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9 A New Family of Boundary-Domain Integral Equations for a Mixed Elliptic BVP with Variable Coefficient

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Abstract

A mixed boundary value problem for the stationary heat transfer partial differential equation with variable coefficient is reduced to some systems of direct segregated parametrix-based Boundary-Domain Integral Equations (BDIEs). We use a parametrix different from the one employed by Mikhailov (2002) and Chkadua, Mikhailov, Natroshvili (2009). Mapping properties of the potential type integral operators appearing in these equations are presented in appropriate Sobolev spaces. We prove the equivalence between the original BVP and the corresponding BDIE system. The invertibility and Fredholm properties of the boundary-domain integral operators are also analysed.

9.1 Preliminaries and the BVP

The domains. Let $\Omega = \Omega^+$ be a bounded simply connected domain, $\Omega^- := \mathbb{R}^3 \setminus \Omega^+$ the complementary (unbounded) subset of Ω. The boundary $S := \partial\Omega$ is simply connected, closed and infinitely differentiable, $S \in \mathcal{C}^\infty$. Furthermore, $S := \overline{S}_N \cup \overline{S}_D$ where both S_N and S_D are non-empty, connected disjoint manifolds of S. The border of these two submanifolds is also infinitely differentiable, $\partial S_N = \partial S_D \in \mathcal{C}^\infty$.

PDE. We consider the following partial differential equation:

$$\mathcal{A}u(x) := \mathcal{A}(x)[u(x)] := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = f(x), \qquad x \in \Omega, \tag{9.1}$$

where $u(x)$ is an unknown function, $a(x) \in \mathcal{C}^\infty$ is a variable coefficient and f is a given function on Ω. It is easy to see that if $a \equiv 1$ then, the operator $\mathcal{A}$ becomes the Laplace operator Δ.

Function spaces. We will use the following function spaces in this paper (see e.g. [5, 3, 4] for more details). Let $\mathcal{D}'(\Omega)$ be the Schwartz distribution space; $W_p^s(\Omega)$ and $W^s(S)$, $s \geq 0$, the Sobolev-Slobodetski spaces; $H^s(\Omega)$ and $H^s(S)$ with $s \in \mathbb{R}$, the Bessel potential spaces; the space $H_K^s(\mathbb{R}^3)$ consisting of all the distributions of $H^s(\mathbb{R}^3)$ whose support is inside of a compact set $K \subset \mathbb{R}^3$; $H_{loc}^s(\overline{\Omega^-})$ the spaces consisting of distributions in $H^s(K)$ for every compact $K \subset \overline{\Omega^-}$, $s \in \mathbb{R}$; $\widetilde{H}^s(S_1) = \{g \in H^s(S) : \mathrm{supp}(g) \in \overline{S_1}\}$; $H^s(S_1) = \{g|_{S_1} : g \in$

$H^s(S)\}$, where the notation $g|_{S_1} = r_{S_1} g$ is used to indicate the restriction of the function g from S to S_1. Note that $H^s_{loc}(\overline{\Omega^-}) = W^s_{2,loc}(\overline{\Omega^-})$ and $H^r(S) = W^r_2(S)$ for $r \geq 0$). We will also make use of the space, see e.g. [2, 1],

$$H^{1,0}(\Omega; \mathcal{A}) := \{u \in H^1(\Omega) : \mathcal{A}u \in L^2(\Omega)\}$$

which is a Hilbert space with the norm defined by

$$\| u \|^2_{H^{1,0}(\Omega;\mathcal{A})} := \| u \|^2_{H^1(\Omega)} + \| \mathcal{A}u \|^2_{L^2(\Omega)}.$$

Traces and conormal derivatives. For a scalar function $w \in H^s(\Omega^\pm)$, $s > 1/2$, the trace operator $\gamma^\pm = \gamma^\pm_S$, acting on w is well defined and $\gamma^\pm w \in H^{s-\frac{1}{2}}(S)$ (see, e.g., [5, 7]). For $u \in H^s(\Omega)$, $s > 3/2$, we can define on S the conormal derivative operator, $T^\pm$, in the trace sense:

$$T^\pm[u(x)] := \sum_{i=1}^{3} a(x) n_i(x) \left(\frac{\partial u}{\partial x_i}\right)^\pm = a(x) \left(\frac{\partial u(x)}{\partial n(x)}\right)^\pm,$$

where $n(x)$ is the exterior unit normal vector to the domain Ω at point $x \in S$.

