribution.

Adopting the terminology used in [3], *risk measures of the first kind* measure the magnitude of deviations from a specific point. These risk measures can be further divided into symmetric

[1] This corresponds to examples of low mean-low risk trade-offs up to high mean – high risk trade-offs.

risk measures and asymmetric risk measures. Symmetric risk measures are calculated in terms of dispersion of results around a pre-specified target. Namely, symmetric risk measures such as variance and mean absolute deviation (MAD) are mathematically convenient while being not so good in representing the reality. Asymmetric risk measures in the other hand quantify risk based on the observation that an investor's true risk is the downside risk. Lower partial moments and central semi-deviations are among the important asymmetric risk measures.

Risk measures of the second kind measure the overall significance of possible losses. These risk measures are concerned only with a certain number of worst outcomes (the left tail), of the distribution. The commonly used risk measures in this category are Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). This section gives brief overview on risk measures (variance, LPM0, VaR and CVaR) that will be used in this work, and review the concept of coherent risk measures.

Variance. Variance is a well-known indicator used in statistics for the spread around the mean of a random variable . The variance of a random variable R_x is defined as its second central moment, the expected value of the square of the deviations of R_x from its own mean;

$$\sigma^2(R_x) = E[R_x - E(R_x))^2]$$

where $E(R_x)$ is the expected value of R_x. The variance of a linear combination of random variables is given as;

$$\sigma^2(aR_1 + bR_2) = a^2\sigma^2(R_1) + b^2\sigma^2(R_2) + 2abCov(R_1, R_2)$$

where R_1, R_2 are random variables, $a, b \in \mathbb{R}$, and $Cov(R_1, R_2)$ is the covariance of R_1 and R_2:

$$Cov(R_1, R_2) = \sigma_{jk} = E[(R_1 - E(R_1))(R_2 - E(R_2))]$$

In portfolio optimisation problems, this property is useful to express the variance of the portfolio return $R_x = x_1R_1 + \ldots + x_nR_n$, as a result from choice $x = (x_1, \ldots, x_n)$ as:

$$\sigma^2(R_x) = \sum_{j=1}^{n}\sum_{k=1}^{n} x_j x_k \sigma_{jk}$$

Thus, variance is expressed as a quadratic function of $x_1, \ldots, x_n$ [8].

Lower Partial Moments (LPM). An asset pricing model using a mean-LPM was first developed by Bawa and Lindenberg [3] and Fishburn [4] in 1977. LPM is a generic name for asymmetric measures that consider a fixed target below which an investor does not want the return to fall. Asymmetric measures provide a more intuitive representation of risk, since upside deviations are not penalised. LPM measures the expected value of deviation below a fixed target value τ.

By letting τ be a predefined target value for the portfolio return R_x, and let $\alpha \geq 0$, the LPM of order α around τ of the random variableR_x with distribution function F is defined as [4]:

$$LPM_\alpha(\tau, R_x) = E\{[max(0, \tau - R_x)]^\alpha\} = \int_{-\infty}^{\tau} (\tau - r)^\alpha dF(r)\,.$$

While τ is a target fixed by decision maker (DM), α is a parameter describing the investor's risk aversion. The larger the α , the more risk-averse is the investor. A decision maker is willing to take a risk in order to minimise the chance that the return falls below τ, provided that the main concern is the failure to meet the target return. For this case, choosing a small α is appropriate. Instead, if small deviations below the target are reasonably harmless when compare to large deviations, the DM prefers to fall lower of τ by small amount. In this case, a larger α is obtained [4].

Value-at-Risk (VaR). One of the most popular quantile-based risk measures is the Value-at-Risk (VaR) [6]. The VaR at confidence level $\alpha \in (0,1)$ in is defined as the $(1-\alpha)$-percentile of the portfolio loss distribution, where α is typically chosen as 0.01 or 0.05. Thus, in the calculation of Value-at-Risk at level α of random variable R_x, we suggest that with probability of at least $(1-\alpha)$, the loss [1] will not exceed VaR. Following definitions presented in [15], the VaR at level α of R_x is defined using the notion of $\alpha-$quantiles:

▶ **Definition 1.** An $\alpha - quantile$ of R_x is a real number r such that

$$P(R_x < r) \leq \alpha \leq P(R_x \leq r)\,.$$

▶ **Definition 2.** The *lower* $\alpha - quantile$ of R_x, denoted by $q_\alpha(R_x)$ is defined as

$$q_\alpha(R_x) = \inf\{r \in \mathbb{R} : F(r) = P(R_x \leq r) \geq \alpha\}\,.$$

▶ **Definition 3.** The *upper* $\alpha - quantile$ of R_x, denoted by $q^\alpha(R_x)$ is defined as

$$q^\alpha(R_x) = \inf\{r \in \mathbb{R} : F(r) = P(R_x \leq r) > \alpha\}\,.$$

▶ **Definition 4.** The *Value-at-Risk at level α of R_x* is defined as the negative of the upper $\alpha-$quantile of $R_x : VaR_\alpha(R_x) = -q^\alpha(R_x)$.

