Envelope-Constrained $\mathcal{H}_\infty$ Filtering with Fading Measurements and Randomly Occurring Nonlinearities: The Finite Horizon Case ⋆

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Abstract

In this paper, the envelope-constrained $\mathcal{H}_\infty$ filtering problem is investigated for a class of discrete time-varying stochastic systems over a finite horizon. The system under consideration involves fading measurements, randomly occurring nonlinearities (RONs) and mixed (multiplicative and additive) noises. A novel envelope-constrained performance criterion is proposed to better quantify the transient dynamics of the filtering error process over the finite horizon. The purpose of the problem addressed is to design a time-varying filter such that both the $\mathcal{H}_\infty$ performance and the desired envelope constraints are achieved at each time step. By utilizing the stochastic analysis techniques combined with the ellipsoid description on the estimation errors, sufficient conditions are established in the form of recursive matrix inequalities (RMIs) reflecting both the envelope information and the desired $\mathcal{H}_\infty$ performance index. The filter gain matrix is characterized by means of the solvability of the deduced RMIs. Finally, a simulation example is provided to show the effectiveness of the proposed filtering design scheme.

Key words: $\mathcal{H}_\infty$ filtering; Finite-horizon filtering; Envelope constraints; Fading measurements; Ellipsoid constraints; Randomly occurring nonlinearities.

1 Introduction

Filtering or state estimation has long been a hot research topic in the areas of communications, control and signal processing. Among various filtering approaches available in the literature, the $\mathcal{H}_\infty$ filtering has gained particular research attention due to its capability of providing a bound for the worst-case estimation error without the need for knowledge of noise statistics. It should be mentioned that most existing results with regard to $\mathcal{H}_\infty$ filtering have been concerned with the time-invariant systems over the infinite-horizon and the corresponding results for time-varying case are relatively few. In reality, almost all practical systems are subject to time-varying parameter variations and should be modeled as time-varying systems (see e.g. [4]). As such, the finite-horizon $\mathcal{H}_\infty$ filtering problem has received extensive research attention in the past few years. For instance, the recursive linear matrix inequality (RLMI) method has been proposed in [10] to effectively solve a finite-horizon filtering $\mathcal{H}_\infty$ problems for a class of nonlinear systems with quantization effects. The backward recursive Riccati difference equation (RDE) approach has been developed in [3] to deal with a distributed $\mathcal{H}_\infty$ state estimation problem over sensor networks.

Because of the ever-increasing popularity of communication networks, more and more control and signal process algorithms are executed over communication links. In networked systems, the limited bandwidth of the communication channel gives rise to various network-induced phenomena such as the transmission delay [2], missing measurements [11], signal quantization [7], randomly occurring nonlinearities (RONs) [10], etc. If not properly handled, these network-induced phenomena could cause performance degradation of the addressed systems. It is worth noting that the phenomena of RONs have been put forward to describe the network-induced nonlinear disturbances. This kind of nonlinear disturbances might stem from some abrupt phenomena such as random failures and repairs of the components, changes in the interconnections of subsystems, modification of the operating point of the linearized model of a nonlinear system. Two typical approaches have been widely adopted to cater for such phenomena: one is to introduce a sequence of random variables obeying the given Bernoulli distribution [10] and the other is to utilize the statistical laws with known expectation and

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secondary moment [17]. The latter description includes some well-studied nonlinearities as special cases such as system with state-dependent multiplicative noises and nonlinear systems with random vectors whose powers depend on the norm of the state. To date, the occurrence of RONs has gained some particular research interest and the corresponding research on more general systems is still ongoing. For example, the $\mathcal{H}_\infty$ filtering issue for time-varying stochastic systems with both RONs and mixed noises (multiplicative and additive noises) has not been adequately investigated, and this constitutes one of the motivations for the present research.

Compared with the well-studied network-induced phenomena such as communication delays and packet dropouts, the research on network-induced channel fading problem is still on its early stage despite the fact that the wireless channels are susceptible to fading effect [5]. Generally speaking, when a signal is transmitted over a wireless channel, it is inevitably subject to some special physical phenomena such as reflection, refraction and diffraction, which lead to the multi-path induced fading or the shadow fading. Fading is often modeled by a time-varying stochastic mathematical model representing the transmitted signal's change in both the amplitude and phase. Some representative models have been investigated, of which the analog erasure channel model, Rice fading channel model and Rayleigh channel model are arguably the most popular ones, see [14] for more details. Up to now, some preliminary results have been reported in the literature concerning the stability analysis, LQG control and Kalman filtering problems with fading measurements, see [5, 8, 15] and the references therein. Unfortunately, the finite-horizon $\mathcal{H}_\infty$ filtering problem for time-varying stochastic systems with fading measurements has attracted little research attention, which remains as an open research issue.