Moreover, for any function $u \in H^{1,0}(\Omega; \mathcal{A})$, we can extend the definition to the canonical conormal derivative $T^\pm u \in H^{-\frac{1}{2}}(\Omega)$, associated with the first Green identity, cf. [2, 5, 7],

$$\langle T^\pm u, w \rangle_S := \pm \int_{\Omega^\pm} [(\gamma^{-1}\omega)\mathcal{A}u + E(u, \gamma^{-1}w)]dx, \text{ for all } w \in H^{\frac{1}{2}}(S), \tag{9.2}$$

where $\gamma^{-1} : H^{\frac{1}{2}}(S) \to H^1(\mathbb{R}^3)$ is a continuous right inverse to the trace operator whereas the function E is defined as

$$E(u, v)(x) := \sum_{i=1}^{n} a(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i},$$

and $\langle , \rangle_S$ represents the L^2-based dual form on S.

Boundary value problem We aim to derive boundary-domain integral equation systems for the following *mixed* boundary value problem. Given $f \in L^2(\Omega)$, $\phi_0 \in H^{\frac{1}{2}}(S_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(S_N)$, we seek a function $u \in H^1(\Omega)$ such that

$$\mathcal{A}u = f \quad \text{in} \quad \Omega, \tag{9.3a}$$
$$r_{S_D}\gamma^+ u = \phi_0 \quad \text{on} \quad S_D, \tag{9.3b}$$
$$r_{S_N} T^+ u = \psi_0 \quad \text{on} \quad S_N, \tag{9.3c}$$

where equation (9.3a) is understood in the weak sense, the Dirichlet condition (9.3b) is understood in the trace sense and the Neumann condition (9.3c) is understood in the functional sense (9.2).

By Lemma 3.4 of [2] (cf. also Theorem 3.9 in [7]), the first Green identity holds for any $u \in H^{1,0}(\Omega;\mathcal{A})$ and $v \in H^1(\Omega)$,

$$\langle T^{\pm}u, \gamma^{+}v\rangle_S := \pm \int_{\Omega} [v\mathcal{A}u + E(u,v)]dx. \tag{9.4}$$

The following assertion is well know and can be proved, e.g., using the Lax-Milgram lemma.

Theorem 9.1.1. *The boundary value problem* (9.3) *has one and only one solution.*

9.2 Parametrices and remainders

Definition 9.2.1. *Let $P(x,y) \in \mathcal{D}'(\Omega)$ be a distribution of two variables. We say that $P(x,y)$ is a parametrix (or Levi function) for the operator $\mathcal{A}(x)$ if*

$$\mathcal{A}(x)[P(x,y)] = \delta(x-y) + R(x,y), \tag{9.5}$$

where $\delta(\cdot)$ is the Dirac delta distribution and remainder $R(x,y)$ possesses a weak (integrable) singularity at $x=y$, i.e., $R(x,y) = \mathcal{O}(|x-y|^{-k})$ with $k<3$.

The notion of parametrix is well known, see e.g. [6] and references therein.

For a given operator $\mathcal{A}$, the parametrix is not unique. For example, the parametrix

$$P^y(x,y) = \frac{1}{a(y)} P_{\Delta}(x-y), \quad x,y \in \mathbb{R}^3,$$

was employed in [6, 1], for the operator $\mathcal{A}$ defined in (9.1), where

$$P_{\Delta}(x-y) = \frac{-1}{4\pi|x-y|}$$

is the fundamental solution of the Laplace operator. The remainder corresponding to the parametrix P_y is

$$R^y(x,y) = \sum_{i=1}^{3} \frac{1}{a(y)} \frac{\partial a(x)}{\partial x_i} \frac{\partial}{\partial x_i} P_{\Delta}(x-y)\,, \quad x,y \in \mathbb{R}^3. \tag{9.6}$$