The minus sign in the definition of VaR is because $q^\alpha(R_x)$ is likely to be negative. Absolute values are considered in reporting this value in term of "loss".

In general, VaR is not a sub additive measure of risk. It means that the risk of a portfolio can be larger than the sum of the individual risks of its components when measured by VaR (see [16] for examples). Furthermore, VaR is difficult to optimize with standard available methods because VaR is not convex with respect to choice of R_x. This is explained in [7] [10] and references therein. Convexity is an important property in optimization because it removes the possibility of a local minimum being different from a global minimum [13].

Conditional Value-at-Risk (CVaR). Conditional Value-at-Risk [11] [12] was proposed as an alternative quantile-based risk measure. It has been gaining interest from practitioners and academics due to its desirable computational and theoretical properties. As shown in [15], we consider that CVaR is approximately equal to the average of losses greater than or equal to VaR at the same α.

▶ **Definition 5.** The CVaR at level α of R_x is defined as:

$$CVaR_\alpha(R_x) = -\frac{1}{\alpha}E(R_x 1_{\{R_x \leq q^\alpha(R_x)\}}\}) - q^\alpha(R_x)[P(R_x \leq q^\alpha(R_x)) - \alpha]\}$$

where

$$1_{Relation} = \begin{cases} 1, & \textit{if } \text{Relation } \textit{is true}; \\ 0, & \textit{if } \text{Relation } \textit{is false}. \end{cases}$$

[1] In our context we refer negative returns as positive losses. Therefeore, any loss related to random variable R_x is represented by a random variable $-R_x$

2.1 CVaR calculation and optimisation

The following results is used in CVaR optimisation; It was proven by Rockafellar and Uryasev in [12]. Let R_x be a random variable depending on a decision vector x that belongs to a feasible set X as defined by 1 and let $\alpha \in (0,1)$. CVaR of the random variable R_x for confidence level α is denoted by the $CVaR_\alpha(x)$. Consider the function:

$$F_\alpha(x,v) = \frac{1}{\alpha}E[-R_x + v]^+ - v,$$

$$[u]^+ = \begin{cases} u, & \textit{if } u \geq 0; \\ 0, & \textit{if } u < 0. \end{cases}$$

Then:

1. As a function of u , F_α is finite and continuous (thus convex) and

   $$CVaR_\alpha(x) = \min_{v\in\mathbb{R}} F_\alpha(x,v)\,.$$

   In addition, the set consisting of the values of v for which the minimum is attained, denoted by $A_\alpha(x)$, is a non-empty, closed and bounded interval that contains $-VaR_\alpha(R_x)$. In some cases, the set may be formed by just one point.
2. Minimising $CVaR_\alpha$ with respect to $x \in X$ is equivalent to minimising F_α with respect to $(x,v) \in X \times \mathbb{R}$:

   $$\min_{x\in X} CVaR_\alpha(x) = \min_{(x,v)\in X\times\mathbb{R}} F_\alpha(x,v)\,.$$

   In addition, a pair (x^*, v^*) minimises the right hand side if and only if x^* minimises the left hand side and $v \in A_\alpha(x^*)$.
3. $CVaR_\alpha(x)$ is convex with respect to x and F_α is convex with respect to (x,v).
   Thus, if the set X of feasible decision vectors is convex, minimising CVaR is a convex optimisation problem.
   In practice, a portfolio return R_x is a discrete random variable because the random returns are usually described by their realisations under various scenarios. This simplifies the calculation and optimisation of CVaR as it makes the optimisation problems above as linear programming problems. Suppose that R_x has m possible outcomes $r_{1x}, \ldots, r_{mx}$ with probabilities $p_1, \ldots, p_m$ with $r_{ix} = \sum_{j=1}^{n} x_j r_{ij}, \forall i \in \{1\ldots m\}$, then:

   $$F_\alpha(x,v) = \frac{1}{\alpha}\sum_{i=1}^{m} p_i[v - r_{ix}]^+ - v = \frac{1}{\alpha}\sum_{i=1}^{m} p_i[v - \sum_{j=1}^{n} x_j r_{ij}]^+ - v\,.$$

   This formulation will be used for the mean-CVaR optimisation model in the next section. In contrast to VaR, the CVaR is a convex function of the portfolio weights $x = (x_1, \ldots, x_n)$. It is obvious that $CVaR_\alpha(x) \geq VaR_\alpha(x)$ for any portfolio $x \in X$. Thus, minimising CVaR can be used to limit the VaR of a portfolio. Furthermore CVaR is known to be a coherent risk measure. We give a review on coherent risk measure in the next subsection.

2.2 Coherent Risk Measures

Consider a set V of random variables representing future returns or net worth of portfolios. The function $\rho : V \mapsto \mathbb{R}$ is said to be a coherent risk measure if it satisfies the following four axioms:

1. **Subadditivity:** For all $v_1, v_2 \in V, \rho(v_1 + v_2) \leq \rho(v_1) + \rho(v_2)$.
2. **Translation Invariance:** For all $v \in V, a \in \mathbb{R}, \rho(v + a) = \rho(v) - a$.
3. **Positive Homogeneity:** For all $v \in V, a \geq 0, \rho(av) = a\rho(v)$.
4. **Monotonicity:** For all $v_1, v_2 \in V$ such that $v_1 \geq v_2, \rho(v_1) \leq \rho(v_2)$.

