Equivalence of QCD in the ϵ-regime and chiral Random Matrix Theory with or without chemical potential

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Abstract

We prove that QCD in the ϵ-regime of chiral Perturbation Theory is equivalent to chiral Random Matrix Theory for zero and both non-zero real and imaginary chemical potential μ. To this aim we prove a theorem that relates integrals over fermionic and bosonic variables to super-Hermitian or super-Unitary groups also called superbosonisation. Our findings extend previous results for the equivalence of the partition functions, spectral densities and the quenched two-point densities. We can show that all k-point density correlation functions agree in both theories for an arbitrary number of quark flavours, for either $\mu = 0$ or $\mu \neq 0$ taking real or imaginary values. This implies the equivalence for all individual k-th eigenvalue distributions which are particularly useful to determine low energy constants from Lattice QCD with chiral fermions.

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1 Introduction

The application of chiral Random Matrix Theory (chRMT) to QCD, initially introduced at zero chemical potential in [1], has lead to many exact analytical results for the Dirac operator spectrum. These have in turn found applications in comparison to Lattice Gauge theory by various groups (for reviews see [2, 3]). The deeper reason for this analytical understanding of the Dirac spectrum in the phase with broken chiral symmetry can be traced back to the relation between chRMT and the epsilon regime of chiral Perturbation Theory ($\epsilon\chi$PT). Being based on global symmetries alone there are three classes of chiral symmetry breaking patterns, corresponding to the three possible anti-unitary symmetries that lead to the unitary, orthogonal and symplectic chRMT [4]. Here, we will only report on the first class containing QCD, which is technically the simplest. Relatively little is known about the equivalence to $\epsilon\chi$PT for gauge theories in the two other classes, see however [5, 6].

In this paper we will show that any correlation function of Dirac operator eigenvalues computed starting from $\epsilon\chi$PT is the same as the corresponding quantity computed from chRMT in the large-N limit of infinite matrices, corresponding to the infinite volume limit of chiral Perturbation Theory. This equivalence holds both with and without chemical potential, being of real or imaginary type. The chRMT we use is the one introduced by Shuryak and Verbaarschot [1] for $\mu = 0$, by Osborn [7] for real $\mu \neq 0$, and by Akemann et al. [8] for imaginary μ where the two latter are two-matrix models. The one-matrix model for real $\mu \neq 0$ introduced earlier by Stephanov [9] can also be mapped to $\epsilon\chi$PT. Being technically more involved we don't give details of our analogous equivalence proof here.

The $\epsilon\chi$PT – chRMT equivalence is established by relating the generating functionals of resolvents of an arbitrary finite degree, with any number of flavours N_f. In $\epsilon\chi$PT these generating functionals are obtained by adding additional fermionic and bosonic quarks, leading to supergroup integrals. These group integrals are shown to be equal to the corresponding quantities in chRMT at large-N, given by the expectation value of ratios of Dirac operator determinants (also called characteristic polynomials in the chRMT language).

The technical tool we use is a theorem we prove also called superbosonisation, which was obtained independently by [10, 11]. After writing the ratios of determinants in terms of superfermionic variables and integrating out the Gaussian random matrices the theorem allows us to map these expressions to the supergroup integrals of $\epsilon\chi$PT. We only take the large-N limit at the very end, without performing any saddle point approximations. Let us emphasise that although there are many applications of RMT in Physics [12], only in few cases an exact map to the underlying microscopic theory has been achieved.

In chRMT there are alternative ways to compute arbitrary k-point density correlations, or individual eigenvalue correlation functions, some without introducing resolvents. One example is the method of orthogonal polynomials [13]. On the other hand the supergroup integrals in $\epsilon\chi$PT are increasingly hard to calculate for $k > 1$. By establishing an equivalence for all generating functionals we can therefore match with all known chRMT correlation functions, were they obtained using resolvents or not. For $\mu = 0$ all generating functionals in chRMT were computed in [14, 15]. For real $\mu \neq 0$ all building blocks for characteristic polynomials in chRMT in the large-N limit follow from [16] and [17]. At imaginary $\mu \neq 0$ so far only the bi-orthogonal polynomial method has been used in chRMT [8].

The explicit results for the k-point density correlation functions can be most conveniently read off directly from [18] for massless, and from [19, 20] for massive correlations at $\mu = 0$, for both massive and massless correlations from [16] at real $\mu \neq 0$, and likewise from [8] for imaginary μ. The corresponding individual eigenvalue correlation functions that require the knowledge of all density correlations were obtained in [21] for $\mu = 0$, in [22] for for real μ and in [23] for imaginary μ.

The following results were previously known about the $\epsilon\chi$PT – chRMT equivalence: partition functions with only fermions were shown to agree for zero [1] and non-zero μ [24, 25, 26, 3]. The equivalence for partition functions with only bosons at $\mu = 0$ follows from [27]. The generating functional of a $(N_f + 1|1)$ supergroup integral leading to the spectral density was computed in [28, 29] including N_f massless fermions. This work was extended to include the quenched two-point density in [30], all for $\mu = 0$. Furthermore, the quenched density at real $\mu \neq 0$ was computed from both $\epsilon\chi$PT [14] and chRMT [7] and found to be in agreement. For imaginary isospin $\mu \neq 0$ the equivalence was established up to the two-point function in [31, 8]. It was pointed out [32] how in principle to compute the distributions of the k-th individual eigenvalue from $\epsilon\chi$PT, using all k-point density correlation functions, in order to reproduce previous chRMT results [21]. The same strategy can be applied for non-zero μ [23, 22].

Our equivalence proof for all k-point density correlation functions thus in particular fills this gap. Indi-

vidual eigenvalue distributions have been used in several groups to compare with Lattice results with exact or approximate chiral symmetry, leading to a well defined gauge field topology.

A further remark concerns universality. On the chRMT side it is well known for $\mu = 0$ that in the microscopic large-N limit the results hold for a much wider class than Gaussian weight functions [18, 33, 19]. For that reason we can restrict ourselves to establish the equivalence for Gaussian chRMT. For $\mu \neq 0$ less is known. Both the model by Stephanov [9] and by Osborn [7] are Gaussian matrix models. As already mentioned above the fermionic partition functions [24, 25, 26, 3] and the quenched density [14, 7] were shown to agree among both matrix models, and with $\epsilon\chi$PT. We can now extend this equivalence to all correlation functions. This agreement suggests universality for chRMT to hold at $\mu \neq 0$ as well in the broken phase.

Finally let us mention that for non-chiral RMT the equivalence to an effective theory for spontaneous flavour symmetry breaking is simpler and has already been fully established at $\mu = 0$ by Szabo [34]. We expect that an extension to $\mu \neq 0$ can be done following the same lines as here.

Our paper is organised as follows: In section 2 the resolvent method is briefly recalled both for Hermitian operators ($\mu = 0$, μ pure imaginary) and non-Hermitian ones (μ real), in order to clarify how the k-point correlation function are obtained from a partially quenched theory. In sections 3, 4 and 5 the proofs of the equivalence for the partially quenched theories for $\mu = 0$, for imaginary μ and real μ are given, respectively. Here we write the averages over ratios of Dirac determinants in chRMT as supervectors integrals, perform the chRMT average and apply our theorem relating to supergroups. This theorem is presented and proved in section 6. More technical remarks on delta functions and on some integrals used are deferred to appendices A and B.

2 The resolvent method

The resolvent method is way to compute the k-point eigenvalue correlation function of an operator distributed according to a given ensemble (in our case these ensembles are QCD or chRMT). Starting from the expectation value of a ratio of characteristic polynomials, this quantity may be considered as the partition function of a theory with an additional number of fermionic and bosonic quarks, called partially quenched QCD (or chRMT). Both supersymmetric and replica methods rely on the resolvent method.

In this section we briefly summarise it's idea for the reader's convenience, referring to the literature for more detailed explanations [35]. Let us emphasise that the resolvent method is substantially different whether one considers a theory with an Hermitian or non-Hermitian operator. QCD with zero or imaginary chemical potential is a theory with an anti-Hermitian Dirac operator, the correspondence with an Hermitian operator is trivial. QCD with real chemical potential is a non-Hermitian theory.

We start by showing the resolvent method for Hermitian theories. The k-point correlation function is defined as the expectation value of the product of k δ-functions:

$$\rho_k(\eta_1, \ldots, \eta_k) \equiv \left\langle \prod_{j=1}^{k} \sum_{\lambda \in \text{e.v.}} \delta(\eta_j - \lambda) \right\rangle \,, \tag{1}$$

where the expectation value is computed according to the ensemble of the operator D whose eigenvalues are λ. Alternative definitions of this quantity may be given, in order to avoid the contact terms obtained through delta functions at coinciding arguments $\eta_i = \eta_j$ [12].

For simplicity we consider first the easiest case, the spectral density ($k = 1$). We define its resolvent as

$$G_1(z) \equiv \left\langle \sum_{\lambda \in \text{e.v.}} \frac{1}{z - \lambda} \right\rangle = \int d\lambda \, \rho_1(\lambda) \frac{1}{z - \lambda}. \tag{2}$$

Here the argument z is taken to lie outside the support of $D \subset \mathcal{R}$. Given that the density is z-independent this integral equation can be inverted as follows [36]:

$$\rho_1(\eta) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \left[G_1(\eta - i\varepsilon) - G_1(\eta + i\varepsilon) \right], \tag{3}$$

taking the discontinuity of the resolvent when crossing the support. An essential ingredient for this result is the fact that all the eigenvalues of an (anti-)Hermitian operator lie on the real (imaginary) axis. This requirements

is not fulfilled by theories with non-Hermitian Dirac operator, invalidating these equations. The next step is to generate the resolvent (2) as a ratio of determinants:

$$G_1(z) = \left\langle Tr\left[\frac{1}{z-D}\right]\right\rangle = \frac{\partial}{\partial z'}\left\langle \frac{\mathcal{D}et\left[D-z'\right]}{\mathcal{D}et\left[D-z\right]}\right\rangle\Big|_{z'=z} . \tag{4}$$

The generalisation to the k-point partition function is conceptually not different, with a bit more attention needed in subtracting eventual higher order poles in z in $\left\langle \prod_{j=1}^{k}\sum_{\lambda\in \text{e.v.}}\frac{1}{z_j-\lambda}\right\rangle$. It can be proved that the k-point correlation function may be generated through the theory with k additional bosonic and fermionic quarks. The corresponding resolvents are defined as

$$G_k(z_1,\ldots,z_k) \equiv \left\langle \prod_{j=1}^{k}\sum_{\lambda\in\text{e.v.}}\frac{1}{z_j-\lambda}\right\rangle = \int\prod_{j=1}^{k}d\lambda_j\frac{1}{z_j-\lambda_j}\rho_k(\lambda_1,\ldots,\lambda_k) . \tag{5}$$

and can be generated as in eq. (4)

$$G_k(z_1,\ldots,z_k) = \left\langle \prod_{j=1}^{k} Tr\left[\frac{1}{z_j-D}\right]\right\rangle = \left(\prod_{j=1}^{k}\frac{\partial}{\partial z'_j}\right)\left\langle \prod_{j=1}^{k}\frac{\mathcal{D}et\left[D-z'_j\right]}{\mathcal{D}et\left[D-z_j\right]}\right\rangle\Big|_{z'_j=z_j\,\forall j} \tag{6}$$