In this paper, for the same operator $\mathcal{A}$ defined in (9.1), we will use another parametrix,

$$P(x,y) = P^x(x,y) = \frac{1}{a(x)} P_{\Delta}(x-y), \quad x,y \in \mathbb{R}^3, \tag{9.7}$$

while the corresponding remainder is

$$\begin{aligned} R(x,y) = R^x(x,y) &= -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{1}{a(x)}\frac{\partial a(x)}{\partial x_i} P_{\Delta}(x,y)\right) \\ &= -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\partial \ln a(x)}{\partial x_i} P_{\Delta}(x,y)\right), \quad x,y \in \mathbb{R}^3. \end{aligned}$$

Note that the both remainders R_x and R_y are weakly singular:

$$R_x(x, y), R_y(x, y) = \mathcal{O}(|x - y|^{-2}).$$

This is due to the smoothness of the variable coefficient a.

9.3 Volume and surface potentials

The volume parametrix-based Newton-type potential and the remainder potential are respectively defined, for $y \in \mathbb{R}^3$, as

$$\mathcal{P}\rho(y) := \int_\Omega P(x, y)\rho(x)\, dx$$
$$\mathcal{R}\rho(y) := \int_\Omega R(x, y)\rho(x)\, dx.$$

The parametrix-based single layer and double layer surface potentials are defined for $y \in \mathbb{R}^3 : y \notin S$, as

$$V\rho(y) := -\int_S P(x, y)\rho(x)\, dS(x),$$

$$W\rho(y) := -\int_S T_x^+ P(x, y)\rho(x)\, dS(x).$$

We also define the following pseudo-differential operators associated with direct values of the single and double layer potentials and with their conormal derivatives, for $y \in S$,

$$\mathcal{V}\rho(y) := -\int_S P(x, y)\rho(x)\, dS(x),$$
$$\mathcal{W}\rho(y) := -\int_S T_x P(x, y)\rho(x)\, dS(x)$$
$$\mathcal{W}'\rho(y) := -\int_S T_y P(x, y)\rho(x)\, dS(x),$$
$$\mathcal{L}^\pm\rho(y) := T_y^\pm W\rho(y).$$

The operators $\mathcal{P}, \mathcal{R}, V, W, \mathcal{V}, \mathcal{W}, \mathcal{W}'$ and $\mathcal{L}$ can be expressed in terms the volume and surface potentials and operators associated with the Laplace operator, as follows

$$\mathcal{P}\rho = \mathcal{P}_\Delta\left(\frac{\rho}{a}\right), \tag{9.8}$$

$$\mathcal{R}\rho = -\nabla\cdot\left[\mathcal{P}_\Delta(\rho)\nabla\ln a\right], \tag{9.9}$$

$$V\rho = V_\Delta\left(\frac{\rho}{a}\right), \tag{9.10}$$

$$\mathcal{V}\rho = \mathcal{V}_\Delta\left(\frac{\rho}{a}\right), \tag{9.11}$$

$$W\rho = W_\Delta\rho - V_\Delta\left(\rho\frac{\partial\ln a}{\partial n}\right), \tag{9.12}$$

$$\mathcal{W}\rho = \mathcal{W}_\Delta\rho - \mathcal{V}_\Delta\left(\rho\frac{\partial\ln a}{\partial n}\right), \tag{9.13}$$

$$\mathcal{W}'\rho = a\mathcal{W}'{}_\Delta\left(\frac{\rho}{a}\right), \tag{9.14}$$

$$\mathcal{L}^{\pm}\rho = \hat{\mathcal{L}}\rho - aT^{\pm}_\Delta V_\Delta\left(\rho\frac{\partial\ln a}{\partial n}\right), \tag{9.15}$$

$$\hat{\mathcal{L}}\rho := a\mathcal{L}_\Delta\rho. \tag{9.16}$$

The symbols with the subscript Δ denote the analogous surface potentials for the constant coefficient case, $a \equiv 1$. Furthermore, by the Liapunov-Tauber theorem, $\mathcal{L}^+_\Delta\rho = \mathcal{L}^-_\Delta\rho = \mathcal{L}_\Delta\rho$.