The four axioms that define coherency were introduced by Artzner et.al [2]. The subadditivity axiom ensures that the risk associated with the sum of two assets cannot be larger than the sum of their individual risk values. This property requires that financial diversification can only reduce risk. Translation invariance means that receiving a sure amount of a reduces the risk quantity by a. Positive homogeneity implies that the risk measure scales proportionally with the size of the investment. Finally, monotonicity implies that when one investment almost surely performs better than another investment, its risk must be smaller.

From all risk measures discussed in this section, only the CVaR is a coherent risk measure. VaR fails to satisfy subadditivity axiom, and the mean-standard deviation risk measure does not satisfy monotonicity axiom.

3 Mean-risk Optimisation Models

In this section we provide the algebraic formulation of mean-risk models that will be used in this work, namely mean-variance, the mean-LPM of target 0 and order 1, and mean-CVaR optimisation modes. At the end of this section we will give a brief equivalence of mean-CVaR optimisation with a model for maximising returns under a CVaR constraint.

3.1 Algebraic form of the mean-risk models

In this subsection we present below the algebraic form of the three mean-risk models used in our computational analysis. We will use the following notation:

Input data

m = the number of (equally probable) scenarios;
n = the number of assets;
r_{ij} = the return of asset j under scenario i; $j = 1 \dots n$, $i = 1 \dots m$;
μ_j = the expected rate of return of asset j; $j = 1 \dots n$;
σ_{kj} = the covariance between returns of asset k and asset j;$k, j = 1 \dots n$;
d = level of targeted return for the portfolio.

The decision variables

x_j = the fraction of the portfolio value invested in asset j,$j = 1 \dots n$.

3.2 The Mean-Variance Model (MV)

$$\min_x \quad \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{kj} x_j x_k$$

subject to:

$$\sum_{j=1}^{n} \mu_j x_j \quad \geq \quad d \quad ; \quad \forall x \in X$$

3.3 The Mean-Expected Downside Risk model (M-LPM0)

For this model, in addition to the decision variables x_j, there are m decision variables, representing the magnitude of negative deviations of the portfolio return from the zero value, for every scenario $i \in \{1 \dots m\}$:

$$y_i = \begin{cases} -\sum_{j=1}^{n} r_{ij}x_j, & \text{if } \sum_{j=1}^{n} r_{ij}x_j \leq 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \min \quad & \frac{1}{m}\sum_{j=1}^{m} y_i \\ \text{subject to:} \quad & \\ & -\sum_{j=1}^{n} r_{ij}x_j \leq y_i \quad ; \quad \forall i \in \{1 \dots m\} \\ & y_i \geq 0 \quad ; \quad \forall i \in \{1 \dots m\} \\ & \sum_{j=1}^{n} \mu_j x_j \geq d \quad ; \quad \forall x \in X \end{aligned}$$

3.4 The Mean-CVaR$_\alpha$ Model (M-CVaR$_\alpha$)

For this model, in addition to the decision variables x_j, there are $m+1$ decision variables. The variable v represents the negative of an α-quantile of the portfolio return distribution. Thus, when solving this model, the optimal value of the variable v may be used as an approximation for VaR$_\alpha$. The other m decision variables represent the magnitude of negative deviations of the portfolio return from the α-quantile, for every scenario $i \in \{1 \dots m\}$:

$$y_i = \begin{cases} -v - \sum_{j=1}^{n} r_{ij}x_j, & \text{if } \sum_{j=1}^{n} r_{ij}x_j \leq -v; \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \min \quad & v + \frac{1}{\alpha m}\sum_{j=1}^{m} y_i \\ \text{subject to:} \quad & \\ & \sum_{j=1}^{n} -r_{ij}x_j - v \leq y_i \quad ; \quad \forall i \in \{1 \dots m\} \\ & y_i \geq 0 \quad ; \quad \forall i \in \{1 \dots m\} \\ & \sum_{j=1}^{n} \mu_j x_j \geq d \quad ; \quad \forall x \in X \end{aligned}$$

4 Mechanism of Option Pricing and Portfolio Management

An option is a financial derivative described as a contract when the holder of the contract is given a right to exercise a deal, but the holder is not obliged to exercise this right. Financial options are traded both on exchanges and in the over-the-counter market.

There are two basic types of options namely calls and puts. A call option gives the holder the right to buy (not the obligation) the underlying asset (stock, real estate, etc.) at a certain price at a specified period of time. A put option gives the holder the right to sell the underlying asset at a certain price at a specified period of time. The price of underlying stated in the contract is known as the exercise price or strike price while the date in the contract is known as the expiration date or maturity [5].

There are four types of participant in option markets, *buyers of calls*, *sellers of calls*, *buyers of puts*, and *seller of puts*. Most common options that are being exercised today are either American options or European options, which differ in period of exercising the option. American options can be exercised at any time from the date of writing up to the expiration date, while European options can only be exercised on the expiration date.