The density correlations are then given by the discontinuities with respect to all arguments (see e.g. [34]):

$$\rho_k(\eta_1,\ldots,\eta_k) = \frac{1}{(2\pi i)^k}\lim_{\varepsilon\to 0^+}\sum_{\{\sigma\},\sigma_j=\pm 1}\left(\prod_{j=1}^{k}\sigma_j\right) G_k\left(\eta_1-i\sigma_1\varepsilon,\ \ldots,\ \eta_k-i\sigma_k\varepsilon\right) . \tag{7}$$

In non-Hermitian theories eigenvalues are no more localised along the real (imaginary) axis, and, as a consequence, the correlation functions cannot be computed through the discontinuity of the resolvent along the support of the eigenvalues. The equation below will be used to generate the two-dimensional δ-function in the complex plane instead:

$$\partial_{z^*}\frac{1}{z} = \pi\delta^2_{\mathbb{C}}(z) . \tag{8}$$

Because the average in eq. (2) is now over complex eigenvalues the resolvent is well-defined everywhere in the complex plane, both in-and outside the support of D. From eq. (8) it follows that for the density we have

$$\rho_1(z) = \frac{1}{\pi}\partial_{z^*}G_1(z) . \tag{9}$$

In other words the resolvent is holomorphic outside the support, and non-holomorphic inside. As before the resolvent may be generated by differentiating the expectation value of a ratio of determinants as given in eq. (4). However, this case has an additional problem due to the fact that the microscopic field theory (such as $\epsilon\chi$PT) requires the introduction of additional bosonic and fermionic conjugated quarks. This phenomenon is due to the Hermiticity requirement [37, 24, 17] for bosonic field theories to be written in terms of convergent integrals, and we will discuss that in more detail in sect. 5.1.

As before the above can be generalised to any k-point function, leading to the introduction of k additional bosonic and fermionic quarks (or to k couples of conjugated bosonic and fermionic quarks, if one needs to consider the microscopic theory), with

$$\rho_k(z_1,\ldots,z_k) = \left(\prod_{j=1}^{k}\frac{1}{\pi}\partial_{z_j^*}\right) G_k(z_1,\ldots,z_k) . \tag{10}$$

Let us mention an alternative method to generate all density correlations - apart from non-resolvent related ones: it generates the two-dimensional δ-function using the identity

$$\delta^2_{\mathbb{C}}(z) = \lim_{\kappa\to 0}\frac{k^2}{\pi(z\cdot z^*+k^2)^2} \tag{11}$$

and hence when considering the sum of the delta function over all the eigenvalues λ of a non-Hermitian matrix D we have (see [38]):

$$\sum_{\lambda\in \text{e.v.}} \delta^2_{\mathbb{C}}(z-\lambda) = \lim_{\kappa\to 0}\frac{1}{\pi} Tr\left[\frac{\kappa^2}{((z-D)(z-D)^\dagger+\kappa^2)^2}\right] . \tag{12}$$

3 Zero chemical potential

We start from the simplest case: the Hermitian chRMT corresponding to QCD with zero chemical potential. In principle this theory could be seen as a particular case of the imaginary μ case studied in sect. 4, setting $\mu = 0$. Since computations are easier here, we present this case separately for pedagogic reasons.

The idea of our approach to prove the equivalence of all spectral properties in the two theories is the following: in both theories the resolvent method allows in principle to compute all k-point correlation function. So without taking the discontinuities, or anti-holomorphic derivatives, it is sufficient to prove that, up to an irrelevant constant, the integrals describing these generating functions of the resolvents coincide. From this knowledge follows that all the k-point correlation functions are equal, and, hence all spectral properties agree. In order to actually compute any given correlation function we may thus choose either theory, and within that theory we may even use any other equivalent method that is simpler.

Hermitian chRMT is a very efficient way to describe the spectral properties of QCD without chemical potential [1, 2]. The equivalent of the QCD Dirac operator in chRMT is given by

$$\mathcal{D}_f + m_f \equiv \begin{pmatrix} m_f \mathbf{1}_{N_+} & iA \\ iA^\dagger & m_f \mathbf{1}_{N_-} \end{pmatrix} , \quad f = 1, \ldots, N_f , \tag{13}$$

where A is a complex $N_+ \times N_-$ random matrix. It describes a fixed topological sector of QCD with a topological charge $\nu = N_+ - N_-$, where ν will be kept fixed while considering the $N_+ + N_- \equiv N \to \infty$ limit.

Thanks to the universality of the microscopic limit of chRMT [18, 33, 19] we can choose a Gaussian weight function for the random matrix A. Since we are interested in finding the spectrum and the k-point correlation function using the resolvent method [39, 28, 29, 34], according to what was said in sect. 2 we consider the more general theory with n_f fermionic quarks and n_b bosonic ones and compute it's partition function:

$$Z_{pq} = \left\langle \frac{\prod_f^{n_f} \mathcal{D}et\left[\mathcal{D}_f + m_f\right]}{\prod_b^{n_b} \mathcal{D}et\left[\mathcal{D}_b + m_b\right]} \right\rangle = \int dA\, e^{-\sigma N Tr\left[A^\dagger A\right]} \frac{\prod_f^{n_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iA \\ iA^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix}}{\prod_b^{n_b} \mathcal{D}et \begin{bmatrix} m_b \mathbf{1}_{N_+} & iA \\ iA^\dagger & m_b \mathbf{1}_{N_-} \end{bmatrix}} . \tag{14}$$

Here dA is the flat measure in the independent entries of the complex matrix A. The quantity defined above is usually called partially-quenched (pq) partition function.

In the following it will be required that $Re(m_b) > 0\, \forall\, b$ in order to have convergent integrals generating the resolvents in the upper half-plane. The result for $Re(m_b) < 0$ may be easily recovered from the equation above[1].

At this point we follow the common procedure [12] of writing the determinants in the numerator in terms of Gaussian fermionic integrals, and the ones in the denominator in terms of bosonic ones we introduce two sets of N_+ and N_- complex-supervectors[2] in $(n_b|n_f)$. In this paper we will use the boson-fermion convention for ordering elements in supervectors. We write these vectors in a matrix form $\psi_{g,\alpha}, \phi_{g,\beta}, \psi^*_{g,\alpha}$ and $\phi^*_{g,\beta}$ where Latin indices run over the $(n_b|n_f)$ superflavours and Greek indices run over the N_+ (N_-) eigenvalues,

$$Z_{pq} = \int dA\, d(\psi,\psi^*,\phi,\phi^*)\, e^{-\sigma N Tr\left[A^\dagger A\right]} \text{Exp}\left[- \sum_{g=-n_f}^{n_b} \begin{pmatrix} \psi^*_{g,\alpha} \\ \phi^*_{g,\beta} \end{pmatrix} \begin{pmatrix} m_g \mathbf{1}_{\alpha,\alpha'} & iA_{\alpha,\beta'} \\ iA^\dagger_{\beta,\alpha'} & m_g \mathbf{1}_{\beta,\beta'} \end{pmatrix} \begin{pmatrix} \psi_{g,\alpha'} \\ \phi_{g,\beta'} \end{pmatrix} \right] . \tag{15}$$

[1] In the literature often a diagonal matrix S with elements $s_i = \pm 1$ is introduced to be able to work in the whole complex plane at once. For simplicity we omitted such a notation.

[2] The anticommuting number part of supervectors and its conjugate are independent Grassmann variables.

Here $d(\psi,\psi^*,\phi,\phi^*)$ is a shorthand notation for the product of the flat measures of the independent entries of the supervectors. The integral above depends on the random matrices only in a Gaussian way:

$$\text{Exp}\left[-\sigma N\ A_{\alpha,\beta}A^*_{\alpha,\beta} - iA_{\alpha,\beta}\left(\psi^*_{g\alpha}\phi_{g,\beta}\right) - iA^*_{\alpha,\beta}\left(\phi^*_{g,\beta}\psi_{g\alpha}\right)\right]\ . \tag{16}$$

We can thus perform the Gaussian integration by completing the squares:

$$\begin{aligned} Z_{pq} \ &\propto \ \int d(\psi,\psi^*,\phi,\phi^*)\ \text{Exp}\left[-Str\left[m_g\cdot\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{g,\alpha} + m_g\cdot\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{g,\beta}\right]\right] \\ &\times\ \text{Exp}\left[-\frac{1}{\sigma N}Str\left[\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{h,\beta}\cdot\sum_\alpha \psi_{h,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]\right]\ . \end{aligned} \tag{17}$$

In the following we introduce the matrix notation $M_{gh} = N\cdot\delta_{gh}m_g$, anticipating the correct scaling later in the large-N limit to obtain N-independent quantities in the thermodynamic limit [41]. This expression is a Gaussian integral in any of the two sets of supervectors. We can easily perform one supervector Gaussian integration $(\phi,\phi^\dagger)$ obtaining a superdeterminant as a result:

$$Z_{pq}\propto\int d(\psi,\psi^*)\,e^{-Str\left[m_g\cdot\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]}\ \mathcal{S}det\left[\frac{1}{N}M_{gh}+\frac{1}{\sigma N}\left(\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{h,\alpha}\right)\right]^{-N_-}\ . \tag{18}$$

The equation above depends on the supervectors only trough the sum of external products. We can now apply the superbosonisation theorem (55),

$$\frac{1}{N_+}\sum_{\alpha=1}^{N_+}\psi_{g,\alpha}\otimes\psi^\dagger_{h,\alpha}\to U_{gh}\ \in\ \hat{Gl}(n_b|n_f) \tag{19}$$

where $\hat{Gl}(n_b|n_f)$ is the maximal Riemannian submanifold of the linear group in the $(n_b|n_f)$ superspace [42]; matrix representations of this manifold are possible, their elements are supermatrices with $Gl(n_b)/U(n_b)\oplus U(n_f)$ as a base manifold[3] and $2\,n_b\cdot n_f$ Grassmann variables. The Haar measure on this manifold is called $d\mu_H(a)$ and may be expressed in terms of usual integrations trough (see eq. (69)):

$$\int_{\hat{Gl}(n_b|n_f)}d\mu_H\ =\int_{H_1=H_1^\dagger}dH_1\theta(H_1)\int_{U(n_f)}d\mu_U(H_2)\,\mathcal{D}et\,[H_2]^{n_f}\int d\Theta\,d\Theta^\dagger\,\mathcal{S}det\begin{bmatrix}H_1 & \Theta^\dagger\\ \Theta & H_2\end{bmatrix}^{n_f-n_b}\ . \tag{20}$$

Here, dH_1 is the flat measure on the Hermitian matrices, $\theta(H_1)$ is the product of the step function in the eigenvalues ($\theta(H_1)>0\iff H_1$ is positive definite), $d\mu_U$ is the Haar measure on unitary matrices H_2 and $d\Theta, d\Theta^\dagger$ is the flat Grassmannian integration in the independent entries of the boson-fermion block[4]. Other parametrisations of $\hat{Gl}(n_b|n_f)$ have been provided for some specific values of $(n_b|n_f)$ [29, 43, 30].

