H_∞ Fault Estimation with Randomly Occurring Uncertainties, Quantization Effects and Successive Packet Dropouts: The Finite-Horizon Case

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Abstract

In this paper, the finite-horizon H_∞ fault estimation problem is investigated for a class of uncertain nonlinear time-varying systems subject to multiple stochastic delays. The randomly occurring uncertainties (ROUs) enter into the system due to the random fluctuations of network conditions. The measured output is quantized by a logarithmic quantizer before being transmitted to the fault estimator. Also, successive packet dropouts (SPDs) happen when the quantized signals are transmitted through an unreliable network medium. Three mutually independent sets of Bernoulli-distributed white sequences are introduced to govern the multiple stochastic delays, ROUs and SPDs. By employing the stochastic analysis approach, some sufficient conditions are established for the desired finite-horizon fault estimator to achieve the specified H_∞ performance. The time-varying parameters of the fault estimator are obtained by solving a set of recursive linear matrix inequalities (RLMIs). Finally, an illustrative numerical example is provided to show the effectiveness of the proposed fault estimation approach.

Keywords

Finite horizon, robust H_∞ fault estimation, multiple stochastic delays, randomly occurring uncertainties, quantization effects, successive packet dropouts

I. Introduction

The past few years have witnessed fruitful research results on the fault detection and fault-tolerant control problems owing to their crucial importance with respect to safety and reliability in engineering practice such as aerospace, automotive and chemical industries [1, 4, 13, 14, 21, 37]. Generally speaking, the purpose of fault detection is to identify when a fault has occurred, and the fault estimation stage aims to estimate the type and size of the faults by using available measurement information. In recent years, the fault estimation problem has gained constant research attention and a number of results have been reported in the literature [18, 24, 36]. For instance, in [36], the fault estimate has been added into the controller to compensate for the unknown real fault. In [18], several multi-objective (e.g. H_2/H_∞ and H_-/H_∞ indices) fault estimation issues have been tackled for time-varying systems in time domain. Very recently, in [24], by recurring to the Krein-space theory, the H_∞ fault estimation issue has been studied and a fault estimator has been designed to achieve the specified performance criterion in terms of the solution to a set of Riccati difference equations.

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Networked control systems (NCSs) have recently gaining much research momentum owing to their appealing advantages as well as wide applications in today's modern industry, and a rich body of literature has appeared with a major focus on those particular phenomena resulting from the limited bandwidth of the communication channels. These network-induced phenomena, if not adequately dealt with, could seriously degrade the system performances. Among others, two frequently investigated network-induced phenomena are packet dropouts [9,23,26,28] and communication delays [2,8,10,17,19,25,30,33–35]. In a networked system, the communication delays are typically random and the delay characteristic varies from sensor to sensor. In other words, it is quite common that a networked system suffers from multiple random delays with different occurrence probabilities [8]. In [23], the successive packet dropouts (SPDs) have been modeled and their impact on the filter performance has been analyzed. Obviously, it is of practical importance to examine how multiple random delays influence the dynamical behavior of the discrete-time networked systems.

Traditionally, in communication community, signals are quantized due to rounding or truncation where the quantizer is essentially a piecewise constant function. The study on quantization problem dates back to early 90s [6] and has received renewed research interest in response to the rapid development of NCSs. Nowadays, the signal quantization is considered as another source for performance degradation of networked systems and a great number of results have been available in the literature, see e.g. [7, 12, 29, 31]. In particular, the quantization has been described by logarithmic types in [11] which can then be converted into the norm-bounded uncertainty. Parameter uncertainties, on the other hand, have long been an important factor that contributes to the complexities of dynamical systems, and the corresponding robust filtering/control problems have attracted considerable research attention in the past few decades [1,5,8,9,15,20,22,32,36]. Nevertheless, in a networked environment, it is a bit too conservative to assume that the uncertainties always occur in a deterministic way. In fact, due to unpredictable changes of the network conditions, the uncertainties may occur randomly with probability laws of certain types and intensity. To account for such a random fashion of parameter uncertainties, the concept of randomly occurring uncertainties (ROUs) has been introduced in [15] and the impact of ROUs on the system behaviors has then been thoroughly examined. Although ROUs have received some initial research attention, it is desirable to consider the simultaneous appearance of ROUs, quantization, successive dropouts and multiple stochastic delays in order to reflect the network-induced phenomena in a more realistic way and this constitutes one of the motivations for the present research.

In reality, it is very often the case that the system dynamics experience constant changes in their structure and parameters caused by a variety of factors such as temperature, changes of the operating point, aging of components, etc. Therefore, time-varying models are of vital importance in engineering practice and, naturally, it is practically significant to design fault estimation schemes directly for time-varying systems. In this case, because of the time-varying nature, one would be more interested in the system's transient performances over a finite period than the traditional steady-state behaviors over the infinite-horizon. It should be pointed out that, in comparison with the numerous literature concerning fault estimation problems over the infinite horizon for time-invariant systems [9, 16, 22, 27, 32], only scattered results have emerged on the finite-horizon fault estimation problems for time-varying systems. This is not surprising because of the following three identified difficulties for finite-horizon fault estimation problems: 1) how to define a reasonable performance criteria such as H_∞ index to evaluate the reliability of a fault estimator; 2) how to analyze the system performance over a finite horizon; and 3) how to design the fault estimator parameters such that the obtained estimator satisfies the defined estimation performance index.

To the best of our knowledge, very little research effort has been made on the fault estimation problems for time-varying networked systems, not to mention the case when the randomly occurring phenomena are also involved in the target plant. With hope to shorten such a gap, in this paper, we are motivated to study the finite-horizon H_∞ fault estimation issue for a class of discrete time-varying stochastic systems with network-induced phenomena. The main novelty lies in three aspects: *1) the plant under consideration is quite comprehensive that covers ROUs, quantization effects, successive packet dropouts, and multiple stochastic delays, hence reflecting the reality more closely; 2) a new finite-horizon H_∞ performance constraint is proposed so as to adequately reflect the effect from the disturbance inputs on the resulting fault estimation systems; 3) a novel fault estimation technique is developed which relies on the forward solution to a set of recursive linear matrix inequalities (RLMIs); and 4) intensive stochastic analysis is conducted to enforce the H-infinity performance for the addressed comprehensive systems in addition to the stochastic stability constraint.*

The rest of this paper is outlined as follows. In Section II, the model addressed in this paper is presented, and some definitions and lemmas are introduced. In Section III, the fault estimation issue is resolved and some sufficient conditions in the form of RLMIs are developed. In Section IV, an illustrative example is provided to demonstrate the effectiveness of the proposed criteria and, finally, conclusions are drawn in Section V.

Notation The notation used here is fairly standard except where otherwise stated. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. $l_2([0,N],\mathbb{R}^n)$ is the n-dimensional vector function's space over $[0,N]$. $[0,N]$ denotes a set of integers ranging from 0 to N. I denotes the identity matrix of compatible dimension. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). A^T represents the transpose of the matrix A. $\|x\|$ describes the Euclidean norm of a vector x. $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x. $\text{diag}\{\dots\}$ stands for a block-diagonal matrix. In symmetric block matrices, the symbol $*$ is used as an ellipsis for terms induced by symmetry. Matrices, if not explicitly specified, are assumed to have compatible dimensions.

II. Problem Formulation and Preliminaries

For presentation clarity, let us start with the following notation:

$$\varphi_{x,k} = \sum_{i=1}^{s} \alpha_{i,k} x_{k-\tau_i} \tag{1}$$

where τ_i $(i = 1, 2, \cdots, s)$ denote the discrete delays that satisfy $0 < \tau_1 < \tau_2 < \cdots < \tau_s$ and occur according to the stochastic variable $\alpha_{i,k}$. Here, s is the number of channels which is fixed. $\alpha_{i,k} \in \mathbb{R}$ $(i = 1, 2, \cdots, s)$ are mutually uncorrelated Bernoulli-distributed white sequences with

$$\text{Prob}\{\alpha_{i,k} = 1\} = \bar{\alpha}_i, \quad \text{Prob}\{\alpha_{i,k} = 0\} = 1 - \bar{\alpha}_i$$

where $\bar{\alpha}_i \in [0, 1]$.