4.1 Value of options

The value of an option can be broken into two components called intrinsic value and time value. *Intrinsic value* is the difference between the value of the underlying and the exercise price. The intrinsic value of an option may be either positive or zero, but it can never be negative. This is because the contract involves no liability on the part of the option holder, where the option holder can walk away without exercising the option.

▶ **Example 6.** For a holder of call option, if the value of underlying (example: stock) is less than the option exercise price, the option is referred as being *out of the money* (OTM). If the value of stock is greater than the exercise price, the option is referred to as being *in the money* (ITM). If the value of the stock is equal to the exercise price, the option is referred to as being at the money (ATM).

Mathematically, value of an option is represented in term of option payoff function. An option payoff function, evaluated as a function of the underlying stock price S_T , at maturity. Consider put and call options with strike price K, the payoff function is given as:

$$V_{put}(S_T) = \max\{0, K - S_T\}, \text{ and } V_{call}(S_T) = \max\{0, S_T - K\},$$

respectively.

▶ **Example 7.** Assume that an investor is holding a portfolio consisting of a stock (long) and a put option on the same stock (long) with strike price K. The payoff function of the portfolio,V_{pf}, is therefore given as

$$V_{pf}(S_T) = S_T + V_{put}(S_T) = \max\{K, S_T\}\,.$$

This payoff function shows that the put option with strike price K secure the portfolio value at maturity from dropping below K.

Risk-neutral valuation [5] is used in our implementation towards incorporating option into the mean-risk portfolio optimisation problems. Thus, we assume that in the long-run, the price on an option will equal to the expected future payoff of the option itself, discounted at a risk-free rate r, over maturity period $T - t$. We denote C_t and P_t as our empirical prices for call and put respectively, so

$$P_t \text{ or } C_t = E[\text{payoffs}]e^{-r(T-t)}\,.$$

4.2 Incorporating option into portfolio optimisation

In the case of stocks, obtaining the parameters r_{ij} necessary in order to implement the optimisation models in 4.1. is straightforward, if we use historical data in order to generate scenarios. In this case, we monitor the prices of the corresponding shares at interval of times equal to the investment period of the portfolio model; we then compute the historical returns and use them as "scenarios", assuming that past will repeat in the future. Alternatively, we can simulate prices for the shares at the end of the investment period, and using the current known prices, to obtain simulations for the returns.

Consider now the case when one of the assets is an option (on a generic underlying asset), with the same maturity as the investment period of the portfolio problem. In this case, obtaining the scenarios for its return at the end of the investment period is different, since the return depends on the price of the underlying asset at expiration.

In this work, we introduce a call and a put option on the FTSE100 index as two additional assets to the existing components of FTSE100.

In this case, we need to simulate the price of FTSE100 at the end of the investment period. We have used historical rates of return for the component assets of FTSE100. To keep consistency, we do the same for the index itself: we monitor the historical rates of return and, using the current (known) price, we simulate prices for FTSE100 at the end of the investment period.

The corresponding pay-offs of the options can then be calculated for any strike price K. The return of the option is calculated then using the current price of the option and the simulated pay-offs. If there is no price available for an option with strike price K and the given maturity, a correct price can be calculated, for example using the arbitrage free / risk neutral pricing using the current risk free rate of return.

To summarise, we use the following method :

1. We compute the historical rates of return for the underlying FTSE 100 for a period of time equal to investment period.
2. The returns from step 1 are used to simulate next period's underlying (FTSE100) value, given its current value:

$$S_{t+1|t} = S_t(1 + r_{t+1})\,.$$

3. Using known strike price for call and put options, K_c and K_p, and one period simulated underlying asset value $S_{(t+1|t)}$, we simulate option payoffs at their maturity $t+1$. Thus the payoffs for call and put are given respectively as:

$$C_{t+1|t} = \max(S_{t+1} - K_c, 0), \text{ and } P_{t+1|t} = \max(K_p - S_{t+1}, 0).$$

4. Using the simulated payoff above, option returns are computed by:

$$r_{t+1|t,c} = \frac{C_{t+1|t}}{C_t} - 1, \text{ and } r_{t+1|t,p} = \frac{P_{t+1|t}}{P_t} - 1$$

5 Computational Results

This section aims to investigate the behaviour of mean-risk models exhibited formulated in Section 3 when used in application of investment portfolio selection. We consider three different mean-risk models, with risk measured by variance (denoted by M-V), the expected downside risk by the lover partial moment of order 1 and target 0 (M-LPM0), and CVaR for 5% confidence level (M-CVaR$_{0.05}$).

We first work with only stocks and we consider the efficient portfolios in the mean-risk models above for several levels or expected return. We investigate their properties, in terms of their composition, the in-sample performance, and the out-of-sample performance. At the

Table 1 The Euclidean distance between efficient portfolios with expected return $d_1 = 0.01$.

Models		Value of D
M-V	M-LPM0	0.245071661
M-V	M-CVaR$_{0.05}$	0.298300895
M-LPM0	M-CVaR$_{0.05}$	0.398379982

same time, the equivalence of M-CVaR$_{0.05}$ and model for maximisation of expected value with CVaR constraint (Max-E) is shown for the same level of alpha.