Using relations (9.10)-(9.16) it is now rather simple to obtain, similar to [1], the mapping properties, jump relations and invertibility results for the parametrix-based surface and volume potentials, provided in Theorems 9.3.1-9.3.8, from the well-known properties of their constant-coefficient counterparts (associated with the Laplace equation).

Theorem 9.3.1. *Let $s \in \mathbb{R}$. Then, the following operators are continuous,*

$$\mathcal{P} : \widetilde{H}^s(\Omega) \longrightarrow H^{s+2}(\Omega), \;\; s \in \mathbb{R}, \tag{9.17}$$

$$\mathcal{P} : H^s(\Omega) \longrightarrow H^{s+2}(\Omega), \;\; s > -\frac{1}{2}, \tag{9.18}$$

$$\mathcal{R} : \widetilde{H}^s(\Omega) \longrightarrow H^{s+1}(\Omega), \;\; s \in \mathbb{R}, \tag{9.19}$$

$$\mathcal{R} : H^s(\Omega) \longrightarrow H^{s+1}(\Omega), \;\; s > -\frac{1}{2}\,. \tag{9.20}$$

Corollary 9.3.2. *Let $s > \frac{1}{2}$, let S_1 a non-empty submanifold of S with smooth boundary. Then, the following operators are compact:*

$$\mathcal{R} : H^s(\Omega) \longrightarrow H^s(\Omega),$$
$$r_{S_1}\mathcal{R} : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(S_1),$$
$$r_{S_1}T^+\mathcal{R} : H^s(S_1) \longrightarrow H^{s-\frac{3}{2}}(S_1).$$

Theorem 9.3.3. *Let $s \in \mathbb{R}$. Then the following operators are continuous:*

$$V : H^s(S) \longrightarrow H^{s+\frac{3}{2}}(\Omega),$$
$$W : H^s(S) \longrightarrow H^{s+\frac{1}{2}}(\Omega).$$

Theorem 9.3.4. *Let $s \in \mathbb{R}$, the following operators are continuous:*

$$\begin{aligned} \mathcal{V} &: H^s(S) \longrightarrow H^{s+1}(S), \\ \mathcal{W} &: H^s(S) \longrightarrow H^{s+1}(S), \\ \mathcal{W}' &: H^s(S) \longrightarrow H^{s+1}(S), \\ \mathcal{L}^{\pm} &: H^s(S) \longrightarrow H^{s-1}(S). \end{aligned}$$

Theorem 9.3.5. *Let $\rho \in H^{-\frac{1}{2}}(S)$, $\tau \in H^{\frac{1}{2}}(S)$. Then the following operators jump relations hold:*

$$\begin{aligned} \gamma^{\pm} V\rho &= \mathcal{V}\rho, \\ \gamma^{\pm} W\tau &= \mp\frac{1}{2}\tau + \mathcal{W}\tau, \\ T^{\pm} V\rho &= \pm\frac{1}{2}\rho + \mathcal{W}'\rho. \end{aligned}$$

Theorem 9.3.6. *Let $s \in \mathbb{R}$, let S_1 and S_2 two non-empty manifolds with smooth boundaries, ∂S_1 and ∂S_2, respectively. Then, the following operators are compact:*

$$\begin{aligned} r_{S_2}\mathcal{V} &: \widetilde{H}^s(S_1) \longrightarrow H^s(S_2), \\ r_{S_2}\mathcal{W} &: \widetilde{H}^s(S_1) \longrightarrow H^s(S_2), \\ r_{S_2}\mathcal{W}' &: \widetilde{H}^s(S_1) \longrightarrow H^s(S_2). \end{aligned}$$

Theorem 9.3.7. *Let S_1 be a non-empty simply connected submanifold of S with infinitely smooth boundary curve, and $0 < s < 1$. Then, the operators*

$$\begin{aligned} r_{S_1}\mathcal{V} &: \widetilde{H}^{s-1}(S_1) \longrightarrow H^s(S_1), \\ \mathcal{V} &: \widetilde{H}^{s-1}(S) \longrightarrow H^s(S) \end{aligned}$$

are invertible.