Nowadays, the envelope-constrained filtering (ECF) technique has been utilized to solve a wide range of practical engineering problems arising in signal processing and communications. Examples include the communication channel equalization problems, the radar and sonar detection problems, and the robust antenna and filter design problems [1]. It should be pointed out that the aim of ECF problems is to find a filter such that the filtering error output stimulated by a specified input signal lies within a desired envelope and the effect from the input noises is also minimized [13]. The specifications of the given envelope can arise either from the standards set by certain regulatory bodies or from the practical design consideration [1]. In the time domain, the ECF issue is usually cast as a finite-dimensional constrained quadratic optimization problem which can be effectively handled by using linear matrix inequality approaches [19]. For example, by utilizing the $\mathcal{H}_\infty$ or $\mathcal{H}_2$ optimal theory, some filter design schemes have been proposed in [12, 18]. Almost all ECF-relevant literature has been concerned with the linear time-invariant systems, and the corresponding investigation on the nonlinear time-varying systems has not received proper research attention for the following two reasons: 1) it is non-trivial to define the envelope constraints for time-varying systems over a finite-horizon; and 2) it is challenging to develop appropriate methodology to analyze the transient dynamics of the filtering error due to the time-varying nature.

Summarizing the discussions made so far, it is of both theoretical significance and practical importance to design an envelope-constrained $\mathcal{H}_\infty$ filter for the time-varying systems with network-induced RON, fading channel and mixed noises. This appears to be a challenging task with three essential difficulties identified as follows: 1) how to define the criterion for the envelope-constrained filtering of a class of time-varying systems with network-induced phenomena? 2) what kind of methods can be developed to solve the addressed envelope-constrained filtering problem over a given finite-horizon? 3) how to examine the impact from the statistical information of both fading measurements and RONs on the filtering performance? It is, therefore, the main motivation of this paper to provide satisfactory answers to the three questions mentioned above and also propose a design scheme of the envelope-constrained $\mathcal{H}_\infty$ filter.

In this paper, we aim to investigate the envelope-constrained $\mathcal{H}_\infty$ filtering problem for a class of discrete time-varying stochastic systems with simultaneous presence of fading measurements, RONs and mixed noises. Some sufficient conditions are established, via intensive stochastic analysis, to guarantee the existence of the desired time-varying filter gains, and then such filter gains is obtained by solving a set of recursive matrix inequalities (RMIs). The main contributions of this paper are outlined as follows: *1) a novel envelope-constrained performance criterion is proposed in order to describe the transient dynamics of the filtering error process; 2) the system under consideration is comprehensive to cover several network-induced phenomena; 3) by utilizing the ellipsoid description on the estimation errors, the given envelope constraints are transformed into a set of matrix inequalities while meeting the specified $\mathcal{H}_\infty$ requirements; and 4) the filter gain matrix is obtained by solving a set of recursive matrix inequalities solvable via standard software package.*

Notation The notation used here is fairly standard except where otherwise stated. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. $l([0,N];\mathbb{R}^n)$ is the space of vector functions over $[0,N]$. I denotes the identity matrix of compatible dimensions. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X-Y$ is positive semi-definite (respectively, positive definite). A^T represents the transpose of A. For matrices $A\in\mathbb{R}^{m\times n}$ and $B\in\mathbb{R}^{p\times q}$, their Kronecker product is a matrix in $\mathbb{R}^{mp\times nq}$ denoted as $A\otimes B$. $\mathbf{1}$ denotes a compatible dimensional column vector with all ones. $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x. $||x||$ describes the Euclidean norm of a vector x. The shorthand diag$\{\cdot\}$ denotes a block diagonal matrix.In symmetric block matrices, the symbol $*$ is used as an ellipsis for terms induced by symmetry.

2 Problem Formulation and Preliminaries

Consider the following discrete time-varying stochastic system defined on $k\in[0,N]$

$$\begin{cases} x_{k+1} = \big(A_k+\xi_k D_k\big)x_k+\alpha_k h(k,x_k)+E_{1,k}v_k \\ \quad y_k = C_k x_k + E_{2,k}v_k \\ \quad z_k = L_k x_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$, $y_k \in \mathbb{R}^{n_y}$ and $z_k \in \mathbb{R}^{n_z}$ are, respectively, the state vector, the measurement output (without fading) and the signal to be estimated. $v_k \in l([0,N];\mathbb{R}^q)$ is the disturbance input, A_k, C_k, D_k, $E_{1,k}$, $E_{2,k}$ and L_k are known time-varying matrices with appropriate dimensions, $\xi_k \in \mathbb{R}$ is a zero-mean random sequence with $\mathbb{E}\{\xi_k^2\} = 1$. The random variable $\alpha_k \in \{0,1\}$, which is uncorrelated to ξ_k, is a Bernoulli distributed white sequence obeying the probability distribution law $\text{Prob}\{\alpha_k = 0\} = 1 - \bar{\alpha}$ and $\text{Prob}\{\alpha_k = 1\} = \bar{\alpha}$ where $\bar{\alpha} \in [0,\ 1]$ is a known scalar. The known nonlinear vector-valued function $h : [0,N] \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is continuous, and satisfies $h(k,0) = 0$ and the sector-bounded condition

$$\begin{aligned}\big[h(k,x) - h(k,y) - \Phi_k(x-y)\big]^T \\ \times \big[h(k,x) - h(k,y) - \Psi_k(x-y)\big] \le 0\end{aligned} \tag{2}$$

for all $x,\ y \in \mathbb{R}^{n_x}$, where Φ_k and Ψ_k are known real matrices with appropriate dimensions.