Finally, we include FTSE 100 index options (put and call), in addition to FTSE100 stocks. We implement the mean-CVaR model with this new universe of assets and compare the performance of the resulting optimal portfolios with that of stocks-only portfolios.

5.1 Data set

The data used for this analysis is drawn from the FTSE100 index. Our investment period is one month. The monthly returns of the 87 stock components of the index from January 2005 until January 2015 were considered. The dataset for the in-sample analysis has 100 time periods from January 2005 until May 2013. For the out-of-sample analysis, the behaviour of the portfolio obtained is examined over the twenty months period of June 2013 until January 2015. The models were implemented in AMPL and solved using CPLEX 12.5 optimisation solver.

5.2 Methodology

The characteristics of efficient portfolios may vary depending on targeted return, d. Based on our data set, the maximum level of asset return is 0.0349 and the minimum is at -0.007323. Thus, for simplicity of implementation and for feasible possible solutions, we chose three different level of d as $d_1 = 0.01$, $d_2 = 0.02$, and $d_3 = 0.03$. We solve the three mean-risk models considered above for every level of expected return d_1, d_2, and d_3. We also give a fair comparison between two equivalent optimisation models: of M-CVaR$_{0.05}$ and Max-E for the same alpha level.

5.3 The composition of the efficient portfolios

The difference of the composition of two portfolios $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is described by using the Euclidean distance between vectors x and y. This quantity is denoted as $D_{(x,y)} = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$.

It is observed that the composition of the M-CVaR$_{0.05}$ efficient portfolios differs substantially from the composition of the other efficient portfolios. Table 1 presents the values of the indicator D for the efficient portfolios with expected return $d_1 = 0.01$. The M-CVaR$_{0.05}$ efficient portfolios have relatively different composition with the other two mean-risk models. The Euclidean distance between the M-CVaR$_{0.05}$ and the M-LPM0 is 0.298 and the distance with the M-V is 0.398. The same features is also observed for the other two levels of expected return.

We also observe the number of assets that constitute the efficient portfolios. It is seen that the model with the risk measure of the first kind, the variance, produced portfolios with the highest number of component assets, while M-CVaR$_{0.05}$ model produce portfolios with the lowest number of component assets. This is consistent with the modelling standard,

Table 2 The number of assets in the composition of mean-risk portfolios.

	M-V	M-LPM0	M-CVaR$_{0.05}$
$d_1 = 0.01$	18	12	9
$d_2 = 0.02$	13	11	8
$d_3 = 0.03$	6	6	4

Table 3 Statistics for the mean-risk efficient distributions with expected value $d_1 = 0.01$.

	M-V	M-LPM0	M-CVaR$_{0.05}$
Median	0.013642802	**0.015415625**	0.011657133
Standard Deviation	**0.033783896**	0.037980983	0.03830069313
Skewness	-0.572809564	-0.699597028	**0.361368572**
Minimum	-0.091575414	-0.13878571	**-0.075281385**
Maximum	0.096340696	0.115140014	**0.1340699**

since diversification is the basis of the mean-variance portfolio theory. The number of assets composition is shown in Table 2. It is also noticed that as the level of portfolio expected return increases, the number of component assets decreases. This is also consistent with the modelling standard that at lower levels of return, an efficient portfolio has more assets in composition in order to lower the risk.

5.4 In-sample analysis

The return distributions of the efficient portfolios are discrete with 100 equally probable outcomes. We analyse these distributions using in sample parameters of standard deviation, skewness, minimum, maximum, and range. We compare sets of three distributions, each having the expected values of d_1, d_2, and d_3.

For a portfolio distribution, it is desirable to have smaller standard deviation and range, and to have larger median, skewness, minimum, and maximum.

Table 3 presents the results obtained for the level of targeted return $d_1 = 0.01$. The M-V efficient portfolio has the lowest standard deviation. The return distribution of the M-CVaR efficient portfolios achieve the best values for the parameters concerned with the left tail of distributions. It has the highest skewness, highest minimum and also highest maximum. Notice that M-CVaR efficient distribution always positively skewed while the other models, the distribution are negatively skewed. Similar results is obtained for other levels of d as shown in Table 4 and 5.

5.5 Out-of-sample analysis

We analysed the performance of the efficient portfolios over the twenty time periods following the last period of in-sample data. The results of the out-of-sample analysis were consistent with those of the in-sample analysis. The consistency is in the sense of the portfolios selected under M-CVaR models were distinct from the other two mean-risk models. This is shown via Figure 1, where it may be noticed that a generally good performance of M-CVaR models are obtained. The similar figures for different in-sample mean returns also shown in Figure 2 and 3.

Figure 1 exhibits the compounded returns of the efficient portfolios with in-sample mean returns $d_2 = 0.02$. It is seen that the M-CVaR efficient portfolio performs better than the

Table 4 Statistics for the mean-risk efficient distributions with expected value $d_2 = 0.02$.