Consider the following class of discrete time-varying stochastic systems defined on $k \in [0,N]$:

$$\begin{cases} x_{k+1} = (A_k + \beta_k \Delta A_k) x_k + h_k(\varphi_{x,k}) + D_{1,k} w_k + F_{1,k} f_k, \\ \tilde{y}_k = C_k x_k + D_{2,k} w_k + F_{2,k} f_k, \\ x_k = \psi_k, \quad -\tau_s \leq k \leq 0, \end{cases} \tag{2}$$

where $x_k \in \mathbb{R}^{n_x}$, $\tilde{y}_k \in \mathbb{R}^{n_y}$ and $f_k \in l_2([0,N],\mathbb{R}^{n_f})$ are the state vector, *the ideal measurement output* and the fault to be estimated. w_k is the exogenous disturbance signal belonging to $l_2([0,N],\mathbb{R}^q)$. $\beta_k \in \mathbb{R}$ is a

Bernoulli-distributed white sequence taking on values of either 0 or 1 with

$$\text{Prob}\{\beta_k = 1\} = \bar{\beta}, \quad \text{Prob}\{\beta_k = 0\} = 1 - \bar{\beta}$$

where $\bar{\beta} \in [0, 1]$ is a known constant. A_k, C_k, $D_{1,k}$, $D_{2,k}$, $F_{1,k}$, $F_{2,k}$ are known, real, time-varying matrices with appropriate dimensions. The parameter uncertainty matrix ΔA_k is a real-valued matrix of the form:

$$\Delta A_k = M_k \mathcal{R}_{1,k} N_k \tag{3}$$

where M_k and N_k are known real matrices with appropriate dimensions, and $\mathcal{R}_{1,k}$ is the unknown time-varying matrix function satisfying

$$\mathcal{R}_{1,k}^T \mathcal{R}_{1,k} \leq I.$$

Moreover, the nonlinear vector-valued function h_k: $\mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ with $h_k(0) = 0$ is assumed to be continuous and satisfies the following condition

$$\|h_k(x)\| \leq \lambda_k \|x\| \tag{4}$$

for all $k \in [0, N]$ and $x \in \mathbb{R}^{n_x}$, where $\lambda_k > 0$ is a known positive scalar.

In a network system, before entering into the fault estimator through a communication channel of limited bandwidth, the signal $\tilde{y}_k$ is first quantized by quantizer $q(\cdot)$ defined by

$$\bar{y}_k := q(\tilde{y}_k) = [q_1(\tilde{y}_{1,k}) \;\; q_2(\tilde{y}_{2,k}) \;\; \cdots \;\; q_{n_y}(\tilde{y}_{n_y,k})]^T. \tag{5}$$

In this paper, the quantizer $q(\cdot)$ is assumed to be of the logarithmic type, that is, the set of quantization levels for each $q_j(\cdot)(1 \leq j \leq n_y)$ is described by

$$\Theta_j = \left\{ \pm\chi_i^{(j)} | \chi_i^{(j)} = \varrho_j^i \chi_0^{(j)}, \; i = 0, \; \pm 1, \; \pm 2, \; \cdots \right\} \cup \{0\}, \quad 0 < \varrho_j < 1, \quad \chi_0^{(j)} > 0.$$

Each of the quantization level corresponds to a segment such that the quantizer maps the whole segment to this quantization level. The logarithmic quantizer $q_j(\cdot)$ is defined as

$$q_j(\tilde{y}_{j,k}) = \begin{cases} \chi_i^{(j)}, & \frac{1}{1+\kappa_j}\chi_i^{(j)} < \tilde{y}_{j,k} \leq \frac{1}{1-\kappa_j}\chi_i^{(j)}, \\ 0, & \tilde{y}_{j,k} = 0, \\ -\chi_i^{(j)}, & -\frac{1}{1-\kappa_j}\chi_i^{(j)} \leq \tilde{y}_{j,k} < -\frac{1}{1+\kappa_j}\chi_i^{(j)}, \end{cases} \tag{6}$$

with $\kappa_j = (1 - \varrho_j)/(1 + \varrho_j)$. By employing the results derived in [11], it follows that $q_j(\tilde{y}_{j,k}) = (1 + \Delta_k^{(j)})\tilde{y}_{j,k}$ with $|\Delta_k^{(j)}| \leq \kappa_j$. Defining $\Delta_k = \text{diag}\{\Delta_k^{(1)}, \; \Delta_k^{(2)}, \; \cdots, \; \Delta_k^{(n_y)}\}$, the measurements after quantization can be expressed as

$$\bar{y}_k = (I + \Delta_k)\tilde{y}_k. \tag{7}$$

Therefore, the quantizing effects have been transformed into sector-bounded uncertainties. In fact, defining $\Gamma = \text{diag}\{\kappa_1, \; \kappa_2, \; \cdots, \kappa_{n_y}\}$ and $\mathcal{R}_{2,k} = \Delta_k \Gamma^{-1}$, we can obtain an unknown real-valued time-varying matrix $\mathcal{R}_{2,k}$ satisfying $\mathcal{R}_{2,k}\mathcal{R}_{2,k}^T = \mathcal{R}_{2,k}^T\mathcal{R}_{2,k} \leq I$.

Remark 1: It is worth mentioning that there are generally two types of quantized communication models, namely, logarithmic quantization [29] and uniform quantization [5]. The differences between these two algorithms are twofold: 1) the logarithmic one provides a non-uniform partition of the state space while the uniform one is concerned with the uniform partition; and 2) the quantization levels of the logarithmic one become finer in the region that is closer to the origin in a logarithmic way and, for the uniform one, the lengths of each two quantization regions are equal. It has been recognized that a logarithmic quantizer is more preferable and practical to be implemented because fewer bits need to be communicated and the quantization error will tend to zero when the signal tends to zero.

In what follows, we assume that an unreliable network medium is present between the physical plant and the fault estimator, and the successive packet dropout phenomenon constitutes another focus of our present research. The measurement received by the fault estimator can be described by

$$y_{f,k} = \delta_k \bar{y}_k + (1-\delta_k) y_{f,k-1} \tag{8}$$

where $y_{f,k} \in \mathbb{R}^{n_y}$ with $y_{f,s} = 0$ $(s < 0)$ is the *actual* signal received by the estimator. $\delta_k \in \mathbb{R}$ is a binary distributed random variable with the following probability:

$$\text{Prob}\{\delta_k = 1\} = \bar{\delta}, \quad \text{Prob}\{\delta_k = 0\} = 1 - \bar{\delta}$$

where $\bar{\delta} \in [0,1]$ is a known constant.

Setting $\tilde{\alpha}_{i,k} := \alpha_{i,k} - \bar{\alpha}_i$, $\tilde{\beta}_k := \beta_k - \bar{\beta}$ and $\tilde{\delta}_k := \delta_k - \bar{\delta}$, we have

$$\mathbb{E}\{\tilde{\alpha}_{i,k}\} = 0, \quad \tilde{\alpha}_i := \mathbb{E}\{\tilde{\alpha}_{i,k}^2\} = \bar{\alpha}_i(1-\bar{\alpha}_i),$$
$$\mathbb{E}\{\tilde{\beta}_k\} = 0, \quad \tilde{\beta} := \mathbb{E}\{\tilde{\beta}_k^2\} = \bar{\beta}(1-\bar{\beta}),$$
$$\mathbb{E}\{\tilde{\delta}_k\} = 0, \quad \tilde{\delta} := \mathbb{E}\{\tilde{\delta}_k^2\} = \bar{\delta}(1-\bar{\delta}).$$

Remark 2: In model (2), the uncertainty ΔA_k term behaves probabilistically owing to the introduction of the random variable β_k, which captures the characteristic of the ROUs. On the other hand, the model presented in (8) has been introduced in [23] to describe the phenomenon of successive packet dropouts (SPDs). For example, if $\delta_k = 1$, one has $y_{f,k} = \bar{y}_k$, which means that the packet dropout phenomenon does not occur; if $\delta_k = 0$, we have $y_{f,k} = y_{f,k-1}$, which means that the measured output at time point k is missing and the received signal at last time is employed to compensate the effect from packet dropouts.

We construct the fault estimator in the following form:

$$\begin{cases} \hat{x}_{k+1} = A_{f,k}\hat{x}_k + B_{f,k} y_{f,k} \\ r_k = L_{f,k}(y_{f,k} - \bar{\delta} C_k \hat{x}_k) \end{cases} \tag{9}$$

where $\hat{x}_k \in \mathbb{R}^{n_x}$ represents the state estimate, $r_k \in l([0,N], \mathbb{R}^{n_f})$ is the fault estimate. $A_{f,k}$, $B_{f,k}$ and $L_{f,k}$ are the estimator gain matrices to be designed.