The result is ($N_+=\nu+N_-$):

$$\begin{aligned} Z_{pq} \ &\propto \ \int_{\hat{Gl}(n_b|n_f)}d\mu_H(U)\ \ \mathcal{S}det\,[U]^{N_+}\,e^{-Str\left[\frac{N_+}{N}MU\right]}\mathcal{S}det\left[\sigma\frac{M}{N_+}+U\right]^{-N_-} \\ &= \ \int_{\hat{Gl}(n_b|n_f)}d\mu_H(U)\,\mathcal{S}det\,[U]^{\nu}\,e^{-Str\left[\frac{N_+}{N}MU\right]}\mathcal{S}det\left[1+\sigma\frac{1}{N_+}M\cdot U^{-1}\right]^{-N_-}\ . \end{aligned} \tag{21}$$

All the equations above hold for finite N. As a last step we can perform the $N\to\infty$ limit. Here we send the masses m_h to zero while keeping M_{gh} fixed. Therefore the $N\to\infty$ merely leads to an expansion of the superdeterminant, without the need to perform any other approximation:

$$\begin{aligned} \lim_{N\to\infty}Z_{pq} \ &= \ \int_{\hat{Gl}(n_b|n_f)}d\mu_H(U)\,\mathcal{S}det\,[U]^{\nu}\,e^{Str\left[-\frac{1}{2}MU-\sigma M\cdot U^{-1}\right]} \\ &\propto \ \int_{\hat{Gl}(n_b|n_f)}d\mu_H(U)\,\mathcal{S}det\,[U]^{\nu}\,e^{-\sqrt{\frac{\sigma}{2}}Str\left[M\cdot U+M\cdot U^{-1}\right]}\ . \end{aligned} \tag{22}$$

[3] The bodies of the boson-boson block and fermion-fermion block belong to a manifold called base manifold.

[4] The Grassmann variables $\Theta_{\alpha,b}$ and $\Theta^\dagger_{b,\alpha}$ are independent real Grassmann variables.

This equation is equivalent to the partially-quenched partition function e.g. in [29], after matching parameters

$$m_i N\sqrt{\frac{\sigma}{2}} = m_i \frac{\Sigma V}{2} \ . \tag{23}$$

We denote by V the volume in χPT and by Σ the chiral condensate, the first low energy constant in χPT. Applying the resolvent method to both eq. (14) in the limit $N \to \infty$, and eq. (22) while keeping eq. (23) finite, we obtain the claimed $\epsilon\chi$PT-chRMT equivalence at $\mu = 0$ of all k-point correlation functions in the microscopic limit.

4 Imaginary chemical potential

The simplest generalisation of the arguments above is obtained by adding imaginary chemical potentials. This is a Hermitian theory too and, hence, the resolvent method may be applied in the very same way as before. The only difference with the former section lies in a different random matrix Dirac operator, containing one more random matrix. This model introduced in [8] follows the idea of [7] in assuming that the chemical potential term in non-diagonal in matrix space. This apparent complication by adding more random variables in fact makes the model simpler: for two different chemical potentials one can go to an eigenvalue basis and use bi-orthogonal polynomials. Below we don't need to diagonalise the matrices as we explicitly integrate them out. The equivalence between chRMT and $\epsilon\chi$PT we show thus holds for any number of different chemical potentials. However, the integrals and density correlations are so far known explicitly only when having two different chemical potentials for any number of flavours.

In principle one could also follows the idea of Stephanov [9] with only one random matrix and the μ-term proportional to unity. However, for technical reasons the computation is more involved and although we don't show it here it leads to the same result.

The Dirac operator for a quark with mass m_f and imaginary chemical potential μ_f is given by [8]:

$$\mathcal{D}_f + m_f \equiv \begin{pmatrix} m_f \mathbf{1}_{N_+} & iA + i\mu_f B \\ iA^\dagger + i\mu_f B^\dagger & m_f \mathbf{1}_{N_-} \end{pmatrix} \ , \tag{24}$$

and likewise for boson masses m_b and chemical potential μ_b. Here A and B are complex $N_+ \times N_-$ random matrices with Gaussian weights. The issue of universality is more subtle here because the matrices A and B will couple after changing variables. We refer to [8] for a more detailed discussion.

We will prove the equivalence of the partition functions for a generic theory with n_b bosons and n_f fermions. The equivalence of the spectra will then follow, applying the resolvent method. As before we will consider only bosonic masses with positive real parts,

$$Z_{pq} = \left\langle \frac{\prod_f^{n_f} \mathcal{D}et\left[\mathcal{D}_f + m_f\right]}{\prod_b^{n_b} \mathcal{D}et\left[\mathcal{D}_b + m_b\right]} \right\rangle = \int dA\, dB\, e^{-\sigma N Tr[AA^\dagger + BB^\dagger]} \frac{\prod_f^{n_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iA + i\mu_f B \\ iA^\dagger + i\mu_f B^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix}}{\prod_b^{n_b} \mathcal{D}et \begin{bmatrix} m_b \mathbf{1}_{N_+} & iA + i\mu_b B \\ iA^\dagger + i\mu_f B^\dagger & m_b \mathbf{1}_{N_-} \end{bmatrix}} \ . \tag{25}$$

From this point on, most of the steps are equivalent to the ones performed in the previous section: we introduce two sets of N_+ and N_- complex-supervectors in $(n_b|n_f)$ to write the ratio of determinants as a Gaussian integral

$$\begin{aligned} Z_{pq} \quad = \quad & \int dA\ dB\ d(\psi,\psi^*,\phi,\phi^*) \mathrm{Exp}\left[-\sigma N \left(Tr\left[A^\dagger A\right] + Tr\left[B^\dagger B\right]\right)\right] \\ & \times \mathrm{Exp}\left[- \sum_{g=-n_f}^{n_b} \begin{pmatrix} \psi^*_{g,\alpha} \\ \phi^*_{g,\beta} \end{pmatrix} \begin{pmatrix} m_g \mathbf{1}_{\alpha,\alpha'} & (iA + i\mu_g B)_{\alpha,\beta'} \\ \left(iA^\dagger + i\mu_g B^\dagger\right)_{\beta,\alpha'} & m_g \mathbf{1}_{\beta,\beta'} \end{pmatrix} \begin{pmatrix} \psi_{g,\alpha'} \\ \phi_{g,\beta'} \end{pmatrix} \right] . \end{aligned} \tag{26}$$

We reduce all the dependence on random matrices in terms of Gaussian functions:

$$\begin{aligned} & \mathrm{Exp}\left[-\sigma N\ A_{\alpha,\beta}A^*_{\alpha,\beta} - iA_{\alpha,\beta}\left(\psi^*_{g\alpha}\phi_{g,\beta}\right) - iA^*_{\alpha,\beta}\left(\phi^*_{g,\beta}\psi_{g\alpha}\right)\right] \\ & \times \mathrm{Exp}\left[-\sigma N\ B_{\alpha,\beta}B^*_{\alpha,\beta} - i\mu_g B_{\alpha,\beta}\left(\psi^*_{g\alpha}\phi_{g,\beta}\right) - i\mu_g B^*_{\alpha,\beta}\left(\phi^*_{g,\beta}\psi_{g\alpha}\right)\right] \ . \end{aligned} \tag{27}$$

Next we can perform the Gaussian integration completing the squares:

$$\begin{aligned} Z_{pq} \quad \propto \quad & \int d(\psi,\psi^*,\phi,\phi^*)\ \mathrm{Exp}\left[-Str\left[m_g\cdot\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{g,\alpha} + m_g\cdot\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{g,\beta}\right]\right] \\ & \times\ \mathrm{Exp}\left[-\frac{1}{\sigma N}Str\left[\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{h,\beta}\cdot\sum_\alpha \psi_{h,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]\right] \\ & \times\ \mathrm{Exp}\left[-\frac{1}{\sigma N}Str\left[\mu_g\cdot\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{h,\beta}\cdot\mu_h\cdot\sum_\alpha \psi_{h,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]\right]. \end{aligned} \tag{28}$$

Here we introduce again the matrix notation $M_{gh}\equiv N\cdot\delta_{gh}m_g$, as well as $B^{(\mu)}_{gh}=\sqrt{\frac{N}{2}}\delta_{gh}\mu_g$, anticipating also the proper scaling of μ with N below. This expression is a Gaussian integral in any of the two set of supervectors. As before we explicitly integrate one set of supervectors and express the remaining external product in terms of an integration over $\hat{Gl}(n_b|n_f)$. The result is:

$$Z_{pq}\propto\int d(\psi,\psi^*)e^{-Str\left[\frac{1}{N}M\cdot\sum_\alpha\psi_\alpha\otimes\psi^\dagger_\alpha\right]}\mathcal{S}det\left[\frac{1}{N}M+\frac{1}{\sigma N}\sum_\alpha\psi_\alpha\otimes\psi^\dagger_\alpha+\frac{2}{\sigma N^2}B^{(\mu)}\cdot\sum_\alpha\psi_\alpha\otimes\psi^\dagger_\alpha\cdot B^{(\mu)}\right]^{-N_-}. \tag{29}$$

We can use again the theorem of section 6 obtaining:

$$Z_{pq}\propto\int_{\hat{Gl}(n_b|n_f)}d\mu_H(U)\,\mathcal{S}det\,[U]^\nu\,e^{-Str\left[\frac{N_+}{N}MU\right]}\mathcal{S}det\left[1+\sigma\frac{1}{N_+}M\cdot U^{-1}+\frac{2}{N}B^{(\mu)}UB^{(\mu)}U^{-1}\right]^{-N_-}. \tag{30}$$

This result is again exact for any finite N. If we now take the large-N limit while keeping M and $B^{(\mu)}$ fixed[5] we obtain finally:

$$\lim_{N\to\infty}Z_{pq}=\int_{\hat{Gl}(n_b|n_f)}\mu_H(U)\,\mathcal{S}det\,[U]^\nu\,e^{-\sqrt{\frac{\sigma}{2}}Str\left[M\cdot U+M\cdot U^{-1}\right]-Str\left[B^{(\mu)}UB^{(\mu)}U^{-1}\right]}$$

where our only approximation has been the expansion of the superdeterminant.

This expression is equivalent to the $\epsilon\chi$PT effective partition [31], where in order to match we use the following relations [26, 3]

$$\begin{aligned} m_iN\sqrt{\frac{\sigma}{2}} =& m_i\frac{\Sigma V}{2}\ , \\ \mu_i\sqrt{N} =& \mu_{D_i}F_\pi\sqrt{V}\ . \end{aligned} \tag{31}$$

Here μ_D is a dimensional constant, instead of the μ used above that is dimensionless. The additional parameter F_π is the pion decay constant, the second low energy constant in the leading order chiral Lagrangian. Thus we have established the $\epsilon\chi$PT-chRMT equivalence for all k-point correlation functions with imaginary chemical potential.