Theorem 9.3.8. *Let S_1 be a non-empty simply connected submanifold of S with infinitely smooth boundary curve, and $0 < s < 1$. Then, the operator*

$$r_{S_1}\hat{\mathcal{L}} : \widetilde{H}^s(S_1) \longrightarrow H^{s-1}(S_1)$$

is invertible and the operator

$$r_{S_1}(\mathcal{L}^{\pm} - \hat{\mathcal{L}}) : \widetilde{H}^s(S_1) \longrightarrow H^{s-1}(S_1)$$

is compact.

9.4 Third Green identities and integral relations

In this section we provide the results similar to the ones in [1] but for our, different, parametrix (9.7).

Let $u, v \in H^{1,0}(\Omega; \mathcal{A})$. Subtracting from the first Green identity (9.4) its counterpart with the swapped u and v, we arrive at the second Green identity, see e.g. [5],

$$\int_{\Omega} (\mathcal{A}(v)u - \mathcal{A}(u)v)\, dx = \int_{S} \left[T^+(v)u - T^+(u)v\right] dS(x). \tag{9.21}$$

Taking now $v(x) := P(x, y)$, we obtain from (9.21) and (9.5) by the standard limiting procedures (cf. [8]) the third Green identity for any function $u \in H^{1,0}(\Omega; \mathcal{A})$:

$$u + \mathcal{R}u - VT^+u + W\gamma^+u = \mathcal{P}\mathcal{A}u \quad \text{in}\, \Omega. \tag{9.22}$$

If $u \in H^{1,0}(\Omega; \mathcal{A})$ is a solution of the partial differential equation (9.3a), then from (9.22) we obtain:

$$u + \mathcal{R}u - VT^+(u) + W\gamma^+u = \mathcal{P}f \;\; in\, \Omega, \tag{9.23}$$

$$\frac{1}{2}\gamma^+u + \gamma^+\mathcal{R}u - \mathcal{V}T^+(u) + \mathcal{W}\gamma^+u = \gamma^+\mathcal{P}f \;\; on\, S, \tag{9.24}$$

$$\frac{1}{2}T^+(u) + T^+\mathcal{R}u - \mathcal{W}'T^+(u) + \mathcal{L}^+\gamma^+u = T^+\mathcal{P}f \;\; on\, S. \tag{9.25}$$

For some distributions f, Ψ and Ψ, we consider a more general, indirect integral relation associated with the third Green identity (9.23):

$$u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}f \;\; \text{in}\ \Omega. \tag{9.26}$$

Lemma 9.4.1. *Let $u \in H^1(\Omega)$, $f \in L_2(\Omega)$, $\Psi \in H^{-\frac{1}{2}}(S)$ and $\Phi \in H^{\frac{1}{2}}(S)$ satisfying the relation (9.26). Then u belongs to $H^{1,0}(\Omega, \mathcal{A})$ and solves the equation $\mathcal{A}u = f$ in Ω, and also the following identity is satisfied,*

$$V(\Psi - T^+u) - W(\Phi - \gamma^+v) = 0 \;\; in\ \Omega. \tag{9.27}$$

Proof. To prove that $u \in H^{1,0}(\Omega; \mathcal{A})$ we take into account that by hypothesis $u \in H^1(\Omega)$, and so there is only left to prove that $\mathcal{A}u \in L^2(\Omega)$.