Our aim in this paper is to find an estimate r_k of the fault f_k, in terms of the *actually* received signal $\{y_{f,k}, 0 \le k \le N\}$, such that the following finite-horizon H_∞ performance constraint is satisfied:

$$\frac{\mathbb{E}\{\sum_{k=0}^{N} ||r_k - f_k||^2\}}{\mathbb{E}\{||x_0 - \hat{x}_0||_{\Theta_0}^2 + s\tau_s \max\limits_{-\tau_s \le i < 0} ||x_i - \hat{x}_i||_{\Theta_1}^2\} + \sum_{k=0}^{N} ||w_k||_{\Theta_2}^2} \le \gamma^2 \tag{10}$$

where $w_k := [v_k^T \quad f_k^T]^T$, γ is a given positive scalar, Θ_0, Θ_1 and Θ_2 are known positive definite weighting matrices. Without loss of generality, the initial state estimates $\hat{x}_k$ $(-\tau_s \le k \le 0)$ are assumed to be zero.

Remark 3: The finite-horizon H_∞ performance index is employed to reflect the effects from the disturbance inputs and the initial states on the dynamics of the fault error $r_k - f_k$. It should be pointed out that, the choices of weight matrixes Θ_0, Θ_1 and Θ_2 are important to adjust the effects from the exogenous disturbances and the initial states on the fault estimator.

For the purpose of simplicity, denote

$$\eta_k := [x_k^T \quad \hat{x}_k^T \quad y_{f,k-1}^T]^T, \ \hbar_k := h_k(\varphi_{x,k}), \ e_k^f := r_k - f_k.$$

From (2), (7), (8), and (9), we obtain the following augmented system

$$\begin{cases} \eta_{k+1} = \bar{\mathcal{A}}_{1,k}\eta_k + \tilde{\beta}_k\Delta\mathcal{A}_{1,k}\eta_k + \tilde{\delta}_k\bar{\mathcal{A}}_{2,k}\eta_k + \mathcal{S}\hbar_k + \bar{\mathcal{B}}_{1,k}\omega_k + \tilde{\delta}_k\bar{\mathcal{B}}_{2,k}\omega_k, \\ e_k^f = \bar{\mathcal{C}}_{1,k}\eta_k + \tilde{\delta}_k\bar{\mathcal{C}}_{2,k}\eta_k + \bar{\mathcal{D}}_{1,k}\omega_k + \tilde{\delta}_k\bar{\mathcal{D}}_{2,k}\omega_k \end{cases} \tag{11}$$

where

$$\begin{aligned}
\bar{\mathcal{A}}_{1,k} &= \mathcal{A}_{1,k} + \bar{\beta}\Delta\mathcal{A}_{1,k} + \bar{\delta}\Delta\mathcal{A}_{2,k}, \quad \bar{\mathcal{A}}_{2,k} = \mathcal{A}_{2,k} + \Delta\mathcal{A}_{2,k}, \quad \mathcal{S} = [I \ 0....0]^T, \\
\mathcal{A}_{1,k} &= \begin{bmatrix} A_k & 0 & 0 \\ \bar{\delta}B_{f,k}C_k & A_{f,k} & (1-\bar{\delta})B_{f,k} \\ \bar{\delta}C_k & 0 & (1-\bar{\delta})I \end{bmatrix}, \quad \Delta\mathcal{A}_{1,k} = \begin{bmatrix} \Delta A_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathcal{A}_{2,k} &= \begin{bmatrix} 0 & 0 & 0 \\ B_{f,k}C_k & 0 & -B_{f,k} \\ C_k & 0 & -I \end{bmatrix}, \quad \Delta\mathcal{A}_{2,k} = \begin{bmatrix} 0 & 0 & 0 \\ B_{f,k}\Delta_k C_k & 0 & 0 \\ \Delta_k \mathcal{C}_k & 0 & 0 \end{bmatrix}, \\
\bar{\mathcal{B}}_{1,k} &= \bar{\delta}\mathcal{B}_k + \mathcal{E}_{1,k} + \bar{\delta}\Delta\mathcal{B}_k, \quad \bar{\mathcal{B}}_{2,k} = \mathcal{B}_k + \Delta\mathcal{B}_k, \\
\mathcal{B}_k &= \begin{bmatrix} 0 & 0 \\ B_{f,k}D_{2,k} & B_{f,k}F_{2,k} \\ D_{2,k} & F_{2,k} \end{bmatrix}, \quad \Delta\mathcal{B}_k = \begin{bmatrix} 0 & 0 \\ B_{f,k}\Delta_k D_{2,k} & B_{f,k}\Delta_k F_{2,k} \\ \Delta_k D_{2,k} & \Delta_k F_{2,k} \end{bmatrix}, \\
\mathcal{E}_{1,k} &= \begin{bmatrix} D_F(k) & 0 & 0 \end{bmatrix}^T, \quad D_F(k) = \begin{bmatrix} D_{1,k} & F_{1,k} \end{bmatrix}^T, \\
\bar{\mathcal{C}}_{1,k} &= \mathcal{C}_{1,k} + \bar{\delta}\Delta\mathcal{C}_k, \quad \bar{\mathcal{C}}_{2,k} = \mathcal{C}_{2,k} + \Delta\mathcal{C}_k, \quad \Delta\mathcal{C}_k = [\ L_{f,k}\Delta_k C_k \quad 0 \quad 0], \\
\mathcal{C}_{1,k} &= [\bar{\delta}L_{f,k}C_k \quad -\bar{\delta}L_{f,k}C_k \quad (1-\bar{\delta})L_{f,k}], \quad \mathcal{C}_{2,k} = [\ L_{f,k}C_k \quad 0 \quad -L_{f,k}], \\
\bar{\mathcal{D}}_{1,k} &= \mathcal{E}_2 + \bar{\delta}\mathcal{D}_k + \bar{\delta}\Delta\mathcal{D}_k, \quad \bar{\mathcal{D}}_{2,k} = \Delta\mathcal{D}_k + \mathcal{D}_k \quad \mathcal{E}_2 = [0 \quad -I], \\
\mathcal{D}_k &= [L_{f,k}D_{2,k} \quad L_{f,k}F_{2,k}], \quad \Delta\mathcal{D}_k = [L_{f,k}\Delta_k D_{2,k} \quad L_{f,k}\Delta_k F_{2,k}].
\end{aligned}$$

III. Main Results

In this section, by resorting to the stochastic analysis technique, some sufficient conditions are derived such that the disturbance rejection attenuation is constrained to a given level by means of the H_∞ performance index.

Denote $\eta_k^* = [\eta_{k-\tau_1}^T \ \eta_{k-\tau_2}^T \ \cdots \ \eta_{k-\tau_s}^T]^T$ and $\tilde{\eta}_k = [\eta_k^T \ \hbar_k^T \ \omega_k^T \ \eta_k^{*T}]^T$. Before proceeding further, we introduce the following lemmas which will be needed for the derivation of our main results.