5 Real chemical potential

Before starting the proof for real chemical potential it's better to spend a few words on the definition of theories with bosonic quarks. It may help understanding both the result and the proof.

5.1 Bosonic partition function with $\mu\neq 0$

The sign problem makes it impossible to write the inverse Dirac operator as a convergent bosonic integral: Hermitisation is needed [45, 46] and a new particle content of the theory comes out [37, 24, 17].

[5] In the theory with real chemical potential in the next section this limit is called weak non-Hermiticity limit [44]. While we inherit the same scaling here our operators are always Hermitian.

Let's consider as an example the simplest case $n_b = 1$, $n_f = 0$, following [17]. The partition function is given by:

$$\left\langle \frac{1}{\mathcal{D}et\left[D + \mu\gamma_0 + m\right]} \right\rangle \ . \tag{32}$$

In the microscopic theory (QCD) the inverse power of the determinant should be drawn from the integration of bosonic quarks. This integration is well defined only if the integral is convergent, that is, if the Hermitian part of the operator is positive (or negative) defined; this requirement is fulfilled when Dirac operator is given by a non-Hermitian (kinetic) part plus a mass term (proportional to the identity operator), but this is not the case for a Dirac operator suffering from the sign problem.

The way to bypass this problem is to add an additional bosonic conjugate quark and a fermionic one with mass $-m^*$ and chemical potential $-\mu$, respectively,

$$\begin{aligned} \left\langle \frac{1}{\mathcal{D}et\left[D + \mu\gamma_0 + m\right]} \right\rangle &= \left\langle \frac{\mathcal{D}et\left[D - \mu\gamma_0 - m^*\right]}{\mathcal{D}et\left[D + \mu\gamma_0 + m\right]\mathcal{D}et\left[D - \mu\gamma_0 - m^*\right]} \right\rangle \\ &\propto \left\langle \frac{\mathcal{D}et\left[D - \mu\gamma_0 - m^*\right]}{\mathcal{D}et\begin{bmatrix} 0 & D + \mu\gamma_0 + m \\ D - \mu\gamma_0 - m^* & 0 \end{bmatrix}} \right\rangle \ . \end{aligned} \tag{33}$$

The operator in the denominator is now anti-Hermitian and hence has only pure imaginary eigenvalues. If we consider the matrix

$$\begin{pmatrix} \varepsilon & D + \mu\gamma_0 + m \\ D - \mu\gamma_0 - m^* & \varepsilon \end{pmatrix} \tag{34}$$

(sending $\varepsilon \to 0$ at the end) we have an operator fulfilling the positivity requirements to have a properly defined theory. This process is usually called Hermitisation [45, 46], and although it could seem as just a mathematical trick this is not the case: the possibility of writing the inverse determinants as convergent integrals is a feature of the underlying theory.

The generalisation to any number n_b of bosonic uncoupled quarks is straightforward: $(N_b|0) \to (N_b{+}N_b^*|N_b^*)$. Despite this regularisation is a feature of the microscopic underlying theory, it may happen that we have properly defined integrals in an effective theory, and that no such Hermitisation is needed there. This is the case of non-Hermitian Random Matrix Theory with uncoupled bosons [47, 16, 17].

5.2 Equivalence for real chemical potential

As already hinted in sect. 2 and explained in sect. 5.1 the resolvent method may require the Hermitisation of the boson-boson part of Dirac operator [37]:

$$\begin{aligned} Z_{pq} &= \left\langle \frac{\prod_i^{n_v} \mathcal{D}et\left[D_i + z_i\right]}{\prod_i^{n_v} \mathcal{D}et\left[D_i + \tilde{z}_i\right]} \prod_f^{N_f} \mathcal{D}et\left[D_f + m_f\right] \right\rangle \\ &= \left\langle \frac{\prod_i^{n_v} \mathcal{D}et\left[D_i + z_i\right]\mathcal{D}et\left[D_i + \tilde{z}_i\right]^*}{\prod_i^{n_v} \mathcal{D}et\left[D_i + \tilde{z}_i\right]\mathcal{D}et\left[D_i + \tilde{z}_i\right]^*} \prod_f^{N_f} \mathcal{D}et\left[D_f + m_f\right] \right\rangle \\ &= \lim_{\varepsilon\to 0} \left\langle \frac{\prod_i^{n_v} \mathcal{D}et\left[D_i + z_i\right]\mathcal{D}et\left[D_i + \tilde{z}_i\right]^*}{\prod_i^{n_v} \mathcal{D}et\left[(D_i + \tilde{z}_i)(D_i^\dagger + \tilde{z}_i^*) + \varepsilon^2\right]} \prod_f^{N_f} \mathcal{D}et\left[D_f + m_f\right] \right\rangle . \end{aligned} \tag{35}$$

The first line is the proper definition of the resolvent, the second that is obtained from the first by a trivial step is the Hermitised version, the last line is the regularised integral. Hermitisation is necessary both for the computation below and for the underlying microscopic theory, but it's not always necessary for making the computation in RMT [47, 17].

In the second line the additional valence quarks are in conjugate pairs $(n_b{+}n_b^*|n_v{+}n_v^*{+}N_f)$ (the replacement $\tilde{z}_i \to z_i$ in the numerator is irrelevant). It will be sufficient to consider the conjugate fermionic quarks as independent quarks with mass $-m^*$ and chemical potential $-\mu$; concerning the bosonic conjugate quark, as already seen in sect. 5.1, there's a deep reason that forces us to treat them simultaneously.

We mention in passing that a purely bosonic theory with bosons in conjugated pairs $(n_b + n_b^*, 0)$ occurs when applying the replica trick or Toda lattice equation [9, 16, 24, 17].

In the following we will prove the equivalence of the partition functions of $\epsilon\chi$PT and chRMT at non zero chemical potential with n_f fermionic quarks with given masses and baryonic potential (m_f, μ_f), and n_b couples of conjugated bosonic quarks, with parameters (m_b, μ_b) and $(-m_b^*, -\mu_b)$. The partially quenched theory with N_f physical quarks, n_v couples of conjugated fermionic quarks and n_v couples of bosonic quarks will result as a special case. The equivalence of the spectra follows applying the resolvent method in both theories.

The case of uncoupled bosons, although not necessary four our purposes here, may be studied anyway in this framework by adding the right number of conjugated bosonic and fermionic quarks as already seen in sect. 5.1.

We will again use the the two-matrix model as was introduced by Osborn [7], but the same calculation can be done also for the Stephanov model. The equivalent of the QCD Dirac operator for a quark with mass m_f and chemical potential μ_f is given by:

$$\mathcal{D}_f + m_f \equiv \begin{pmatrix} m_f \mathbf{1}_{N_+} & iA + \mu_f B \\ iA^\dagger + \mu_f B^\dagger & m_f \mathbf{1}_{N_-} \end{pmatrix} \ . \tag{36}$$

where A and B are complex $N_+ \times N_-$ random matrices.

The partition function equivalent to partially quenched QCD is:

$$Z_{pq} = \int dA\ dB\ w(A)\ w(B)\ \frac{\prod_f^{n_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iA + \mu_f B \\ iA^\dagger + \mu_f B^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix}}{\prod_b^{n_b} \mathcal{D}et \begin{bmatrix} m_b \mathbf{1}_{N_+} & iA + \mu_b B \\ iA^\dagger + \mu_b B^\dagger & m_b \mathbf{1}_{N_-} \end{bmatrix} \mathcal{D}et \begin{bmatrix} -m_b^* \mathbf{1}_{N_+} & iA - \mu_b B \\ iA^\dagger - \mu_b B^\dagger & -m_b^* \mathbf{1}_{N_-} \end{bmatrix}} \ , \tag{37}$$

with a Gaussian weight function

$$w(X) = \mathrm{Exp}\left[-\sigma N Tr\left[X^\dagger X\right]\right] \ . \tag{38}$$

In order to write the inverse determinants as bosonic Gaussian integrals we perform the same anti-Hermitisation as in eq. (34), obtaining an anti-Hermitian matrix apart from an ε times the identity. The regularised denominator is:

$$\begin{aligned} &\mathcal{D}et \left[\begin{pmatrix} m_b \mathbf{1}_{N_+} & iA + \mu_b B \\ iA^\dagger + \mu_b B^\dagger & m_b \mathbf{1}_{N_-} \end{pmatrix} \cdot \begin{pmatrix} -m_b^* \mathbf{1}_{N_+} & iA - \mu_b B \\ iA^\dagger - \mu_b B^\dagger & -m_b^* \mathbf{1}_{N_-} \end{pmatrix} + \varepsilon^2 \mathbf{1}_{N_+ + N_-} \right] = \\ &= \mathcal{D}et \left[\begin{pmatrix} \varepsilon \mathbf{1}_{N_+} & 0 & m_b \mathbf{1}_{N_+} & iA + \mu_b B \\ 0 & \varepsilon \mathbf{1}_{N_-} & iA^\dagger + \mu_b B^\dagger & m_b \mathbf{1}_{N_-} \\ -m_b^* \mathbf{1}_{N_+} & iA - \mu_b B & \varepsilon \mathbf{1}_{N_+} & 0 \\ iA^\dagger - \mu_b B^\dagger & -m_b^* \mathbf{1}_{N_-} & 0 & \varepsilon \mathbf{1}_{N_-} \end{pmatrix} \right] \\ &= \mathcal{D}et \left[\begin{pmatrix} \epsilon + \frac{m - m^*}{2} & iA & \frac{m + m^*}{2} & \mu B \\ iA^\dagger & \epsilon + \frac{m - m^*}{2} & \mu B^\dagger & \frac{m + m^*}{2} \\ -\frac{m + m^*}{2} & -\mu B & \epsilon - \frac{m - m^*}{2} & -iA \\ -\mu B^\dagger & -\frac{m + m^*}{2} & -iA^\dagger & \epsilon - \frac{m - m^*}{2} \end{pmatrix} \right] \end{aligned} \tag{39}$$