	M-V	M-LPM0	M-CVaR$_{0.05}$
Median	0.026989273	**0.019547821**	0.027136642
Standard Deviation	**0.045539089**	0.048306959	0.053504299
Skewness	-1.068379489	-0.541590574	**0.106153237**
Minimum	-0.169911236	-0.169067343	**-0.147248853**
Maximum	0.113089218	0.132563297	**0.189886327**

Table 5 Statistics for the mean-risk efficient distributions with expected value $d_3 = 0.03$.

	M-V	M-LPM0	M-CVaR$_{0.05}$
Median	0.034052445	**0.025471156**	0.033057945
Standard Deviation	**0.077244906**	0.080794207	0.084842553
Skewness	-0.577646499	-0.307744989	**0.199328725**
Minimum	-0.254267371	-0.269452992	**-0.213015378**
Maximum	0.186538415	0.200218938	**0.274484094**

other two portfolios. Statistical parameters for out-of-sample results also show that M-CVaR had the highest mean. However, note that the out-of-sample performance for $d_3 = 0.03$ is fairly different from our expected performance. Further investigations are required to explain this behaviour in the near future.

5.6 Introducing index options in the universe of assets

In this subsection, we consider in addition to the stocks described in 5.1, a put and a call option on FTSE 100 with maturity one month and with strike price $K = 6583.1$, which is equal to the current price of FTSE 100. We compute the prices for these options using risk free valuation and simulate the rates of return as described in section 4.3. We test the performance on two mean-risk models, the M-V and the M-CVaR$_{0.05}$, for d_1, d_2, and d_3. Based on portfolio composition, it is interesting that both models include index put option as one of the components in the efficient portfolio for every level of targeted returns, with range of weights from 1.2% to 2.9% of overall wealth. The call option is however not selected at any level of targeted returns. Table 6 summarizes these figures.

Another out-of-sample finding is on the risk measures comparison between portfolio with stock only (S-portfolios) versus portfolio with stocks and options (OS-portfolios). It is observed that for every cases of d, in the case of M-CVaR$_{0.05}$, risk associated with OS-portfolios always lower than of S-portfolios. For example of $d = 0.01$, the average loss in the worst 5% cases is only 2.34% of initial investment of OS-portfolios. At the same level, the average loss fo S-portfolios is nearly doubles at 5.63%. We present the values in Table 7.

We also vary our strike price K to different levels such as at the average index point level and the minimum index point level over the simulated index points. Interestingly, the portfolio composition gives similar results as we present here; (1) It is observed that no call option will be selected to the portfolio in all cases, and (2) the risk measures for OS-portfolios are lower than of S-portfolios at every level of d.

To continue with the implementation of options into our portfolio optimisation problems, a backtesting is conducted to see the accuracy of our portfolio in predicting actual results.

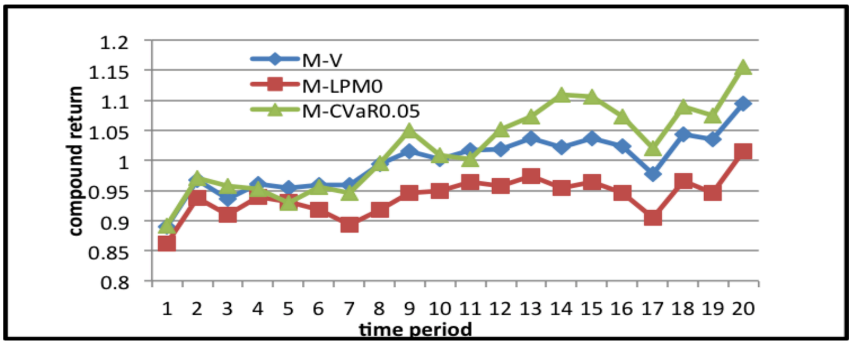

Figure 1 The out-of-sample performance of the mean-risk efficient portfolios with in-sample mean returns $d_2 = 0.02$

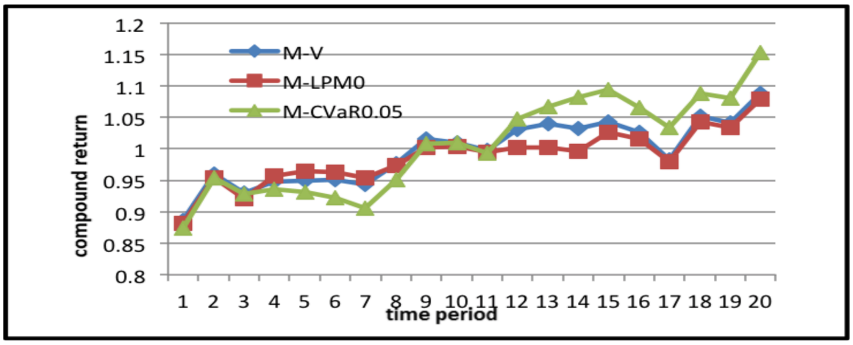

Figure 2 The out-of-sample performance of the mean-risk efficient portfolios with in-sample mean returns $d_1 = 0.01$

5.7 Backtesting

The purpose of backtesting is to see how the optimized portfolios would have performed in reality. Using the same data drawn from FTSE100 as before, we use the optimized weights of each stocks (and options) to evaluate the next period's expected return.