Due to (9.8), (9.10) and (9.12), equation (9.26) implies

$$\begin{aligned} u =& \mathcal{P}f - \mathcal{R}u + V\Psi - W\Phi \\ =& \mathcal{P}_{\Delta}\left(\frac{f}{a}\right) - \mathcal{R}u + V_{\Delta}\left(\frac{\Psi}{a}\right) - W_{\Delta}\Phi + V_{\Delta}\left(\frac{\partial \ln a}{\partial n}\,\Phi\right). \end{aligned} \tag{9.28}$$

We note that $\mathcal{R}u \in H^2(\Omega)$ due to the mapping properties (9.20). Moreover, V_{Δ} and W_{Δ} in (9.28) are harmonic potentials, while $\mathcal{P}_{\Delta}$ is the Newtonian potential for the Laplacian, i.e. $\Delta\mathcal{P}_{\Delta}\left(\frac{f}{a}\right) = \frac{f}{a}$. Consequently, $\Delta u = \frac{f}{a} - \Delta\mathcal{R}u \in L^2(\Omega)$. Hence, $\mathcal{A}u \in L^2(\Omega)$ and $u \in H^{1,0}(\Omega; \mathcal{A})$.

Since $u \in H^{1,0}(\Omega;\mathcal{A})$, the third Green identity (9.23) is valid for the function u, and we proceed subtracting (9.23) from (9.26) to obtain

$$W(\gamma^+ u - \Phi) - V(T^+ u - \Psi) = \mathcal{P}(\mathcal{A}u - f). \tag{9.29}$$

Recalling again the relations (9.8), (9.10) and (9.12), and applying the Laplace operator to both sides of (9.29) we obtain

$$\mathcal{A}u - f = 0, \tag{9.30}$$

i.e., u solves (9.3a). Finally, substituting (9.30) into (9.29), we prove (9.27). □

Lemma 9.4.2. *Let* $\Psi^* \in H^{-\frac{1}{2}}(S)$. *If*

$$V\Psi^*(y) = 0, \qquad y \in \Omega \tag{9.31}$$

then $\Psi^*(y) = 0$.

Proof. Taking the trace of (9.31)gives:

$$\mathcal{V}\Psi^*(y) = \mathcal{V}_\triangle\left(\frac{\Psi^*}{a}\right)(y) = 0, \qquad y \in \Omega,$$

from where the result follows due to the invertibility of the operator $\mathcal{V}_\triangle$ (cf. Theorem 9.3.7). □

9.5 BDIE system for the mixed problem

We aim to obtain a segregated boundary-domain integral equation system for mixed BVP (9.3). To this end, let the functions $\Phi_0 \in H^{\frac{1}{2}}(S)$ and $\Psi_0 \in H^{-\frac{1}{2}}(S)$ be respective continuations of the boundary functions $\phi_0 \in H^{\frac{1}{2}}(S_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(S_N)$ to the whole S. Let us now represent

$$\gamma^+ v = \Phi_0 + \phi, \quad T^+(v) = \Psi_0 + \psi \text{ on } S, \tag{9.32}$$

where $\phi \in \widetilde{H}^{\frac{1}{2}}(S_N)$ and $\psi \in \widetilde{H}^{-\frac{1}{2}}(S_D)$ are unknown boundary functions.

To obtain one of the possible boundary-domain integral equation systems we employ (9.23) in the domain Ω and (9.24) on S, substituting there $\gamma^+ u = \Phi_0 + \phi$ and $T^+ u = \Psi_0 + \psi$ and further considering the unknown functions ϕ and ψ as formally independent (segregated) of u in Ω. Consequently, we obtain the following system $M12$ of two equations for three unknown functions,

$$u + \mathcal{R}u - V\psi + W\phi = F_0 \qquad in\ \ \Omega, \tag{9.33a}$$

$$\frac{1}{2}\phi + \gamma^+\mathcal{R}u - \mathcal{V}\psi + \mathcal{W}\phi = \gamma^+ F_0 - \Phi_0 \qquad on\ \ S, \tag{9.33b}$$

where

$$F_0 = \mathcal{P}f + V\Psi_0 - W\Phi_0. \tag{9.34}$$

We remark that F_0 belongs to the space $H^1(\Omega)$ in virtue of the mapping properties of the surface and volume potentials, see Theorems 9.3.1 and 9.3.3.