We introduce two sets of N_+ and N_- complex supervectors in $(2n_b|n_f)$ to write the ratio of determinants as a Gaussian integral

$$\begin{aligned} Z_{pq} \ = \ & \int dA\ dB\ d(\psi, \psi^*, \phi, \phi^*) \mathrm{Exp}\left[-\sigma N \left(Tr\left[A^\dagger A\right] + Tr\left[B^\dagger B\right]\right)\right] \\ & \times \mathrm{Exp}\left[- \sum_f^{n_f} \begin{pmatrix} \psi_{f,\alpha}^* \\ \phi_{f,\beta}^* \end{pmatrix} \begin{pmatrix} m_f \mathbf{1}_{\alpha,\alpha'} & (iA + \mu_f B)_{\alpha,\beta'} \\ \left(iA^\dagger + \mu_f B^\dagger\right)_{\beta,\alpha'} & m_f \mathbf{1}_{\beta,\beta'} \end{pmatrix} \begin{pmatrix} \psi_{f,\alpha'} \\ \phi_{f,\beta'} \end{pmatrix} \right] \\ & \times \mathrm{Exp}\left[- \sum_b^{n_b} \begin{pmatrix} \psi_{2b-1,\alpha}^* \\ \phi_{2b-1,\beta}^* \\ \psi_{2b,\alpha}^* \\ \phi_{2b,\beta}^* \end{pmatrix} \begin{pmatrix} \epsilon + \frac{m_b - m_b^*}{2} & iA_{\alpha\beta'} & \frac{m_b + m_b^*}{2} & \mu_b B_{\alpha\beta'} \\ iA_{\beta\alpha'}^\dagger & \epsilon + \frac{m_b - m_b^*}{2} & \mu_b B_{\beta\alpha'}^\dagger & \frac{m_b + m_b^*}{2} \\ -\frac{m_b + m_b^*}{2} & -\mu_b B_{\alpha\beta'} & \epsilon - \frac{m_b - m_b^*}{2} & -iA_{\alpha\beta'} \\ -\mu_b^* B_{\beta\alpha'}^\dagger & -\frac{m_b + m_b^*}{2} & -iA_{\beta\alpha'}^\dagger & \epsilon - \frac{m_b - m_b^*}{2} \end{pmatrix} \begin{pmatrix} \psi_{2b-1,\alpha'} \\ \phi_{2b-1,\beta'} \\ \psi_{2b,\alpha'} \\ \phi_{2b,\beta'} \end{pmatrix} \right] . \end{aligned} \tag{40}$$

As before we have reduced all the dependence on random matrices in terms of Gaussian functions:

$$\begin{aligned}
&\mathrm{Exp}\left[-\sigma N\ A_{\alpha,\beta}A^*_{\alpha,\beta} - iA_{\alpha,\beta}\left(\sum_f \psi^*_{f,\alpha}\phi_{f,\beta} + \sum_b \left(\psi^*_{2b-1,\alpha}\phi_{2b-1,\beta} - \psi^*_{2b,\alpha}\phi_{2b,\beta}\right)\right)\right.\\
&\left.-iA^*_{\alpha,\beta}\left(\sum_f \phi^*_{f,\beta}\psi_{f,\alpha} + \sum_b \left(\phi^*_{2b-1,\beta}\psi_{2b-1,\alpha} - \phi^*_{2b,\beta}\psi_{2b,\alpha}\right)\right)\right] \qquad (41)\\
&\times\ \mathrm{Exp}\left[-\sigma N\ B_{\alpha,\beta}B^*_{\alpha,\beta} - B_{\alpha,\beta}\left(\sum_f \mu_f\psi^*_{f,\alpha}\phi_{f,\beta} + \sum_b \mu_b\left(\psi^*_{2b-1,\alpha}\phi_{2b,\beta} - \psi^*_{2b,\alpha}\phi_{2b-1,\beta}\right)\right)\right.\\
&\left.-B^*_{\alpha,\beta}\left(\sum_f \mu_f\phi^*_{f,\beta}\psi_{f,\alpha} + \sum_b \mu_b\left(-\phi^*_{2b,\beta}\psi_{2b-1,\alpha} + \phi^*_{2b-1,\beta}\psi_{2b,\alpha}\right)\right)\right]\ ,
\end{aligned}$$

and we can perform the Gaussian integration completing the squares:

$$\begin{aligned}
&\mathrm{Exp}\left[-\frac{1}{\sigma N}\sum_{\alpha,\beta}\psi^*_{g,\alpha}\Gamma^A_{gl}\phi_{l,\beta}\phi^*_{m,\beta}\Gamma^A_{m,n}\psi_{n,\alpha} + \frac{2}{\sigma N^2}\sum_{\alpha,\beta}\psi^*_{g,\alpha}\Gamma^B_{gl}\phi_{l,\beta}\phi^*_{m,\beta}\Gamma^B_{m,n}\psi_{n,\alpha}\right] = \qquad (42)\\
=\ &\mathrm{Exp}\left[-\frac{1}{\sigma N}Str\left[\Gamma^A\sum_\alpha\left(\phi_\alpha\otimes\phi^\dagger_\alpha\right)\Gamma^A\sum_\beta\left(\psi_\beta\otimes\psi^\dagger_\beta\right) - \frac{2}{N}\Gamma^B\sum_\alpha\left(\phi_\alpha\otimes\phi^\dagger_\alpha\right)\Gamma^B\sum_\beta\left(\psi_\beta\otimes\psi^\dagger_\beta\right)\right]\right]\ .
\end{aligned}$$

Here we have introduced the $(2n_b|n_f)\times(2n_b|n_f)$ supermatrices

$$\Gamma^A = \begin{pmatrix} \left.\begin{matrix}1 & 0\\ 0 & -1\end{matrix}\right\}\times n_b & 0\\ 0 & \mathbf{1}_{n_f}\end{pmatrix}\ ,\quad \Gamma^B = \sqrt{\frac{N}{2}}\begin{pmatrix} \left.\begin{matrix}0 & \mu_b\\ -\mu_b & 0\end{matrix}\right\}\times n_b & 0\\ 0 & \left.\mu_f\right\}\times n_f\end{pmatrix}\ , \qquad (43)$$

and the mass matrix:

$$M = N\begin{pmatrix} \left.\begin{matrix}\varepsilon + \frac{m_b - m^*_b}{2} & \frac{m_b + m^*_b}{2}\\ -\frac{m_b + m^*_b}{2} & \varepsilon - \frac{m_b - m^*_b}{2}\end{matrix}\right\}\times n_b & 0\\ 0 & \left.m_f\right\}\times n_f\end{pmatrix}\ , \qquad (44)$$

anticipating their N-dependence below. We can rewrite the partition function (40):

$$\begin{aligned}
Z_{pq} \ &=\ \int d\left(\psi,\psi^*\phi,\phi^*\right)\mathrm{Exp}\left[-Str\left[\frac{1}{N}M\cdot\sum_\beta\psi_\beta\otimes\psi^\dagger_\beta\right]\right] \qquad (45)\\
&\times\ \mathrm{Exp}\left[-Str\left[\sum_\alpha\phi_\alpha\otimes\phi^\dagger_\alpha\cdot\left(\frac{1}{N}M + \frac{1}{\sigma N}\Gamma^A\sum_\beta\psi_\beta\otimes\psi^\dagger_\beta\Gamma^A - \frac{2}{\sigma N^2}\Gamma^B\sum_\beta\psi_\beta\otimes\psi^\dagger_\beta\Gamma^B\right)\right]\right].
\end{aligned}$$

From this point on the procedure is the same as before: we integrate explicitely the sets of supervectors ϕ,ϕ^* and use the superbosonisation theorem

$$\begin{aligned}
Z_{pq} \ &=\ \int d\left(\psi,\psi^*\right)\mathrm{Exp}\left[-Str\left[\frac{1}{N}M\cdot\sum_\beta\psi_\beta\otimes\psi^\dagger_\beta\right]\right] \qquad (46)\\
&\times\ \mathcal{S}det\left[\frac{1}{N}M + \frac{1}{\sigma N}\Gamma^A\sum_\beta\psi_\beta\otimes\psi^\dagger_\beta\Gamma^A - \frac{2}{\sigma N^2}\Gamma^B\sum_\beta\psi_\beta\otimes\psi^\dagger_\beta\Gamma^B\right]^{-N_-}\\
&\propto\ \int_{\hat{G}l(2n_b|n_f)} d\mu_H(U)\mathcal{S}det\left[U\right]^{N_+}\ e^{-Str\left[\frac{N_+}{N}M\cdot U\right]}\mathcal{S}det\left[\frac{1}{N}M + \frac{N_+}{\sigma N}\Gamma^A\cdot U\cdot\Gamma^A - \frac{2N_+}{\sigma N^2}\Gamma^B\cdot U\cdot\Gamma^B\right]^{-N_-}.
\end{aligned}$$

This result is valid for finite-N. Once performing the $N \to \infty$ weak non-Hermiticity limit [44], keeping M fixed as well as Γ^B, the following result is obtained

$$\lim_{N\to\infty} Z_{pq} = \int_{\hat{Gl}(2n_b|n_f)} d\mu_H(U) \mathcal{S}det\,[U]^\nu \,\mathrm{Exp}\left[Str\left[-\sqrt{\frac{\sigma}{2}} M\left(U + \Gamma^A \cdot U^{-1} \cdot \Gamma^A\right) + \Gamma^A \Gamma^B U \Gamma^B \Gamma^A U^{-1}\right]\right]. \tag{47}$$

Rotating all the matrices under the transformation $X \to T \cdot X \cdot T^\dagger$ with

$$T = \begin{pmatrix} \left.\begin{matrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{matrix}\right\} \times n_b & 0 \\ 0 & \mathbf{1}_{n_f} \end{pmatrix}, \tag{48}$$

we write the result in generalising the notation already used in literature [24, 17]:

$$\lim_{N\to\infty} Z_{pq} = \int_{\hat{Gl}(2n_b|n_f)} d\mu_H(U) \mathcal{S}det\,[U]^\nu \,\mathrm{Exp}\left[Str\left[-\sqrt{\frac{\sigma}{2}} \hat{M}\left(U + I \cdot U^{-1} \cdot I\right) + B_+^{(\mu)} U B_-^{(\mu)} U^{-1}\right]\right], \tag{49}$$

where

$$\begin{aligned}
\hat{M} &\equiv T^\dagger \cdot \tilde{M} \cdot T = \begin{pmatrix} \left.\begin{matrix} \varepsilon N & -i m_b N \\ -i m_b^* N & \varepsilon N \end{matrix}\right\} \times n_b & 0 \\ 0 & \left. m_f N \right\} \times n_f \end{pmatrix} \\
B_+^{(\mu)} &\equiv -T^\dagger \cdot \Gamma^A \cdot \Gamma^B \cdot T = \begin{pmatrix} \left.\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}\right\} \times n_b & 0 \\ 0 & \mathbf{1}_{n_f} \end{pmatrix} \\
B_-^{(\mu)} &\equiv -T^\dagger \cdot \Gamma^B \cdot \Gamma^A \cdot T = \begin{pmatrix} \left.\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix}\right\} \times n_b & 0 \\ 0 & \mathbf{1}_{n_f} \end{pmatrix} \\
I &\equiv -T^\dagger \cdot \Gamma^A \cdot T = \begin{pmatrix} \left.\begin{matrix} 0 & -i \\ i & 0 \end{matrix}\right\} \times n_b & 0 \\ 0 & \mathbf{1}_{n_f} \end{pmatrix}.
\end{aligned} \tag{50}$$

As before we can recover the usual $\epsilon\chi$PT expressed in term of Σ, F_π and chemical potential using eq. (31), thus proving the $\epsilon\chi$PT-chRMT equivalence for real $\mu \neq 0$. The expression above is a generalisation of the formulas of $\epsilon\chi$PT with one pair of bosons [24, 17], and of $\epsilon\chi$PT with N_f fermions $\epsilon\chi$PT [35]. The existence of two different matrices $B^{(\mu)}$ is due to the fact that the covariant derivative has a different behaviour on bosonic and fermionic quarks in $\epsilon\chi$PT [17].