Lemma 1: (Boyd et al. [3]) Let $M = M^T$ and W and V be real matrices of appropriate dimensions with V satisfying $V^TV \leq I$. Then, $M + UVW + W^TV^TU^T \leq 0$, if and only if there exists a positive scalar $\varrho > 0$ such that $M + \varrho^{-1}UU^T + \varrho W^TW < 0$ or equivalently

$$\begin{bmatrix} M & U & \varrho W^T \\ * & -\varrho I & 0 \\ * & * & -\varrho I \end{bmatrix} < 0.$$

Lemma 2: For the given symmetric positive definite matrices $\mathcal{P}_k$ and $\mathcal{Q}_k$, the following cost function

$$V_k = \eta_k^T \mathcal{P}_k \eta_k + \sum_{i=1}^{s} \sum_{j=k-\tau_i}^{k-1} \eta_j^T \mathcal{Q}_j \eta_j \tag{12}$$

satisfies

$$\mathbb{E}\{\Delta V_k\} := \mathbb{E}\{V_{k+1} - V_k\} = \mathbb{E}\{\tilde{\eta}_k^T \Pi_1^k \tilde{\eta}_k\} \tag{13}$$

where

$$\begin{aligned}
\Pi_1^k &= \begin{bmatrix} \Pi_{11}^k & \Pi_{12}^k & \Pi_{13}^k & 0 \\ * & \Pi_{22}^k & \Pi_{23}^k & 0 \\ * & * & \Pi_{33}^k & 0 \\ * & * & * & \Pi_{44}^k \end{bmatrix}, \\
\Pi_{11}^k &= \bar{\mathcal{A}}_{1k}^T \mathcal{P}_{k+1} \bar{\mathcal{A}}_{1k} + \tilde{\beta} \Delta\mathcal{A}_{1k}^T \mathcal{P}_{k+1} \Delta\mathcal{A}_{1k} + \tilde{\delta} \bar{\mathcal{A}}_{2k}^T \mathcal{P}_{k+1} \bar{\mathcal{A}}_{2k} - \mathcal{P}_k + s\mathcal{Q}_k, \\
\Pi_{12}^k &= \bar{\mathcal{A}}_{1k}^T \mathcal{P}_{k+1} \mathcal{S}, \quad \Pi_{13}^k = \bar{\mathcal{A}}_{1k}^T \mathcal{P}_{k+1} \bar{\mathcal{B}}_{1k} + \tilde{\delta} \bar{\mathcal{A}}_{2k}^T \mathcal{P}_{k+1} \bar{\mathcal{B}}_{2k}, \\
\Pi_{22}^k &= \mathcal{S}^T \mathcal{P}_{k+1} \mathcal{S}, \quad \Pi_{23}^k = \mathcal{S}^T \mathcal{P}_{k+1} \bar{\mathcal{B}}_{1k}, \quad \Pi_{33}^k = \bar{\mathcal{B}}_{1k}^T P_{k+1} \bar{\mathcal{B}}_{1k} + \tilde{\delta} \bar{\mathcal{B}}_{2k}^T P_{k+1} \bar{\mathcal{B}}_{2k}, \\
\Pi_{44}^k &= -\mathrm{diag}\big\{\mathcal{Q}_{k-\tau_1}, \mathcal{Q}_{k-\tau_2}, \ \cdots \ , \mathcal{Q}_{k-\tau_s}\big\}.
\end{aligned}$$

Proof: By calculating the difference of the first term in V_k along the trajectory of the system (11) and taking the mathematical expectation, we have

$$\begin{aligned}
& \mathbb{E}\{\eta_{k+1}^T \mathcal{P}_{k+1} \eta_{k+1} - \eta_k^T \mathcal{P}_k \eta_k\} \\
= {} & \mathbb{E}\Big\{ \big(\bar{\mathcal{A}}_{1k}\eta_k + \tilde{\beta}_k \Delta\mathcal{A}_{1k}\eta_k + \tilde{\delta}_k \bar{\mathcal{A}}_{2k}\eta_k + \mathcal{S}\hbar_k + \bar{\mathcal{B}}_{1k}\omega_k + \tilde{\delta}_k \bar{\mathcal{B}}_{2k}\omega_k\big)^T \mathcal{P}_{k+1} \\
& \times \big(\bar{\mathcal{A}}_{1k}\eta_k + \tilde{\beta}_k \Delta\mathcal{A}_{1k}\eta_k + \tilde{\delta}_k \bar{\mathcal{A}}_{2k}\eta_k + \mathcal{S}\hbar_k + \bar{\mathcal{B}}_{1k}\omega_k + \tilde{\delta}_k \bar{\mathcal{B}}_{2k}\omega_k\big) - \eta_k^T \mathcal{P}_k \eta_k \Big\} \\
= {} & \mathbb{E}\Big\{ \eta_k^T \bar{\mathcal{A}}_{1k}^T \mathcal{P}_{k+1} \bar{\mathcal{A}}_{1k}\eta_k + 2\eta_k^T \bar{\mathcal{A}}_{1k}^T \mathcal{P}_{k+1} \mathcal{S}\hbar_k + 2\eta_k^T \bar{\mathcal{A}}_{1k}^T \mathcal{P}_{k+1} \bar{\mathcal{B}}_{1k}\omega_k \\
& + \tilde{\beta}\eta_k^T \Delta\mathcal{A}_{1k}^T \mathcal{P}_{k+1} \Delta\mathcal{A}_{1k}\eta_k + \tilde{\delta}\eta_k^T \bar{\mathcal{A}}_{2k}^T \mathcal{P}_{k+1} \bar{\mathcal{A}}_{2k}\eta_k + 2\tilde{\delta}\eta_k^T \bar{\mathcal{A}}_{2k}^T \mathcal{P}_{k+1} \bar{\mathcal{B}}_{2k}\omega_k \\
& + \hbar_k^T \mathcal{S}^T P_{k+1} \mathcal{S}\hbar_k + 2\hbar_k^T \mathcal{S}^T P_{k+1} \bar{\mathcal{B}}_{1k}\omega_k + \omega_k^T \bar{\mathcal{B}}_{1k}^T P_{k+1} \bar{\mathcal{B}}_{1k}\omega_k \\
& + \tilde{\delta}\omega_k^T \bar{\mathcal{B}}_{2k}^T P_{k+1} \bar{\mathcal{B}}_{2k}\omega_k - \eta_k^T \mathcal{P}_k \eta_k \Big\}.
\end{aligned} \tag{14}$$

On the other hand, it is not difficult to show that

$$\begin{aligned}
& \mathbb{E}\Big\{ \sum_{i=1}^{s} \sum_{j=k+1-\tau_i}^{k} \eta_j^T \mathcal{Q}_j \eta_j - \sum_{i=1}^{s} \sum_{j=k-\tau_i}^{k-1} \eta_j^T \mathcal{Q}_j \eta_j \Big\} \\
= {} & \mathbb{E}\Big\{ \sum_{i=1}^{s} \Big(\sum_{j=k+1-\tau_i}^{k} \eta_j^T \mathcal{Q}_j \eta_j - \sum_{j=k-\tau_i}^{k-1} \eta_j^T \mathcal{Q}_j \eta_j \Big) \Big\} \\
= {} & \sum_{i=1}^{s} \mathbb{E}\Big\{ \eta_k^T \mathcal{Q}_k \eta_k - \eta_{k-\tau_i}^T Q_{k-\tau_i} \eta_{k-\tau_i} \Big\} \\
= {} & \mathbb{E}\Big\{ s\eta_k^T \mathcal{Q}_k \eta_k - \eta_k^{*T} \mathrm{diag}\big\{\mathcal{Q}_{k-\tau_1}, \mathcal{Q}_{k-\tau_2}, \ \cdots \ , \mathcal{Q}_{k-\tau_s}\big\} \eta_k^* \Big\}.
\end{aligned} \tag{15}$$

Obviously, it follows from (14) and (15) that the equality (13) holds, which completes the proof. ■

Next, let us proceed with the $\mathcal{H}_\infty$ performance of the augmented system (11), i.e., presenting sufficient conditions under which the performance index is achieved for a given estimator.

Theorem 1: Consider the nonlinear system (2) in the presence of ROUs, quantization effects, successive packet dropouts as well as multiple stochastic delays. For the given positive scalar $\gamma > 0$, positive definite matrices $\Theta_i > 0$ $(i = 0, 1, 2)$ and fault estimator parameters $A_f(k)$, $B_f(k)$ and $L_f(k)$ in (9), the augmented system (11) satisfies the desired H_∞ performance requirement defined in (10) if there exist a family of positive scalars $\{\varepsilon_k\}_{k\in[0,N]}$ and two sequences of positive definite matrices $\{\mathcal{P}_k\}_{k\in[0,N+1]}$, $\{\mathcal{Q}_k\}_{k\in[-\tau_s,N]}$ satisfying the initial conditions

$$\mathcal{P}_0 \;\leq\; \gamma^2\Theta_0, \quad \mathcal{Q}_k \;\leq\; \gamma^2\Theta_1, \quad k = -\tau_s, -\tau_s + 1, \cdots, -1 \tag{16}$$

and the following recursive matrix inequality

$$\Pi_2^k = \begin{bmatrix} \bar{\Pi}_{11}^k & \Pi_{12}^k & \bar{\Pi}_{13}^k & 0 \\ * & \bar{\Pi}_{22}^k & \Pi_{23}^k & 0 \\ * & * & \bar{\Pi}_{33}^k & 0 \\ * & * & * & \bar{\Pi}_{44}^k \end{bmatrix} < 0, \tag{17}$$