The backtest plan that we use in this study follows the following Figure 4.

We analysed the performance of our strategy by comparing the backtest results of portfolio with stock only (S-portfolios) versus portfolio with stocks and options (OS-portfolios). The backtest is performed over 12-month period (until t=12) by looking at the growth rate of both portfolios. Results show that the growth of OS-portfolios is higher until the 8th month. In these time periods, we observed that options are beneficial to the portfolios, while the S-portfolios suffer from bigger losses or lower returns. For the last 4-month period we can see that the growth of S-portfolios growing stronger. In these periods, it is made clear that there are some scenarios when buying put options is just a cost of insurance for a portfolio against huge losses. Figure 5 exhibit the growth that we mentioned here.

The expected return for each period is shown in the following Table 8 , by comparing the returns for the S-portfolios with the OS-portfolios. It is observed that the maximum return of S-portfolio is 9.3% and the maximum for the OS-portfolio is at 8.1%. Whereas, the minimum for S-portfolio is the loss of 10.9% while the minimum for OS-portfolio is only a loss of 3.1%. We interpret the overall performance of the two portfolios by looking at the average of these expected returns and their standard deviations. It is clearly seen that the average expected returns of S-portfolios is slightly higher of 0.5% against the returns

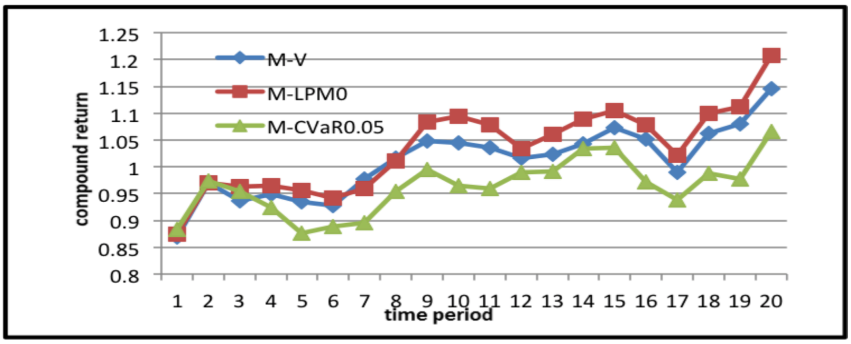

Figure 3 The out-of-sample performance of the mean-risk efficient portfolios with in-sample mean returns $d_3 = 0.03$

Table 6 The number of assets in the composition of mean-risk portfolios with weight of index put option.

	M-V		M-CVaR$_{0.05}$	
	Number of assets	Weight of put	Number of assets	Weight of put
$d_1 = 0.01$	19	0.0124823	17	0.0217767
$d_2 = 0.02$	15	0.0149846	13	0.0235232
$d_3 = 0.03$	6	0.020194	6	0.02948

of OS-portfolios. However, the deviation measure of S-portfolios is s higher than deviation measure of OS-portfolios by 2.4%. These indicate that we have a better performance of CVaR efficient portfolios in which index options are included.

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6 Conclusion

We consider three mean-risk models, where the risk measures are variance, the lower partial moment of oder 1 and target 0 and CVaR at confidence level 95%. They are conceptually measuring risk very differently, as variance is a symmetric risk measure that penalises deviations from mean on either side of the distribution. Lower partial moments are asymmetric risk measures that penalise only negative deviations from a fixed target, while CVaR is a tail/quantile risk measure that only looks at a pre-specified percentage of worst case losses.

We implement the risk models in AMPL using a dataset drawn from FTSE100 with 87 stocks. For each mean-risk models, we consider the efficient portfolios at expected rate of return 1%, 2% and 3% respectively – corresponding to low mean-low risk, medium mean-medium risk, high mean-high risk trade-offs. We observed that the mean-variance portfolios are the most diversified while the mean-CVaR efficient portfolios the least diversified. The

Table 7 Risk measure quantities for M-CVaR$_{0.05}$ for efficient portfolio with and without options.

	M-CVaR$_{0.05}$	
	S-portfolio	OS-portfolio
$d_1 = 0.01$	5.63%	2.34%
$d_2 = 0.02$	8.05%	2.67%
$d_3 = 0.03$	13.42%	8.54%