6 Superbosonisation theorem

The aim of this section is to find a way to express integrals of functions of external products of supervectors in terms of integrals over a smaller space. In formulas we are going to study

$$\int d(\psi,\psi^*)\; f\left(\sum_k \psi_k \otimes \psi_k^\dagger\right), \tag{51}$$

where $\psi_k, \psi_k^\dagger$ are complex vectors of n_b bosons and n_f fermions, $d(\psi,\psi^*)$ is a shorthand notation for $\prod_k^N d\psi_k\, d\psi_k^*$ and f is a function defined over (n_b, n_f) supermatrices.

The main idea is to embed $\sum_k \psi_k \otimes \psi_k^\dagger$ in some manifold where we can define a δ-function and where computations are feasible. At a merely symbolic level we have:

$$\int d(\psi,\psi^*)\; f\left(\sum_k \psi_k \otimes \psi_k^\dagger\right) = \int_{\mathcal{M}} da\, f(a) \int d(\psi,\psi^*)\; \delta_{\mathcal{M}}\left(a - \sum_k \psi_k \otimes \psi_k^\dagger\right). \tag{52}$$

In order to give a meaning to the equation above we have to specify which is the manifold $\mathcal{M}$ we use and which is the measure we use on it (the δ-function depends on it). The use of a δ-function requires f to be continuous with respect to the metric chosen.

A good choice is to take as $\mathcal{M}$ in eq. (52) the manifold of super-Hermitian matrices. One could ask whether or not a δ-function may be used dealing with matrices like external product of supervectors [11]; in appendix A we show that such a δ may be properly defined. The δ-function may be written using an extension of the usual Fourier-transform representation of δ-function on Hermitian matrices (see appendix A).

A crucial ingredient in this proof will be the possibility to flip the order of integration. Changing the order between commuting and anti-commuting variables is not a problem, but problems can arise when considering the case of two commuting variables. In order to see where these problems come from, and how to avoid them, let's consider a simple example, the one of a single commuting number vector of length 1:

$$\begin{aligned}\int_{\mathbb{C}} d^2z\, f(z\cdot z^*) &= \int_{\mathbb{C}} d^2z \int_{\mathbb{R}} dx\, f(x)\,\delta(x-zz^*) \\ &= \int_{\mathbb{R}} dx\, f(x) \int_{\mathbb{C}} d^2z\,\delta(x-zz^*) \\ &= \int_{\mathbb{R}} dx\, f(x) \int_{\mathbb{C}} d^2z \int_{\mathbb{R}} dy\, e^{iy(x-zz^*)}\ .\end{aligned} \tag{53}$$

The first change of variable is always allowed when considering converging integrals. At this point we need to flip the order of integration of y and z, z^*, and this is an illicit step since the integrals are not converging. This problem may be avoided considering a real quantity $\eta > 0$

$$\begin{aligned}\int_{\mathbb{C}} d^2z\, f(zz^*) &= \int_{\mathbb{C}} d^2z\, f(zz^*) e^{(\eta-\eta)zz^*} \\ &= \int_{\mathbb{R}} dx\, f(x) e^{\eta x} \int_{\mathbb{C}} d^2z \int_{\mathbb{R}} \frac{dy}{2\pi}\, e^{iy(x-zz^*)} e^{-\eta zz^*} \\ &= \int_{\mathbb{R}} dx\, f(x) e^{\eta x} \int_{\mathbb{R}} \frac{dy}{2\pi}\, e^{iyx} \int_{\mathbb{C}} d^2z e^{-(\eta+iy)zz^*} \\ &= \int_{\mathbb{R}} dx\, f(x) e^{\eta x} \int_{\mathbb{R}} \frac{dy}{2\pi}\, e^{iyx} \frac{(-i)\pi}{y-i\eta} \\ &= \int_{\mathbb{R}} dx\, f(x) e^{\eta x}\, e^{-\eta x}\pi\,\theta(x) = \pi \int_0^\infty dx\, f(x)\ .\end{aligned} \tag{54}$$

and all the steps are mathematically rigorous any time we can apply a δ-distribution as a functional to the function $f(x)e^{\eta x}$. The symbol $\theta(x)$ indicates the step function.

This easy example is conceptually not too different from the proof of the following theorem.

Theorem 1 (Superbosonisation) *Let f be a function defined on the $(n_b|n_f)$ supermatrices, then the following identity holds*

$$\int \prod_k^N d\psi_k d\psi_k^*\, f\left(\sum_k \psi_k \otimes \psi_k^\dagger\right) \propto \int_{\hat{G}l(n_b|n_f)} d\mu_H(U)\ \mathcal{S}det\,[U]^N\ \ f(U) \tag{55}$$

whenever the integral on the l.h.s. is well defined. The first integration is performed over N complex supervectors in $(n_b|n_f)$ and $d\mu_H(U)$ denotes the Haar measure over $\hat{G}l(n_b|n_f)$.

The idea of expressing integrals of a function of an external product $\sum_k \psi_k \otimes \psi_k^\dagger$ in terms of an integral of the same function over a simpler space has been widely used in physics, for external product of Grassmannian vectors [48] (the term "bosonisation" comes from this kind of application), commuting number vectors [27] and supervectors [49, 50, 51, 52]. This (super)bosonisation may be seen as application of the Riesz Representation Theorem[6]. However, up to our knowledge, a graded version of this theorem is lacking.

[6]This theorem ensures that a bounded linear functional of functions over a locally compact Hausdorff space may be computed like an integration of that function over that space using a proper measure [53].

By coincidence the superbosonisation theorem has been independently developed at the very same time[7] of an analogous theorem on superbosonisation by Littelmann, Sommers and Zirnbauer [10, 11]. Anyway the proof they give is different from ours, theirs is an algebraic proof, ours uses analysis instruments. The main difference is that we base our proof on the existence of a δ-function fulfilling eq. (52), and that collaboration develops a powerful apparatus in order not to use such an equation [10]. For this reason we have chosen to show in appendix A all the details concerning the mathematical rigorousness of our definition.

Proof: As already hinted above we write the l.h.s. of eq. (55) introducing an additional integration over super-Hermitian matrices:

$$\begin{aligned}\int d(\psi,\psi^*)\, f\left(\sum_k \psi_k\otimes\psi_k^\dagger\right) &= \int_{H=H^\dagger} dH\, f(H)\, e^{\eta Str[H]} \int d(\psi,\psi^*)\, \delta\left(\sum_k \psi_k\otimes\psi_k^\dagger - H\right) e^{-\eta\sum_k \psi_k^\dagger\cdot\psi_k} \\ &\propto \int_{H=H^\dagger} dH\, f(H)\, e^{\eta Str[H]} \int d(\psi,\psi^*)\, e^{-\eta\sum_k \psi_k^\dagger\cdot\psi_k} \int_{F=F^\dagger} dF\, e^{iF\left(H-\sum_k \psi_k\otimes\psi_k^\dagger\right)} \\ &= \int_{H=H^\dagger} dH\, f(H)\, e^{\eta Str[H]} \int_{F=F^\dagger} dF\, e^{iFH}\, \mathcal{S}det\left[\eta+iF\right]^{-N} \end{aligned} \tag{56}$$

where $d(\psi,\psi^*)\equiv\prod_k^N d\psi_k d\psi_k^*$ and $d\psi_k d\psi_k^* = \prod_b^{n_b} d^2_{\mathbb{C}}\psi_{k,b}\prod_f^{n_f} d\psi_{k,f}d\psi_{k,f}^*$. We denote by $\psi^\dagger=\psi^{*\,T}$, and $\psi\otimes\psi^\dagger$ is the external product and $\psi^\dagger\cdot\psi$ the scalar product in $(n_b|n_f)$. The measure dH is the flat measure in the independent entries of the super-Hermitian matrix.

From now on let's focus on the second integral. This quantity reminds of a quantity already computed in [52], unfortunately we cannot use this result because it does not take care about the boundary terms arising from the diagonalisation of super-Hermitian matrices (Efetov-Wegner terms). In the following we will never change the Z-gradings of the integration manifold, and hence no boundary terms will arise [54].

Writing F and H in terms of blocks, $dF = dF_1\, dF_2\, d\Phi\, d\Phi^\dagger$

$$F=\begin{pmatrix} F_1 & \Phi^\dagger \\ \Phi & F_2\end{pmatrix}\ , \qquad H=\begin{pmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2\end{pmatrix}\ , \tag{57}$$

we obtain:

$$\begin{aligned}\mathcal{I} &\equiv \int_{F=F^\dagger} dF\, e^{iFH}\, \mathcal{S}det\left[F-i\eta\right]^{-N} \qquad (58)\\ &= \int dF_1\, dF_2\, d\Phi\, d\Phi^\dagger\ \left(\frac{\mathcal{D}et\left[F_2-i\eta-\Phi(F_1-i\eta)^{-1}\Phi^\dagger\right]}{\mathcal{D}et\left[F_1-i\eta\right]}\right)^N \mathrm{Exp}\left[iTr\left[F_1H_1+\Phi^\dagger\Theta\right]-iTr\left[H_2F_2+\Phi\Theta^\dagger\right]\right]\end{aligned}$$

As a first step we note that if we diagonalise the matrix F_1 and make an analytic continuation in the eigenvalues we see that the only poles in the expression above are the ones where at least one eigenvalue of F_1 is $i\eta$. Considering the integration on the eigenvalues as a complex contour integral and closing the integral in the upper semicircle (or lower one, depending on the signature of the matrix H_1 [8]) we have non vanishing contributions only if every path of the eigenvalues winds the pole in $i\eta$. We can consider this integral as a contour integral around $i\eta$ whenever $\Theta(H_1)>0$, otherwise it's zero. Since there are no poles apart from the ones in $i\eta$ we can fix the contour integral as we prefer. We choose the modulus of the eigenvalues as equal to 1, hence we have that the matrix $F_1-i\eta$ may be analytically continued obtaining a unitary matrix.

$$\begin{aligned}\mathcal{I} &= \Theta(H_1)e^{-\eta Tr[H_1]}\oint_{U(n_b)} dF_1 \int dF_2\, d\Phi\, d\Phi^\dagger\ \left(\frac{\mathcal{D}et\left[F_2-i\eta-\Phi F_1^{-1}\Phi^\dagger\right]}{\mathcal{D}et\left[F_1\right]}\right)^N \\ &\quad\times \mathrm{Exp}\left[i\left[F_1H_1+\Phi^\dagger\Theta\right]-i\left[H_2F_2+\Phi\Theta^\dagger\right]\right]\ . \end{aligned} \tag{59}$$

[7] This work was already presented by one of the authors in "QCD in extreme conditions", Frascati 6-8 August 2007

[8] The matrices H_1 and H_2 are complex number Hermitian matrices.