for all $0 \leq k \leq N$, where

$$\begin{aligned} \bar{\Pi}_{11}^k &= \Pi_{11}^k + \bar{\mathcal{C}}_{1k}^T\bar{\mathcal{C}}_{1k} + \tilde{\delta}\bar{\mathcal{C}}_{2k}^T\bar{\mathcal{C}}_{2k}, \quad \bar{\Pi}_{13}^k = \Pi_{13}^k + \bar{\mathcal{C}}_{1k}^T\bar{\mathcal{D}}_{1k} + \tilde{\delta}\bar{\mathcal{C}}_{2k}^T\bar{\mathcal{D}}_{2k}, \\ \bar{\Pi}_{22}^k &= \Pi_{22}^k - \varepsilon_k I, \quad \bar{\Pi}_{33}^k = \Pi_{33}^k + \bar{\mathcal{D}}_{1k}^T\bar{\mathcal{D}}_{1k} + \tilde{\delta}\bar{\mathcal{D}}_{2k}^T\bar{\mathcal{D}}_{2k} - \gamma^2\Theta_2, \\ \bar{\Pi}_{44}^k &= \Pi_{44}^k + \varepsilon_k\lambda_k^2\Lambda, \quad \Lambda = \big[\; \bar{\alpha}_i\bar{\alpha}_j I \;\big]_{s\times s} - \mathrm{diag}\{\tilde{\alpha}_1 I, \tilde{\alpha}_2 I, \cdots, \tilde{\alpha}_s I\}. \end{aligned}$$

Proof: In order to analyze the H_∞ performance of the system (11), define the following cost function

$$\mathcal{J}(k) = \eta_{k+1}^T\mathcal{P}_{k+1}\eta_{k+1} - \eta_k^T\mathcal{P}_k\eta_k + \sum_{i=1}^{s}\Big(\sum_{j=k+1-\tau_i}^{k}\eta_j^T\mathcal{Q}_j\eta_j - \sum_{j=k-\tau_i}^{k-1}\eta_j^T\mathcal{Q}_j\eta_j\Big). \tag{18}$$

Denoting $\Lambda_k = [\; \alpha_1(k)I, \;\alpha_2(k)I, \;\cdots, \;\alpha_s(k)I \;]$, it can be readily verified from (4) that the nonlinear function $\hbar_k$ satisfies

$$\hbar_k^T\hbar_k - \lambda_k^2\eta_k^{*T}(\Lambda_k^T\Lambda_k)\eta_k^* \leq 0. \tag{19}$$

Substituting the above inequality into (18) results in

$$\mathbb{E}\{\mathcal{J}(k)\} \leq \mathbb{E}\Big\{\tilde{\eta}_k^T\Pi_1^k\tilde{\eta}_k - \varepsilon_k\big\{\hbar_k^T\hbar_k - \lambda_k^2\eta_k^{*T}(\Lambda_k^T\Lambda_k)\eta_k^*\big\}\Big\}. \tag{20}$$

On the other hand, it follows from (11) that

$$\begin{aligned} \mathbb{E}\{\|e_k^f\|^2\} &= \mathbb{E}\Big\{\big(\bar{\mathcal{C}}_{1k}\eta_k + \tilde{\delta}_k\mathcal{C}_{2k}\eta_k + \bar{\mathcal{D}}_{1k}\omega_k + \tilde{\delta}_k\bar{\mathcal{D}}_{2k}\omega_k\big)^T \\ &\quad \times \big(\bar{\mathcal{C}}_{1k}\eta_k + \tilde{\delta}_k\mathcal{C}_{2k}\eta_k + \bar{\mathcal{D}}_{1k}\omega_k + \tilde{\delta}_k\bar{\mathcal{D}}_{2k}\omega_k\big)\Big\} \\ &= \mathbb{E}\Big\{\eta_k^T\bar{\mathcal{C}}_{1k}^T\bar{\mathcal{C}}_{1k}\eta_k + 2\eta_k^T\bar{\mathcal{C}}_{1k}\bar{\mathcal{D}}_{1k}\omega_k + \tilde{\delta}\eta_k^T\bar{\mathcal{C}}_{2k}^T\bar{\mathcal{C}}_{2k}\eta_k \\ &\quad + 2\tilde{\delta}\eta_k^T\bar{\mathcal{C}}_{2k}^T\bar{\mathcal{D}}_{2k}\omega_k + \omega_k^T\bar{\mathcal{D}}_{1k}^T\bar{\mathcal{D}}_{1k}\omega_k + \tilde{\delta}\omega_k^T\bar{\mathcal{D}}_{2k}^T\bar{\mathcal{D}}_{2k}\omega_k\Big\}. \end{aligned} \tag{21}$$

Furthermore, adding the zero term $\mathbb{E}\{||e_k^f||^2-\gamma^2||w_k||^2_{\Theta_2}-\big(||e_k^f||^2-\gamma^2||w_k||^2_{\Theta_2}\big)\}$ to $\mathbb{E}\{\mathcal{J}(k)\}$ yields

$$\begin{aligned}\mathbb{E}\{\mathcal{J}(k)\}\leq\ &\mathbb{E}\Big\{\tilde{\eta}_k^T\Pi_1^k\tilde{\eta}_k-\varepsilon_k\hbar_k^T\hbar_k+\lambda_k^2\varepsilon_k\eta_k^{*T}\Lambda\eta_k^*+\eta_k^T\bar{\mathcal{C}}_{1k}^T\bar{\mathcal{C}}_{1k}\eta_k\\&+2\eta_k^T\bar{\mathcal{C}}_{1k}\bar{\mathcal{D}}_{1k}\omega_k+\tilde{\delta}\eta_k^T\bar{\mathcal{C}}_{2k}^T\bar{\mathcal{C}}_{2k}\eta_k+2\tilde{\delta}\eta_k^T\bar{\mathcal{C}}_{2k}^T\bar{\mathcal{D}}_{2k}\omega_k+\omega_k^T\bar{\mathcal{D}}_{1k}^T\bar{\mathcal{D}}_{1k}\omega_k\\&+\tilde{\delta}\omega_k^T\bar{\mathcal{D}}_{2k}^T\bar{\mathcal{D}}_{2k}\omega_k-\gamma^2||w_k||^2_{\Theta_2}\Big\}-\mathbb{E}\Big\{||e_k^f||^2-\gamma^2||w_k||^2_{\Theta_2}\Big\}\\=\ &\mathbb{E}\Big\{\tilde{\eta}_k^T\Pi_2^k\tilde{\eta}_k\Big\}-\mathbb{E}\Big\{||e_k^f||^2-\gamma^2||w_k||^2_{\Theta_2}\Big\}.\end{aligned}\tag{22}$$

Summing up (22) on both sides from 0 to N with respect to k, we obtain

$$\begin{aligned}\sum_{k=0}^{N}\mathbb{E}\{\mathcal{J}(k)\}&=\mathbb{E}\{V_{N+1}\}-\mathbb{E}\{V_0\}\\&\leq\mathbb{E}\Big\{\sum_{k=0}^{N}\tilde{\eta}_k^T\Pi_2^k\tilde{\eta}_k\Big\}-\mathbb{E}\Big\{\sum_{k=0}^{N}\big(||e_k^f||^2-\gamma^2||w_k||^2_{\Theta_2}\big)\Big\}\end{aligned}\tag{23}$$

which implies that

$$\begin{aligned}\mathbb{E}\{V_{N+1}\}\leq\ &-\mathbb{E}\Big\{\sum_{k=0}^{N}\big(||e_k^f||^2-\gamma^2||w_k||^2_{\Theta_2}\big)\Big\}+\mathbb{E}\{V_0\}\\=\ &-\mathbb{E}\Big\{\sum_{k=0}^{N}\big(||e_k^f||^2-\gamma^2||w_k||^2_{\Theta_2}\big)\Big\}+\mathbb{E}\Big\{\eta_0^T\mathcal{P}_0\eta_0+\sum_{i=1}^{s}\sum_{j=-\tau_i}^{-1}\eta_j^T\mathcal{Q}_j\eta_j\Big\}\\=\ &-\mathbb{E}\Big\{\sum_{k=0}^{N}\big(||e_k^f||^2-\gamma^2||w_k||^2_{\Theta_2}\big)-\gamma^2\Big(\eta_0^T\Theta_0\eta_0+s\tau_s\max_{-\tau_s\leq i<0}\eta_i^T\Theta_1\eta_i\Big)\Big\}\\&+\mathbb{E}\Big\{\eta_0^T\mathcal{P}_0\eta_0+\sum_{i=1}^{s}\sum_{j=-\tau_i}^{-1}\eta_j^T\mathcal{Q}_j\eta_j-\gamma^2\Big(\eta_0^T\Theta_0\eta_0+s\tau_s\max_{-\tau_s\leq i<0}\eta_i^T\Theta_1\eta_i\Big)\Big\}.\end{aligned}\tag{24}$$

According to the above inequality and the condition (16), it is easy to find that the H_∞ performance index (10) holds, which completes the proof. ■

Based on the analysis results with a given fault estimator, we are now ready to handle the fault estimator design issue. In the following theorem, sufficient conditions are provided for the existence of the desired fault estimators.