The system $M12$, given by (9.33a)-(9.33b) can be written in matrix notation as

$$\mathcal{M}^{12}\mathcal{X} = \mathcal{F}^{12}, \tag{9.35}$$

where $\mathcal{X}$ represents the vector containing the unknowns of the system,

$$\mathcal{X} = (u, \psi, \phi)^\top \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N), \tag{9.36}$$

the right hand side vector is

$$\mathcal{F}^{12} := [F_0, \gamma^+ F_0 - \Psi_0]^\top \in H^1(\Omega) \times H^{\frac{1}{2}}(S),$$

and matrix operator $\mathcal{M}^{12}$ is defined by:

$$\mathcal{M}^{12} = \begin{bmatrix} I + \mathcal{R} & -V & W \\ \gamma^+\mathcal{R} & -\mathcal{V} & \dfrac{1}{2}I + \mathcal{W} \end{bmatrix}. \tag{9.37}$$

We note that the mapping properties of the operators involved in the matrix imply the continuity of the operator

$$\mathcal{M}^{12} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(S).$$

Theorem 9.5.1 (BDIE-BVP Equivalence)**.** *Let $f \in L_2(\Omega)$. Let $\Phi_0 \in H^{\frac{1}{2}}(S)$ and $\Psi_0 \in H^{-\frac{1}{2}}(S)$ be some fixed extensions of $\phi_0 \in H^{\frac{1}{2}}(S_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(S_N)$ respectively.*

1. *If some $u \in H^1(\Omega)$ solves the BVP* (9.3)*, then the triple $(u, \psi, \phi)^\top \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N)$ where*

$$\phi = \gamma^+ u - \Phi_0, \qquad \psi = T^+ u - \Psi_0, \qquad \textit{on } S, \tag{9.38}$$

   solves the BDIE system $M12$.

2. *If a triple $(u, \psi, \phi)^\top \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N)$ solves the BDIE system then u solves the BVP and the functions ψ, ϕ satisfy* (9.38)*.*

3. *The system $M12$ is uniquely solvable.*

Proof. First, let us prove item 1. Let $u \in H^1(\Omega)$ be a solution of the boundary value problem (9.3) and let ϕ, ψ be defined by (9.38). Then, due to (9.3b) and (9.3c), we have

$$(\psi, \phi) \in \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N).$$

Then, it immediately follows from the third Green identities (9.23) and (9.24) that the triple (u, ϕ, ψ) solves BDIE system $\mathcal{M}^{12}$.

Item 2. Let the triple $(u, \psi, \phi)^\top \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N)$ solve the BDIE system. Taking the trace of the equation (9.33a) and substract it from the equation (9.33b), we obtain

$$\phi = \gamma^+ u - \Phi_0 \quad \text{on } S. \tag{9.39}$$

This means that the first condition in (9.38) is satisfied. Now, restricting equation (9.39) to S_D, we observe that ϕ vanishes as $supp(\phi) \subset S_N$. Hence, $\phi_0 = \Phi_0 = \gamma^+ u$ on S_D and consequently, the Dirichlet condition of the BVP (9.3b) is satisfied.

We proceed using the Lemma 9.4.1 in the first equation of the system $\mathcal{M}^{12}$, (9.33a), with $\Psi = \psi + \Psi_0$ and $\Phi = \phi + \Phi_0$ which implies that u is a solution of the equation (9.3a) and also the following equality:

$$V(\Psi_0 + \psi - T^+ u) + W(\Phi_0 + \phi - \gamma^+ u) = 0 \text{ in } \Omega. \tag{9.40}$$

In virtue of (9.39), the second term of the previous equation vanishes. Hence,

$$V(\Psi_0 + \psi - T^+ u) = 0 \text{ in } \Omega. \tag{9.41}$$

Now, in virtue of Lemma 9.4.2 we obtain

$$\Psi_0 + \psi - T^+ u = 0 \text{ on } S. \tag{9.42}$$

Since ψ vanishes on S_N, we have $\Psi_0 = \psi_0$, and equation (9.42) implies that u satisfies the Neumann condition (9.3c).