The notation $\oint_{U(n_b)} dF_1$ stands for an integral over the manifold $U(n_b)$ considering as a measure the analytic continuation of the flat measure of Hermitian matrices. It is related to the Haar measure $d\mu_U$ integration by:

$$\oint_{U(n_b)} dF_1 = \int_{U(n_b)} d\mu_U(F_1)\, \mathcal{D}et\left[F_1\right]^{n_b} \ . \tag{60}$$

The relation between these two integrals is the same as between the two integrals below describing a circutation around zero, with $z = e^{i\theta}$:

$$\int_0^{2\pi} d\theta = \oint_{|z|=1} dz\, \frac{1}{z} \neq \oint_{|z|=1} dz \tag{61}$$

Eq. (60) can be derived diagonalising the matrix $F_1 = U \cdot \mathsf{f} \cdot U^\dagger$ where $\mathsf{f}_j = e^{i\theta_j}$:

$$\begin{aligned}
\oint_{U(n_b)} dF_1 &\equiv \prod_i \oint d\mathsf{f}_i \Delta^2(\{\mathsf{f}\}) \int dU \\
&= \prod_i \oint \frac{d\mathsf{f}_i}{\mathsf{f}_i} \mathcal{D}et\left[\mathsf{f}\right]\ \Delta(\{\mathsf{f}\})\Delta\left(\{\mathsf{f}^{-1}\}\right) \mathcal{D}et\left[\mathsf{f}\right]^{n_b-1} \int dU \\
&= \prod_i \int_0^{2\pi} d\theta_i \left|\Delta(\{e^{i\theta}\})\right|^2 \mathcal{D}et\left[\mathsf{f}\right]^{n_b} \int dU \\
&\equiv \int_{U(n_b)} d\mu_U(F_1)\, \mathcal{D}et\left[F_1\right]^{n_b} \ .
\end{aligned} \tag{62}$$

We consider now that

$$\begin{aligned}
\int_{A=A^\dagger} dA\ \mathcal{D}et\left[A\right]^N e^{-iTr[AH_2]} &= \int_{A=A^\dagger} dA\ \mathcal{D}et\left[A-i\eta\right]^N e^{-iTr[(A-i\eta)H_2]} \\
&= \int_{A=A^\dagger} dA\ \mathcal{D}et\left[A-i\eta-\Phi^\dagger F_1^{-1}\Phi\right]^N e^{-iTr\left[(A-i\eta-\Phi F_1^{-1}\Phi^\dagger)H_2\right]} \ .
\end{aligned} \tag{63}$$

The first equality comes from the analyticity of the integrand in the diagonal entries of the matrix and the second comes from the analogue of the contour invariance in superanalysis [55], applied to the real and imaginary parts of the Hermitian matrix entries. Applying this equivalence we obtain

$$\begin{aligned}
\mathcal{I} &= \Theta(H_1) e^{\eta Tr[H_2]-\eta Tr[H_1]} \oint_{U(n_b)} dF_1 \int dF_2\, d\Phi\, d\Phi^\dagger\ \left(\frac{\mathcal{D}et\left[F_2\right]}{\mathcal{D}et\left[F_1\right]}\right)^N \\
&\quad \times \mathrm{Exp}\left[iTr\left[F_1H_1+\Phi^\dagger\Theta\right] - iTr\left[H_2F_2+\Phi\Theta^\dagger\right] - iTr\left[H_2\Phi F_1^{-1}\Phi^\dagger\right]\right] \ .
\end{aligned} \tag{64}$$

The subsequent step is to perform the $\Phi, \Phi^\dagger$ integration, using

$$\begin{aligned}
Tr\left[\Phi^\dagger\Theta\right] - Tr\left[\Phi\Theta^\dagger + H_2\Phi F_1^{-1}\Phi^\dagger\right] &= -Tr\left[\Theta\Phi^\dagger + \Phi\Theta^\dagger + H_2\Phi F_1^{-1}\Phi^\dagger\right] \\
&= -Tr\left[\Theta F_1 F_1^{-1}\Phi^\dagger + H_2\Phi\Theta^\dagger H_2^{-1} + H_2\Phi F_1^{-1}\Phi^\dagger\right] \\
&= -Tr\left[(\Theta F_1 + H_2\Phi)\cdot\left(F_1^{-1}\Phi^\dagger + \Theta^\dagger H_2^{-1}\right) - \Theta F_1\Theta^\dagger H_2^{-1}\right] .
\end{aligned} \tag{65}$$

We can transform the fermionic variables of integration $\Phi \to \tilde{\Phi} \equiv H_2\Phi + \Theta F_1$ and $\Phi^\dagger \to \tilde{\Phi}^\dagger \equiv F_1^{-1}\Phi^\dagger + \Theta^\dagger H_2^{-1}$ and thus perform the Gaussian integration,

$$\int d\Phi\, d\Phi^\dagger \mathrm{Exp}\left[-iTr\left[H_2\Phi F_1^{-1}\Phi^\dagger\right] + iTr\left[\Theta F_1\Theta^\dagger H_2^{-1}\right]\right] = \frac{\mathcal{D}et\left[H_2\right]^{n_b}}{\mathcal{D}et\left[F_1\right]^{n_f}} e^{iTr\left[\Theta F_1\Theta^\dagger H_2^{-1}\right]} \int d\tilde{\Phi}\, d\tilde{\Phi}^\dagger e^{-iTr\left[\tilde{\Phi}\tilde{\Phi}^\dagger\right]}, \tag{66}$$

where the last integration is just a constant. Getting back to eq. (64) we have:

$$\begin{aligned}
\mathcal{I} &\propto \Theta(H_1)\mathcal{D}et\left[H_2\right]^{n_b}\ e^{-\eta Str[H]} \oint_{U(n_b)} dF_1 \int dF_2\, \frac{\mathcal{D}et\left[F_2\right]^N}{\mathcal{D}et\left[F_1\right]^{N+n_f}} e^{iTr[F_1H_1]-iTr[H_2F_2]+iTr\left[\Theta F_1\Theta^\dagger H_2^{-1}\right]} \qquad (67) \\
&= \Theta(H_1)\mathcal{D}et\left[H_2\right]^{n_b}\ e^{-\eta Str[H]} \int_{F_2=F_2^\dagger} dF_2\, \mathcal{D}et\left[F_2\right]^N e^{-iTr[H_2F_2]} \oint_{U(n_b)} dF_1 \frac{1}{\mathcal{D}et\left[F_1\right]^{N+n_f}} e^{i\left[F_1\left(H_1-\Theta^\dagger H_2^{-1}\Theta\right)\right]}.
\end{aligned}$$

Let us consider again the whole eq. (56). We can now perform the integration in F_1 and F_2, using the integrals in appendix B

$$\begin{aligned}
\int d(\psi,\psi^*)\, f\left(\sum_k \psi_k \otimes \psi_k^\dagger\right) &\propto \int_{H_1=H_1^\dagger} dH_1\, \Theta(H_1) \int_{H_2=H_2^\dagger} dH_2\, \mathcal{D}et\left[H_2\right]^{n_b} \int d\Theta\, d\Theta^\dagger\ f(H) \\
&\quad \times \int_{F_2=F_2^\dagger} dF_2 \mathcal{D}et\left[F_2\right]^N e^{-iTr[H_2F_2]} \oint_{U(n_b)} dF_1 \frac{1}{\mathcal{D}et\left[F_1\right]^{N+n_f}} e^{i\left[F_1\left(H_1-\Theta^\dagger H_2^{-1}\Theta\right)\right]} \\
&\propto \int_{H_1=H_1^\dagger} dH_1\, \Theta(H_1) \int_{H_2=H_2^\dagger} dH_2\, \mathcal{D}et\left[H_2\right]^{n_b} \int d\Theta\, d\Theta^\dagger\ f(H) \\
&\quad \times \int_{F_2=F_2^\dagger} dF_2\, \mathcal{D}et\left[F_2\right]^N e^{-iTr[H_2F_2]} \mathcal{D}et\left[H_1-\Theta^\dagger H_2^{-1}\Theta\right]^{N+n_f-n_b} \\
&\propto \int_{H_1=H_1^\dagger} dH_1 \Theta(H_1) \oint_{U(n_f)} dH_2 \int d\Theta\, d\Theta^\dagger\, \frac{\mathcal{D}et\left[H_1-\Theta^\dagger H_2^{-1}\Theta\right]^{N+n_f-n_b}}{\mathcal{D}et\left[H_2\right]^{N+n_f-n_b}} f(H) \\
&= \int_{H_1=H_1^\dagger} dH_1 \Theta(H_1) \oint_{U(n_f)} dH_2 \int d\Theta\, d\Theta^\dagger\, \mathcal{S}det \begin{bmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2 \end{bmatrix}^{N+n_f-n_b} f(H)\ . \qquad (68)
\end{aligned}$$

The last step missing is to find a relation between the measure used in the equation above and the Haar measure. This relation is:

$$\int_{H_1=H_1^\dagger} dH_1 \Theta(H_1) \oint_{U(n_f)} dH_2 \int d\Theta\, d\Theta^\dagger\, \mathcal{S}det \begin{bmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2 \end{bmatrix}^{n_f-n_b} = \int_{\hat{G}l(n_b|n_f)} \mu_H \begin{pmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2 \end{pmatrix}. \qquad (69)$$

It is proved in the following. The manifolds we are integrating are the same, in fact the $H_1 = H_1^\dagger > 0$ is equivalent to the boson-boson [40] base manifold of $\hat{Gl}(n_b|n_f)$ that is $Gl(n_b)/U(n_b)$ [42, 29]. The analytic continuation of the eigenvalues of $H_2 = H_2^\dagger$ making a circutation around zero is equivalent to the fermion-fermion part $U(n_f)$ [42].

We have now to consider the integration measures in eq. (69). The point is that the measure on the l.h.s. is the Haar measure of super-Hermitian matrices (a group closed under addition) computed on the analytically continued manifold. It is induced by the flat metric:

$$Str\left[dU \cdot dU\right]\ , \qquad (70)$$

but the Haar measure on the r.h.s. is the analytical continuation of the Haar measure on super-Unitary matrices [42, 40, 29] (a group closed under multiplication), induced by the metric [29]:

$$Str\left[dU \cdot dU^{-1}\right] = Str\left[U^{-1}dU \cdot U^{-1}dU\right]. \qquad (71)$$

In order to find a relation between these two matrices one has to compute the Berezinean of the transformation [55, 42]:

$$\mathfrak{B} = \mathcal{S}det_{ij,mn}\left[\frac{(U^{-1}\,dU)_{ij}}{dU_{mn}}\right]. \qquad (72)$$

Since this is not a change of variables (the differentials dU are the same) and hence no Z-gradings are changed no boundary (Efetov-Wegner) terms arise in the superintegration. We write this Berezinean matrix in the block structure (Latin indices stay for bosons and Greek for fermions):

$$\begin{aligned}
\frac{U^{-1}\,dU}{dU} &= \begin{pmatrix} bb/bb & bb/ff & bb/fb & bb/bf \\ ff/bb & ff/ff & ff/fb & ff/bf \\ fb/bb & fb/ff & fb/fb & fb/bf \\ bf/bb & bf/ff & bf/fb & bf/bf \end{pmatrix} \\
&= \begin{pmatrix} U^{-1}_{ab}\otimes \mathbf{1}_{n_b} & 0 & U^{-1}_{a\beta}\otimes \mathbf{1}_{n_b} & 0 \\ 0 & U^{-1}_{\alpha\beta}\otimes \mathbf{1}_{n_f} & 0 & U^{-1}_{\alpha b}\otimes \mathbf{1}_{n_f} \\ U^{-1}_{\alpha b}\otimes \mathbf{1}_{n_b} & 0 & U^{-1}_{\alpha\beta}\otimes \mathbf{1}_{n_b} & 0 \\ 0 & U^{-1}_{a\beta}\otimes \mathbf{1}_{n_f} & 0 & U^{-1}_{ab}\otimes \mathbf{1}_{n_f} \end{pmatrix}. \qquad (73)
\end{aligned}$$

Computing the superdeterminant of the matrix above we obtain

$$\mathfrak{B} = \mathcal{S}det_{ij,mn}\left[\frac{(U^{-1}\,dU)_{ij}}{dU_{mn}}\right] = \mathcal{S}det\,[U]^{n_f-n_b}\,. \tag{74}$$

As a consequence we have:

$$d\mu_H\begin{pmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2\end{pmatrix} = dH_1\,dH_2\,d\Theta\,d\Theta^\dagger\,\mathcal{S}det\begin{bmatrix} H_1 & \Theta^\dagger \\ \Theta & H_2\end{bmatrix}^{n_f-n_b}\,, \tag{75}$$

that is eq. (69). Together with eq. (68) we obtain the proof of our theorem eq. (55). ■

7 Conclusions

In the present paper we prove that χPT in the ε-regime and chRMT have the very same spectral properties in the microscopic (weak non-Hermiticity) limit. The equivalence holds for zero, imaginary and real chemical potentials.