Theorem 2: Consider the nonlinear system (2) in the presence of ROUs, quantization effects, successive packet dropouts as well as multiple stochastic delays. For the given positive scalar $\gamma>0$ and positive definite matrices $\Theta_i>0$ $(i=0,1,2)$, the augmented system (11) satisfies the desired H_∞ performance requirement in (10) if there exist families of positive scalars $\{\varepsilon_k,\ \varrho_{1k},\ \varrho_{2k}\}_{k\in[0,N]}$, positive definite matrices $\{P_k,\bar{P}_k\}_{k\in[0,N+1]}$ and $\{Q_k,\bar{Q}_k\}_{k\in[-\tau_s,N]}$, and real-valued matrices $\{Y_{1,k},\ Y_{2,k},\ L_{f,k}\}_{k\in[0,N]}$ satisfying the initial conditions

$$\mathcal{P}_0\ \leq\ \gamma^2\Theta_0,\quad \mathcal{Q}_j\ \leq\ \gamma^2\Theta_1,\quad j=-\tau_s,-\tau_s+1,\cdots,-1\tag{25}$$

and the following recursive matrix inequality

$$\begin{bmatrix} \Xi_{11}^k & \Xi_{12}^k & 0 & 0 & \varrho_{1,k}\tilde{N}_k^{1T} & \varrho_{2,k}\tilde{N}_k^{2T} \\ * & \Xi_{22}^k & \bar{\Upsilon}_{17}^k & \bar{\Upsilon}_{18}^k & 0 & 0 \\ * & * & -\varrho_{1,k}I & 0 & 0 & 0 \\ * & * & * & -\varrho_{2,k}I & 0 & 0 \\ * & * & * & * & -\varrho_{1,k}I & 0 \\ * & * & * & * & * & -\varrho_{2,k}I \end{bmatrix} < 0 \tag{26}$$

where

$$\begin{aligned}
\Xi_{11}^k &= \mathrm{diag}\{-\mathcal{P}_k + s\mathcal{Q}_k,\ -\varepsilon_k I,\ -\gamma^2\Theta_2,\ \bar{\Pi}_{44}^k\}, \quad \mathcal{P}_k = \mathrm{diag}\{P_k, P_k, \bar{P}_k\}, \\
\Xi_{12}^k &= \begin{bmatrix} (\bar{\Upsilon}_{12}^k)^T & (\bar{\Upsilon}_{13}^k)^T & 0 & (\bar{\Upsilon}_{15}^k)^T & (\bar{\Upsilon}_{16}^k)^T \end{bmatrix}, \quad \mathscr{S}_k = [\, P_k\ 0\ \cdots\ 0\,]^T, \\
\Xi_{22}^k &= \mathrm{diag}\{-\mathcal{P}_{k+1},\ -\mathcal{P}_{k+1},\ -\mathcal{P}_{k+1},\ -I,\ -I\}, \quad \mathcal{Q}_k = \mathrm{diag}\{Q_k, Q_k, \bar{Q}_k\} \\
\bar{\Upsilon}_{12}^k &= \begin{bmatrix} \mathscr{A}_{1,k} & \mathscr{S}_k & \bar{\delta}\mathscr{B}_k + \mathscr{D}_k & 0 \end{bmatrix}, \quad \bar{\Upsilon}_{13}^k = \begin{bmatrix} \sqrt{\tilde{\delta}}\mathscr{A}_{2,k} & 0 & \sqrt{\tilde{\delta}}\mathscr{B}_k & 0 \end{bmatrix}, \\
\bar{\Upsilon}_{15}^k &= \begin{bmatrix} \mathcal{C}_{1k} & 0 & \mathcal{E}_2 + \bar{\delta}\mathcal{D}_k & 0 \end{bmatrix}, \quad \bar{\Upsilon}_{16}^k = \begin{bmatrix} \sqrt{\tilde{\delta}}\mathcal{C}_{2,k} & 0 & \sqrt{\tilde{\delta}}\mathcal{D}_k & 0 \end{bmatrix}, \\
\bar{\Upsilon}_{17}^k &= [\bar{\beta}\bar{\Upsilon}_{19}^k \quad 0 \quad \sqrt{\tilde{\beta}}\bar{\Upsilon}_{19}^k \quad 0 \quad 0]^T, \quad \mathcal{Y}_k = [\, 0 \quad Y_{2,k}^T \quad P_{k+1}\,]^T \\
\bar{\Upsilon}_{18}^k &= [\bar{\delta}\mathcal{Y}_k^T \quad \sqrt{\tilde{\delta}}\mathcal{Y}_k^T \quad 0 \quad \bar{\delta}L_{f,k}^T \quad \sqrt{\tilde{\delta}}L_{f,k}^T]^T, \quad \bar{\Upsilon}_{19}^k = [\, M_k^T P_{k+1} \quad 0 \quad 0\,]^T, \\
\tilde{\mathcal{N}}_k^1 &= \begin{bmatrix} [N_k \quad 0 \quad 0] & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathcal{N}}_k^2 = \begin{bmatrix} \Gamma[C_k \quad 0 \quad 0] & 0 & \Gamma[D_{2,k}\ F_{2,k}] & 0 \end{bmatrix}, \\
\mathscr{A}_{1,k} &= \begin{bmatrix} P_{k+1}A_k & 0 & 0 \\ \bar{\delta}Y_{2,k}C_k & Y_{1,k} & (1-\bar{\delta})Y_{2,k} \\ \bar{\delta}P_{k+1}C_k & 0 & (1-\bar{\delta})P_{k+1} \end{bmatrix}, \quad \mathscr{B}_k = \begin{bmatrix} 0 & 0 \\ Y_{2,k}D_{2,k} & Y_{2,k}F_{2,k} \\ P_{k+1}D_{2,k} & P_{k+1}F_{2,k} \end{bmatrix}, \\
\mathscr{A}_{2,k} &= \begin{bmatrix} 0 & 0 & 0 \\ Y_{2,k}C_k & 0 & -Y_{2,k} \\ P_{k+1}C_k & 0 & -P_{k+1} \end{bmatrix}, \quad \mathscr{D}_k = \begin{bmatrix} P_{k+1}D_{1,k} & P_{k+1}F_{1,k} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Furthermore, if (26) holds, then the other two parameters of the fault estimator in the form of (9) are given by $A_{f,k} = P_{k+1}^{-1}Y_{1,k}$ and $B_{f,k} = P_{k+1}^{-1}Y_{2,k}$.