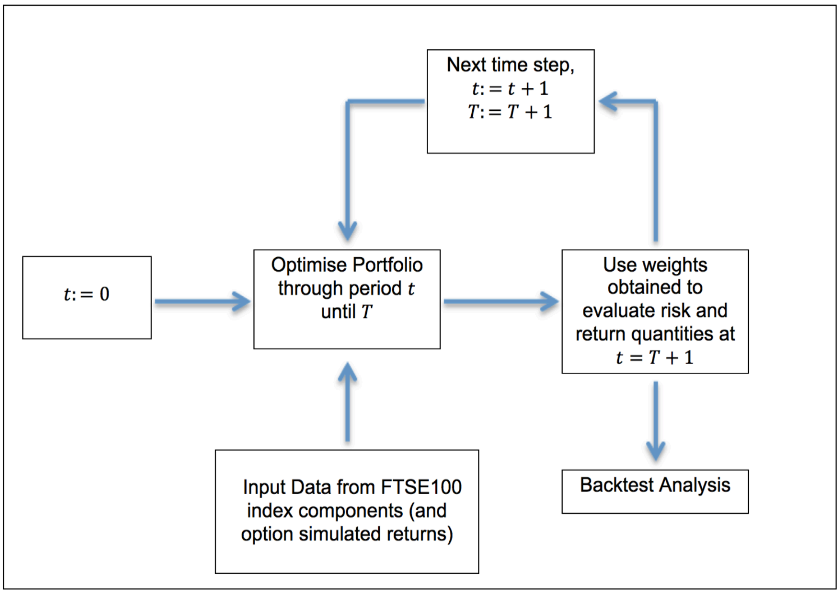

Figure 4 The Backtesting Plan.

in-sample characteristics of their return distributions are somewhat as expected, considering the nature of the risk measures. The mean-CVaR efficient portfolios have the best left tail, with the highest minimum and highest skewness. The out-of-sample performance, evaluated on the next 12 months following the last price observations, show a generally better performance of mean-CVaR efficient portfolios, although this is not consistent across.

We include FTSE100 index options (calls and puts) in the mean-variance and mean-CVaR portfolio optimisation. The maturity of the options is the same with the investment period (one month in our computational work) and we experimented with different strike prices; for each of these we calculated the price of the option under the risk neutral valuation and generated the scenarios for options returns. For each of the strike prices considered, the optimal portfolio weight of the call is zero hence there is no investement in it, while in the majority of cases, the optimal weight of the put option was around 2% of the portfolio value.

This is somewhat expected since the return of the portfolio of stocks, even though actively constructed via risk minimisation is positively correlated with index's return. An investment in a portfolio of stocks from FTSE100 (long positions) is somewhat assuming that the price of FTSE100 is on increase. The put option acts hence as a type of insurance for the cases when the FTSE100 prices are on the decrease.

The computational results are interesting not because the risk (either measured by variance or CVaR) is decreased with the introduction of option; this is natural since the option is simply an additional asset which can only improve or keep the same risk value.

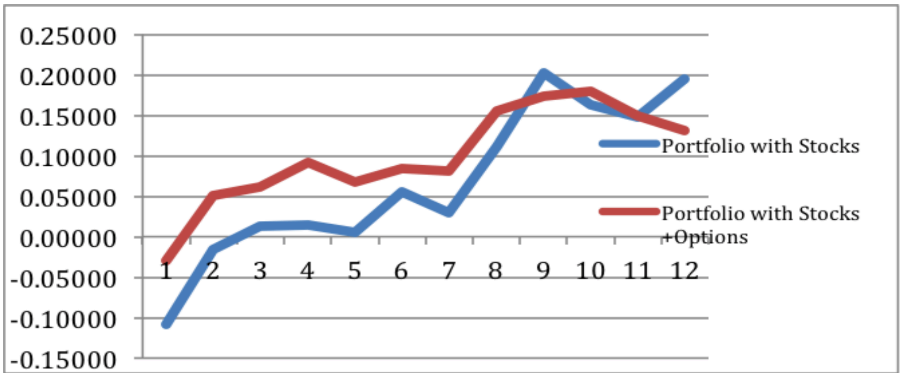

Figure 5 Portfolio Growth Comparison between Stock Portfolio (S-portfolio) and Stock+Options Portfolio (OS-portfolio).

Table 8 Expected Returns for S-portfolios and OS-portfolios.

	Expected Returns	
Backtest Periods	S-portfolio	OS-portfolio
1	**-0.10889**	-0.03016
2	**0.09329**	**0.08102**
3	0.02801	0.01105
4	0.00209	0.03019
5	-0.00969	-0.02504
6	0.05007	0.01787
7	-0.02598	-0.00431
8	0.08269	0.07479
9	0.09147	0.01838
10	-0.03970	0.00561
11	-0.01483	**-0.03051**
12	0.04622	-0.01729
Average	**0.01623**	0.01097
Standard deviation	0.060752874	**0.037197654**

The results are interesting because the decrease in risk is substantial. For example, with the CVaR minimisation at 2% in sample expected rate of return, the optimal CVaR in the case of stocks only is 8.05%, while in the case of stocks + put is 2.67%, for only 2.35% of the portfolio value invested in the put (refer Table 6 and 7).

The backtesting results show that the portfolios composed of stocks and options had substantially different realised returns, compared with the stocks only portfolio. It is somewhat expected that the stocks only portfolio has an average slightly better returns there will be cases when the put has zero pay off and -100% return. However, the realised returns of the portfolios including the option have a better minimum– the lowest realised rate of return is 8%, as compared to -11% in the case of stocks only portfolios – and also much lower standard deviation. As a conclusion, adding a put index option in addition to stocks, in order to actively create a portfolio, can substantially reduce the risk at a relatively low cost. As a future research work, we would like to consider the case when short positions are considered, including writing call options.

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