This result is achieved by proving the equivalence (up to irrelevant overall constant) of the partition function of the partially quenched chRMT, with n_b bosonic and n_f fermionic quarks, to an integral defined over the maximal Riemannian subspace of the linear group of $(n_b|n_f)$ supermatrices for any finite N. Performing the $N\to\infty$ limit of this integral one obtains the usual partially quenched $\epsilon\chi$PT. An essential ingredient of this computation is a superbosonisation theorem connecting integrals of external products of supervectors to integrals over $\hat{Gl}(n_b|n_f)$. An explicit parametrisation of integrals over $\hat{Gl}(n_b|n_f)$ is provided as well. Using the computations done in the purely imaginary case one can deduce the form that $\epsilon\chi$PT has with non zero chemical potential and bosons not occurring in conjugate pairs.

The equivalence we have established by matching the generating functionals of density correlations holds also in cases where the densities are not yet known explicitly. Only for at most two different chemical potentials all densities have been derived so far. Our results also shed some light on universality in chRMT with chemical potential by matching the Gaussian one- and two-matrix model of Stephanov and Osborn, respectively. This gives hope to establish a further reaching universality beyond Gaussian RMT.

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A Defining δ-functions on super-Hermitian matrices

The question is whether or not one can define a δ-distribution such that

$$f\left(\sum_k \psi_k\otimes\psi_k^\dagger\right) = \int dM\, f(M)\delta\left(M-\sum_k\psi_k\otimes\psi_k^\dagger\right). \tag{76}$$

In order to clarify the idea we take the same simple but nontrivial example as in [11], that is $k=1$, $(n_b|n_f)=(0|2)$. The external product is a 2×2 matrix, it's terms are only nilpotent commuting numbers ψ_i and ψ_j^* for $i,j=1,2$. We can consider this matrix as belonging to a superanalytic continuation of a manifold of real dimension 4 (at least). We could take as a such manifold $U(2)$, $Gl(2)/U(2)$, Hermitian matrices or real matrices. Despite in the rest of the work we have used Hermitian matrices, for this simple example we will use the one with the simplest notation, real matrices:

$$a\equiv\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\equiv\psi\otimes\psi^\dagger = \begin{pmatrix}\psi_1\psi_1^* & \psi_1\psi_2^*\\ \psi_2\psi_1^* & \psi_2\psi_2^*\end{pmatrix}\,. \tag{77}$$

Once considering the external product of vectors like an element of a commuting number manifold, the well definiteness of the δ-distribution is inherited from the one of the base manifold by superanalyticity in the real parameters of the manifold. In formulas:

$$f(a) \equiv f(a_{11}, a_{12}, a_{21}, a_{22}) = \int dm\, \delta(m-a)\, f(m_{11}, m_{12}, m_{21}, m_{22}) \ , \tag{78}$$

where the integration manifold is the base manifold (real matrices) $\int dm \equiv \int_{-\infty}^{\infty} \prod_{i,j=1}^{2} dm_{ij}$, and the δ distribution is the one of the base manifold too, $\delta(m-a) \equiv \prod_{i,j=1}^{2} \delta(m_{ij} - a_{ij})$.

The last thing to be verified is that the superanalytic continuation of the δ-distribution is properly defined; this was already shown in [55]. Given a quantity $x = x_B + x_S$, where the first part is the body of the number, and the second is the nilpotent part. Given a function with a sufficient number of derivatives in x_B such that $f(x_B + x_S) = \sum_{n=0}^{\infty} f^{(n)}(x_B) \frac{x_S^n}{n!}$ is properly defined, we have:

$$\begin{aligned} \int_{-\infty}^{+\infty} dy\, \delta(y - x_B - x_S)\, f(y) &\equiv \int_{-\infty}^{+\infty} dy \sum_{n=0}^{\infty} \delta^{(n)}(y - x_B) \frac{(-x_S)^n}{n!} f(y) \\ &= \sum_{n=0}^{\infty} \frac{(x_S)^n}{n!} \int_{-\infty}^{+\infty} dy\, \delta(y - x_B)\, f^{(n)}(x_B) = f(x_B + x_S) \ . \end{aligned} \tag{79}$$

Completing the discussion on the example above we can write explicitely $\delta(m-a)$:

$$\delta(m-a) = \delta(m) - \sum_{i,j=1}^{2} \psi_i \psi_j^* \partial_{m_{ij}} \delta(m) + \psi_1 \psi_1^* \psi_2 \psi_2^* (\partial_{m_{11}} \partial_{m_{22}} + \partial_{m_{12}} \partial_{m_{21}}) \delta(m) \tag{80}$$

where m is a real number 2×2 matrix.

The generalisation for what is done above to eq. (76) is straightforward: one considers the external product as an element belonging to the superanalytic continuation of a manifold. The manifold we choose is the one of super-Hermitian matrices: the boson-boson and the fermion-fermion block are Hermitian matrices, and the δ is defined as the superanalytic continuation of the product of δ's over the real and imaginary parts of the independent elements of the matrices; boson-fermion and fermion-boson blocks are made of independent Grassmann integration variables and δ may be represented as:

$$\delta(\theta - \tilde{\theta}) \propto (\theta - \tilde{\theta}) \propto \int d\xi\, e^{i\xi(\theta - \tilde{\theta})}. \tag{81}$$

As an exponential representation of δ is allowed both for commuting and anticommuting variables, such a representation of δ-function on super-Hermitian matrices is allowed too:

$$\begin{aligned} f(a) &= \int_{M=M^\dagger} dM\, f(M)\, \delta(M-a) \\ &\propto \int_{M=M^\dagger} dM\, f(M) \int_{F=F^\dagger} dF\, e^{iStr[F(M-a)]} \end{aligned} \tag{82}$$

where dM and dF stand for the flat measure over the independent entries of super-Hermitian matrices:

$$dM \equiv \prod_{i=1}^{n_b} \prod_{\alpha=1}^{n_f} dM_{i,\alpha}\, dM_{\alpha,i} \prod_{i=1}^{n_b} dM_{i,i} \prod_{i>j} d\mathbb{R}e\, M_{i,j}\, d\mathbb{I}m\, M_{i,j} \prod_{\alpha=1}^{n_f} dM_{\alpha,\alpha} \prod_{\alpha,\beta} d\mathbb{R}e\, M_{\alpha,\beta}\, d\mathbb{I}m\, M_{\alpha,\beta} \ . \tag{83}$$

B The integrals in eq. (67)

B.1 Boson-boson block

Let's start from

$$\begin{aligned}
&\oint_{U(n_b)} dF_1 \, \frac{1}{\mathcal{D}et\,[F_1]^{N+n_f}} \mathrm{Exp}\left[iTr\left[F_1\tilde{H}\right]\right] \; = \\
&= \int_{U(n_b)} d\mu_U(F_1)\, \mathcal{D}et\,[F_1]^{-N-n_f+n_b}\, \mathrm{Exp}\left[iTr\left[F_1\tilde{H}\right]\right] \\
&= \mathcal{D}et\left[\tilde{H}\right]^{N+n_f-n_b} \int_{U(n_b)} d\mu_U(F_1)\, \mathcal{D}et\left[F_1\tilde{H}\right]^{-N-n_f+n_b} \mathrm{Exp}\left[iTr\left[F_1\tilde{H}\right]\right] \; ,
\end{aligned} \tag{84}$$

where for simplicity of notation we use $\tilde{H} = H_1 - \Theta H_2^{-1}\Theta^\dagger$. The integral in the last line may be performed using the character expansion. This is a particular case of a more general integral computed in [56]; the result is:

$$\sum_r \frac{\alpha_r^{(0)}}{d_r} \alpha_r^{(N+n_f-n_b)} \chi_r(0) \tag{85}$$

where the sum is over the irreducible representations of $Gl(n_b)$. The quantity $\chi_r(0)$ is different from zero for all the representation apart from the trivial one. The result is just a constant[9].

$$\oint_{U(n_b)} dF_1 \frac{1}{\mathcal{D}et\,[F_1]^{N+n_f}} e^{i\left[F_1\tilde{H}\right]} \propto \mathcal{D}et\left[\tilde{H}\right]^{N+n_f-n_b} . \tag{86}$$

This result, together with the argument of analytic continuation of Hermitian matrices, gives an alternative way for computing the Ingham-Siegel integral as was done in [27].

B.2 Fermion-fermion block

The fermion-fermion block integral has already been computed by Guhr [52]. The result is:

$$\int_{F_2=F_2^\dagger} dF_2\, \mathcal{D}et\,[F_2]^N\, \mathrm{Exp}\left[-iTr\,[F_2H_2]\right] \propto \prod_j \delta^{(N+n_f-1)}(h_j) \; . \tag{87}$$

Here we don't have any problem with possible Efetov-Wegner terms arising from these diagonalisation: both F_2 and H_2 are complex Hermitian matrices and no Z-grading will be changed during diagonalisation [54].

If we integrate this functional with a given function analytic in the matrix entries, using the Cauchy integral formula we obtain:

$$\begin{aligned}
\int d\mu_U(U) \prod_j \int dh_j\, \Delta(\{h\})^2\, g(UhU^\dagger) \prod_j \delta^{(N+n_f-1)}(h_j) \quad &\propto \quad \int d\mu_U(U) \oint \prod_j dh_j \frac{1}{h_j^{N+n_f}}\, \Delta(\{h\})^2\, g(UhU^\dagger) \\
&= \quad \oint_{U(N_f)} dH_2 \; \frac{1}{\mathcal{D}et\,[H_2]^{N+n_f}} \; g(H_2) \; .
\end{aligned} \tag{88}$$

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