Proof: First, (17) can be rewritten as

$$\begin{aligned}
\Pi_2^k = \Xi_{11}^k &+ (\Upsilon_{12}^k)^T\mathcal{P}_{k+1}\Upsilon_{12}^k + (\Upsilon_{13}^k)^T\mathcal{P}_{k+1}\Upsilon_{13}^k \\
&+ (\Upsilon_{14}^k)^T\mathcal{P}_{k+1}\Upsilon_{14}^k + (\Upsilon_{15}^k)^T\Upsilon_{15}^k + (\Upsilon_{16}^k)^T\Upsilon_{16}^k < 0
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
\Upsilon_{12}^k &= \begin{bmatrix} \bar{\mathcal{A}}_{1,k} & \mathcal{S} & \bar{\mathcal{B}}_{1,k} & 0 \end{bmatrix}, \quad \Upsilon_{13}^k = \begin{bmatrix} \sqrt{\tilde{\delta}}\bar{\mathcal{A}}_{2,k} & 0 & \sqrt{\tilde{\delta}}\bar{\mathcal{B}}_{2,k} & 0 \end{bmatrix}, \\
\Upsilon_{14}^k &= \begin{bmatrix} \sqrt{\tilde{\beta}}\Delta\mathcal{A}_{1,k} & 0 & 0 & 0 \end{bmatrix}, \quad \Upsilon_{15}^k = \begin{bmatrix} \bar{\mathcal{C}}_{1,k} & 0 & \bar{\mathcal{D}}_{1,k} & 0 \end{bmatrix}, \\
\Upsilon_{16}^k &= \begin{bmatrix} \sqrt{\tilde{\delta}}\bar{\mathcal{C}}_{2,k} & 0 & \sqrt{\tilde{\delta}}\bar{\mathcal{D}}_{2,k} & 0 \end{bmatrix}.
\end{aligned}$$

Then, by exploiting the Schur Complement Lemma, the above inequality is equivalent to

$$\Pi_3^k := \begin{bmatrix} \Xi_{11}^k & (\tilde{\Upsilon}_{12}^k + \Delta\Upsilon_{12}^k)^T \mathcal{P}_{k+1} & (\tilde{\Upsilon}_{13}^k + \Delta\Upsilon_{13}^k)^T \mathcal{P}_{k+1} & \Upsilon_{14}^{kT}\mathcal{P}_{k+1} & (\bar{\Upsilon}_{15}^k + \Delta\Upsilon_{15}^k)^T & (\bar{\Upsilon}_{16}^k + \Delta\Upsilon_{16}^k)^T \\ * & -\mathcal{P}_{k+1} & 0 & 0 & 0 & 0 \\ * & * & -\mathcal{P}_{k+1} & 0 & 0 & 0 \\ * & * & * & -\mathcal{P}_{k+1} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0$$

where

$$\begin{aligned} \tilde{\Upsilon}_{12}^k &= \begin{bmatrix} \mathcal{A}_{1,k} & \mathcal{S} & \bar{\delta}\mathcal{B}_k + \mathcal{E}_1 & 0 \end{bmatrix}, \quad \tilde{\Upsilon}_{13}^k = \begin{bmatrix} \sqrt{\tilde{\bar{\delta}}}\mathcal{A}_{2,k} & 0 & \sqrt{\tilde{\bar{\delta}}}\mathcal{B}_k & 0 \end{bmatrix}, \\ \Delta\Upsilon_{12}^k &= \begin{bmatrix} \bar{\beta}\Delta\mathcal{A}_{1,k} + \bar{\delta}\Delta\mathcal{A}_{2,k} & 0 & \bar{\delta}\Delta\mathcal{B}_k & 0 \end{bmatrix}, \quad \Delta\Upsilon_{13}^k = \begin{bmatrix} \sqrt{\tilde{\bar{\delta}}}\Delta\mathcal{A}_{2,k} & 0 & \sqrt{\tilde{\bar{\delta}}}\Delta\mathcal{B}_k & 0 \end{bmatrix}, \\ \Delta\Upsilon_{15}^k &= \begin{bmatrix} \bar{\delta}\Delta\mathcal{C}_k & 0 & \bar{\delta}\Delta\mathcal{D}_k & 0 \end{bmatrix}, \quad \Delta\Upsilon_{16}^k = \begin{bmatrix} \sqrt{\tilde{\bar{\delta}}}\Delta\mathcal{C}_k & 0 & \sqrt{\tilde{\bar{\delta}}}\Delta\mathcal{D}_k & 0 \end{bmatrix}. \end{aligned}$$

Note that Π_3^k can be decomposed as follows:

$$\begin{aligned} \Pi_3^k := \breve{\Pi}_3^k + \Delta\Pi_3^k = & \begin{bmatrix} \Xi_{11}^k & \tilde{\Upsilon}_{12}^{kT}\mathcal{P}_{k+1} & \tilde{\Upsilon}_{13}^{kT}\mathcal{P}_{k+1} & 0 & \bar{\Upsilon}_{15}^{kT} & \bar{\Upsilon}_{16}^{kT} \\ * & -\mathcal{P}_{k+1} & 0 & 0 & 0 & 0 \\ * & * & -\mathcal{P}_{k+1} & 0 & 0 & 0 \\ * & * & * & -\mathcal{P}_{k+1} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix} \\ & + \begin{bmatrix} 0 & \Delta\Upsilon_{12}^{kT}\mathcal{P}_{k+1} & \Delta\Upsilon_{13}^{kT}\mathcal{P}_{k+1} & \Upsilon_{14}^{kT}\mathcal{P}_{k+1} & \Delta\Upsilon_{15}^{kT} & \Delta\Upsilon_{16}^{kT} \\ * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}. \end{aligned}$$

Furthermore, one has

$$\begin{aligned} \Delta\Pi_3^k = & \begin{bmatrix} 0 \\ \tilde{\mathcal{M}}_k^1 \end{bmatrix} \mathcal{R}_{1,k} \begin{bmatrix} \tilde{\mathcal{N}}_k^1 & 0 \end{bmatrix} + \begin{bmatrix} \tilde{\mathcal{N}}_k^1 & 0 \end{bmatrix}^T \mathcal{R}_{1,k}^T \begin{bmatrix} 0 \\ \tilde{\mathcal{M}}_k^1 \end{bmatrix}^T \\ & + \begin{bmatrix} 0 \\ \tilde{\mathcal{M}}_k^2 \end{bmatrix} \mathcal{R}_{2,k} \begin{bmatrix} \tilde{\mathcal{N}}_k^2 & 0 \end{bmatrix} + \begin{bmatrix} \tilde{\mathcal{N}}_k^2 & 0 \end{bmatrix}^T \mathcal{R}_{2,k}^T \begin{bmatrix} 0 \\ \tilde{\mathcal{M}}_k^2 \end{bmatrix}^T \end{aligned}$$

where

$$\tilde{\mathcal{M}}_k^1 = [\bar{\beta}(\mathcal{P}_{k+1}\mathcal{M}_k)^T \quad 0 \quad \sqrt{\tilde{\beta}}(\mathcal{P}_{k+1}\mathcal{M}_k)^T \quad 0 \quad 0]^T,$$
$$\tilde{\mathcal{M}}_k^2 = [\bar{\delta}(\mathcal{P}_{k+1}\tilde{\mathcal{B}}_k)^T \quad \sqrt{\tilde{\delta}}(\mathcal{P}_{k+1}\tilde{\mathcal{B}}_k)^T \quad 0 \quad \bar{\delta}L_{f,k}^T \quad \sqrt{\tilde{\delta}}L_{f,k}^T]^T,$$
$$\mathcal{M}_k = [\, M_k^T \quad 0 \quad 0\,]^T,\ \tilde{\mathcal{B}}_k = [\, 0 \quad B_{f,k}^T \quad I\,]^T.$$

In terms of the above equality and Lemma 1, it is easy to find that (27) holds if the following inequality

$$\begin{bmatrix} \Xi_{11}^k & \Xi_{12}^k & 0 & 0 & \varrho_{1,k}\tilde{N}_k^{1T} & \varrho_{2,k}\tilde{N}_k^{2T} \\ * & \Xi_{22}^k & \tilde{\mathcal{M}}_k^1 & \tilde{\mathcal{M}}_k^2 & 0 & 0 \\ * & * & -\varrho_{1,k}I & 0 & 0 & 0 \\ * & * & * & -\varrho_{2,k}I & 0 & 0 \\ * & * & * & * & -\varrho_{1,k}I & 0 \\ * & * & * & * & * & -\varrho_{2,k}I \end{bmatrix} < 0 \tag{28}$$

is true. Defining $Y_{1,k} := P_{k+1}A_{f,k}$ and $Y_{2,k} := P_{k+1}B_{f,k}$, it is not difficult to see that (28) is equivalent to (26). Finally, based on Theorem 1, the desired H_∞ performance requirement of the augmented system (11) is guaranteed, which completes the proof. ■

Remark 4: For the stochastic time-varying model (2) under consideration in this paper, there are five main aspects which complicate the design of the fault estimator, i.e. ROUs, quantization effects, successive packet dropouts, multiple stochastic delays and nonlinearities. In Theorem 2, sufficient conditions, which include all of the information on these five aspects, are established for a finite-horizon fault estimator to satisfy the prescribed H_∞ performance requirement. The corresponding solvability conditions for the desired fault estimator gains are expressed in terms of the feasibility of a series of recursive linear matrix inequalities (RLMIs). Note that the RLMIs provided in Theorem 2 are time-varying and non-strict, which depend on both the variable matrices at the current time $\mathcal{P}_k$ and $\mathcal{Q}_k$ and the variable matrices at the next time $\mathcal{P}_{k+1}$ and $\mathcal{Q}_{k+1}$. In addition, the solution of the RLMI is also dependent on the choices of initial conditions. Compared to the traditional static (non-recursive) LMIs, our developed algorithm would enjoy the advantage of less conservatism since more information about the system state is employed.