Since every solution of the BVP is a solution of the BDIEs $\mathcal{M}^{12}$, then the BDIEs has at least one solution. However, every solution of the homogeneous BDIEs can, by item 2, be related with solution of the homogeneous BVP, which can be only the trivial solution. This implies that the homogeneous BDIE solution can be only trivial, which completes the proof of item 3. □

Lemma 9.5.1. $(F_0, \gamma^+ F_0 - \Phi_0) = 0$ *if and only if* $(f, \Phi_0, \Psi_0) = 0$

Proof. It is trivial that if $(f, \Phi_0, \Psi_0) = 0$ then $(F_0, \gamma^+ F_0 - \Phi_0) = 0$. Conversely, supposing that $(F_0, \gamma^+ F_0 - \Phi_0) = 0$, then taking into account equation (9.34) and applying Lemma 9.4.1 with $F_0 = 0$ as u, we deduce that $f = 0$ and $V\Psi_0 - W\Phi_0 = 0$ in Ω. Now, the second equality, $\gamma^+ F_0 - \Phi_0 = 0$, implies that $\Phi_0 = 0$ on S and applying Lemma 9.4.2 gives $\Psi_0 = 0$ on S. □

Theorem 9.5.2. *The operator*

$$\mathcal{M}^{12} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(S),$$

is invertible.

Proof. Let $\mathcal{M}_0^{12}$ be the matrix operator defined by

$$\mathcal{M}_0^{12} := \begin{bmatrix} I & -V & W \\ 0 & -\mathcal{V} & \frac{1}{2}I \end{bmatrix}. \tag{9.43}$$

The operator $\mathcal{M}_0^{12}$ is also bounded due to the mapping properties of the operators involved. Furthermore, the operator

$$\mathcal{M}^{12} - \mathcal{M}_0^{12} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(S)$$

is compact due to the compact mapping properties of the operators $\mathcal{R}$ and $\mathcal{W}$, (cf. Theorem 9.3.2 and Theorem 9.3.6).

Let us prove that the operator $\mathcal{M}_0^{12}$ is invertible.For this purpose, we consider the following system with arbitrary right hand side $\widetilde{F} = [\widetilde{F_1}, \widetilde{F_2}]^\top \in H^1(\Omega) \times H^{\frac{1}{2}}(S)$ and let $\mathcal{X} = (u, \psi, \phi)^\top \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N)$ be the vector of unknowns

$$\mathcal{M}_0^{12}\mathcal{X} = \widetilde{F}. \tag{9.44}$$

Writing (9.44) component-wise,

$$u - V\psi + W\phi = \widetilde{F_1}, \qquad in\ \ \Omega, \tag{9.45a}$$

$$\frac{1}{2}\phi - \mathcal{V}\psi = \widetilde{F_2}, \qquad on\ \ S. \tag{9.45b}$$

Equation (9.45b) restricted to S_D gives:

$$-r_{S_D}\mathcal{V}\psi = r_{S_D}\widetilde{F_2}. \tag{9.46}$$

Due to the invertibility of the operator $\mathcal{V}$ (cf. Lemma 9.3.7), equation (9.46) is uniquely solvable on S_D. Equation (9.46) means that $(\mathcal{V}\psi + \widetilde{F_2}) \in \widetilde{H}^{\frac{1}{2}}(S_N)$. Thus, the unique solvability of (9.46) implies that ϕ is also uniquely determined by the equation

$$\phi = (2\mathcal{V}\psi + 2\widetilde{F_2}) \in \widetilde{H}^{\frac{1}{2}}(S_N). \tag{9.47}$$

Consequently, u also is uniquely determined by the first equation (9.45a) of the system.

$$u = V\psi - W\phi + \widetilde{F_1}.$$

Furthermore, since $V\psi$, $W\phi \in H^1(\Omega)$, we have $u \in H^1(\Omega)$.

Thus, the operator $\mathcal{M}_0^{12}$ is invertible and the operator $\mathcal{M}^{12}$ is a zero index Fredholm operator due to the compactness of the operator $\mathcal{M}^{12} - \mathcal{M}_0^{12}$. Hence the Fredholm property and the injectivity of $\mathcal{M}^{12}$ due item 3 of to Lemma 9.5.1 imply the invertibility of operator $\mathcal{M}^{12}$. □

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