IV. A Numerical Example

In this section, a numerical example is presented to illustrate the effectiveness of the proposed design scheme of finite-horizon H_∞ fault estimator for discrete time-varying nonlinear systems (2) with ROUs, quantization effects, successive packet dropouts as well as multiple stochastic delays. The corresponding parameters are given as follows

$$\begin{aligned} A_k &= \begin{bmatrix} 0.24 & -0.18 \\ 0.36 & 0.20+0.07\sin(k) \end{bmatrix},\quad C_k = \begin{bmatrix} 0.13 \\ -0.13 \end{bmatrix}^T,\quad D_{1,k} = \begin{bmatrix} 0.08 \\ -0.05 \end{bmatrix}^T, \\ D_{2,k} &= 0.12,\ F_{1,k} = \begin{bmatrix} -0.28 \\ -0.40 \end{bmatrix}^T,\ F_{2,k} = -0.12,\ M_k = \begin{bmatrix} 0.05 \\ -0.10 \end{bmatrix},\ N_k = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}^T. \end{aligned}$$

Let the nonlinear vector-valued function $h_k(x)$ be

$$h_k(x) = [\ 0.75\sin(x_1) \quad 0.75\sin(x_2)\ ]^T$$

where x_i $(i = 1, 2)$ denotes the i-th element of the vector x. The probabilities of delays, ROUs and packet dropouts are, respectively, taken as $\alpha_1 = 0.15$, $\alpha_2 = 0.05$, $\beta = 0.85$ and $\delta = 0.85$. The time-delays τ_1 and τ_2 are 1 and 3. The parameters of the logarithmic quantizer are $\chi_0^{(i)} = 0.005$ and $\varrho_i = 0.9$ $(i = 1, 2, \cdots, n_y)$. In addition, in this example, the H_∞ performance level γ, the time-horizon N and three positive definite matrices Θ_0, Θ_1 and Θ_2 in (10) are, respectively, 0.98, 25, $10I$, $20I$ and $10I$. By using the Matlab software, a set of solutions to RLMIs in Theorem 2 is obtained and the fault estimator gain matrices are shown in TABLE I.

TABLE I

ESTIMATOR PARAMETERS

k	$A_f(k)$	$B_f(k)$	$L_f(k)$
0	$\begin{bmatrix} 0.0139 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	-2.7273
1	$\begin{bmatrix} 0.4120 & 0.0078 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.0281 \\ 0 \end{bmatrix}$	-3.0422
2	$\begin{bmatrix} 0.4478 & 0.0205 \\ -0.0241 & -0.4507 \end{bmatrix}$	$\begin{bmatrix} 0.0646 \\ 0.0997 \end{bmatrix}$	-3.3395
3	$\begin{bmatrix} 0.4534 & 0.0181 \\ -0.0222 & -0.4295 \end{bmatrix}$	$\begin{bmatrix} 0.0691 \\ 0.1085 \end{bmatrix}$	-3.4315
4	$\begin{bmatrix} 0.4865 & 0.0020 \\ -0.0025 & -0.4625 \end{bmatrix}$	$\begin{bmatrix} 0.0742 \\ 0.1104 \end{bmatrix}$	-3.0205
$\vdots$	$\vdots$	$\vdots$	$\vdots$
24	$\begin{bmatrix} 0.4604 & -0.0181 \\ 0.0220 & -0.4392 \end{bmatrix}$	$\begin{bmatrix} 0.1218 \\ 0.1932 \end{bmatrix}$	-3.2930
25	$\begin{bmatrix} 0.4598 & -0.0103 \\ 0.0125 & -0.4488 \end{bmatrix}$	$\begin{bmatrix} 0.1180 \\ 0.1922 \end{bmatrix}$	-3.2920

In the simulation, the exogenous disturbance inputs are selected as

$$w_k = 5\sin(k), \quad v_k = 0.8\cos(0.7k), \quad \xi_k = 0.48\cos(0.2k).$$

The simulation results are shown in Fig. 1 and Fig. 2, where Fig. 1 plots the measurement outputs and the *actual* signals received by the estimator, and Fig. 2 depicts the fault f_k and its estimate r_k. The simulation results have confirmed that the designed estimator performs very well over a finite time horizon.

V. CONCLUSIONS

In this paper, the H_∞ finite-horizon fault estimation issue has been investigated for discrete time-varying stochastic systems with nonlinearities, quantization effects, ROUs, successive dropouts as well as multiple stochastic delays. The last three phenomena have been governed by some Bernoulli-distributed white sequences with known conditional probabilities. By recurring to the intensive stochastic analysis techniques, some sufficient conditions have been established for the existence of the desired H_∞ fault estimator. Then, all the estimator parameters have been designed simultaneously by solving a set of RLMIs. Finally, a numerical simulation example has been exploited to demonstrate the effectiveness of the fault estimation scheme presented in this paper.

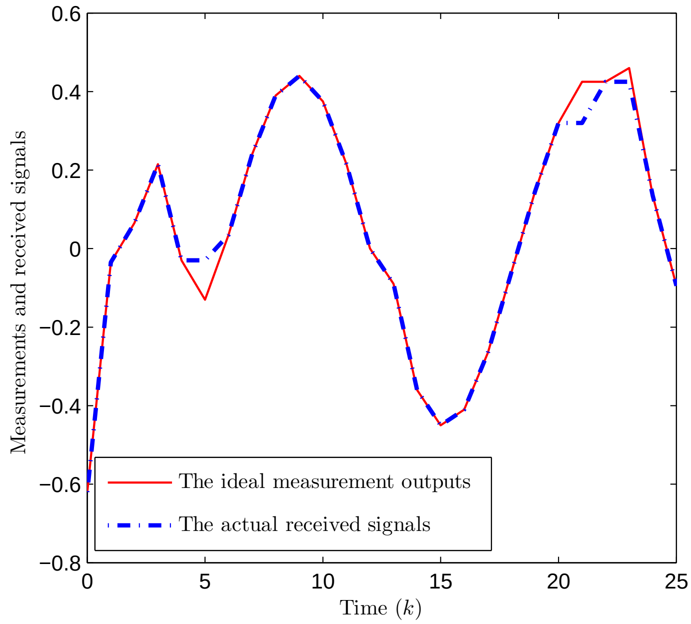

Fig. 1. Measurements and received signals.

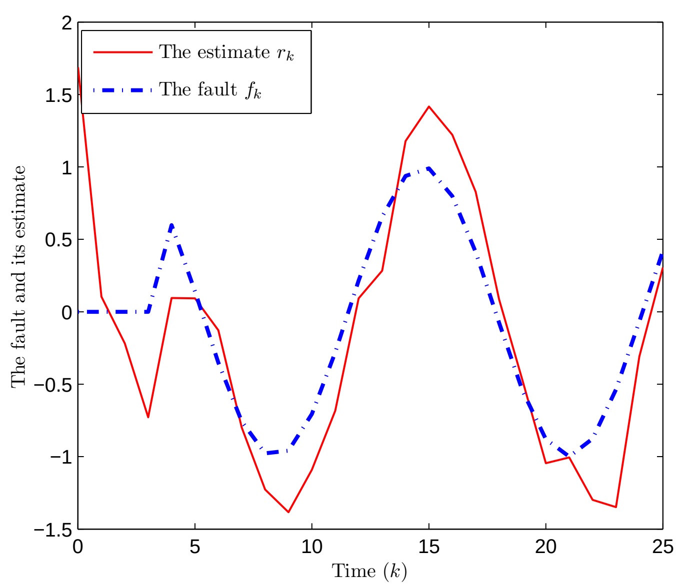

Fig. 2. The fault f_k and its estimate r_k.

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