Letting the number of paths denoted by ℓ be given, the *actually* received signal by the filter is of the following form

$$\tilde{y}_k = \sum\nolimits_{s=0}^{\ell_k} \vartheta_k^s y_{k-s} + E_{3,k} w_k \tag{3}$$

where $\ell_k = \min\{\ell,\ k\}$, ϑ_k^s $(s = 0,1,\cdots,\ell_k)$ are the mutually independent channel coefficients having probability density functions $f(\vartheta_k^s)$ on the interval $[0,1]$ with known mathematical expectations $\bar{\vartheta}^s$ and variances $\tilde{\vartheta}^s$, and $w_k \in l([0,N];\mathbb{R})$ is an external disturbance.

For the purpose of simplicity, for $-\ell \le i \le -1$, we assume that $C_i = 0$, $y_i = 0$ and $[v_i^T\ \ w_i^T] = 0$. Based on the *actually* received signal $\tilde{y}_k$, the following filter is constructed

$$\begin{cases} \hat{x}_{k+1} = A_k \hat{x}_k + \bar{\alpha} h(k,\hat{x}_k) + K_k \Big(\tilde{y}_k - \sum\limits_{s=0}^{\ell} \bar{\vartheta}^s C_{k-s}\hat{x}_{k-s}\Big) \\ \hat{z}_k = L_k \hat{x}_k \end{cases} \tag{4}$$

where $\hat{x}_k \in \mathbb{R}^{n_x}$ is the estimated state, $\hat{z}_k \in \mathbb{R}^{n_z}$ represents the estimated output, and K_k is the time-varying filter gain matrix to be designed.

Let the state estimation error be $e_k = x_k - \hat{x}_k$ and the output estimation error be $\tilde{z}_k = z_k - \hat{z}_k$. Then, the dynamics of the filtering errors can be obtained from (1) and (4) as follows

$$\begin{cases} e_{k+1} = A_k e_k + \xi_k D_k x_k + \bar{\alpha}(h(k,x_k) - h(k,\hat{x}_k)) \\ \qquad + (\alpha_k - \bar{\alpha}) h(k,x_k) - \sum\nolimits_{s=0}^{\ell} \bar{\vartheta}^s K_k C_{k-s} e_{k-s} \\ \qquad - \sum\nolimits_{s=0}^{\ell} (\vartheta_k^s - \bar{\vartheta}^s) K_k C_{k-s} x_{k-s} + E_{1,k} v_k \\ \qquad - \sum\nolimits_{s=0}^{\ell} \vartheta_k^s K_k E_{2,k-s} v_{k-s} - K_k E_{3,k} w_k \\ \tilde{z}_k = L_k e_k. \end{cases} \tag{5}$$

Furthermore, denoting $\eta_k = [x_k^T\ \ e_k^T]^T$, $\zeta_k = [v_k^T\ \ w_k^T]^T$ and $h_k = \big[h^T(k,x_k)\ \ \ h^T(k,x_k) - h^T(k,\hat{x}_k)\big]^T$, we have the following augmented system

$$\begin{cases} \eta_{k+1} = \mathcal{A}_k \eta_k + \bar{\alpha} h_k + (\alpha_k - \bar{\alpha})\mathcal{S}_1 h_k \\ \qquad + \xi_k \mathcal{D}_k \mathcal{S}_1 \eta_k - \sum\nolimits_{s=0}^{\ell} (\vartheta_k^s - \bar{\vartheta}^s)\mathcal{K}_k \mathcal{C}_{k-s}\mathcal{S}_3 \eta_{k-s} \\ \qquad - \sum\limits_{s=1}^{\ell} \bar{\vartheta}^s \big(\mathcal{K}_k \mathcal{C}_{k-s}\mathcal{S}_2 \eta_{k-s} - \mathcal{K}_k \mathcal{E}_{k-s}\mathcal{S}_3 \zeta_{k-s}\big) \\ \qquad - \sum\nolimits_{s=0}^{\ell} (\vartheta_k^s - \bar{\vartheta}^s)\mathcal{K}_k \mathcal{E}_{k-s}\mathcal{S}_3 \zeta_{k-s} + \mathcal{F}_k \zeta_k \\ \tilde{z}_k = \mathcal{L}_k \eta_k \end{cases} \tag{6}$$

where

$$\mathcal{A}_k = \text{diag}\{A_k, A_k - \bar{\vartheta}^0 K_k C_k\},\ \mathcal{C}_k = \text{diag}\{C_k,\ C_k\},$$
$$\mathcal{L}_k = [\,0 \quad L_k\,],\ \mathcal{D}_k = \text{diag}\{D_k,\ D_k\},$$
$$\mathcal{K}_k = \text{diag}\{K_k,\ K_k\},\ \mathcal{S}_2 = \text{diag}\{0,\ I\},$$
$$\mathcal{S}_1 = \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix},\ \mathcal{S}_3 = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix},$$
$$\mathcal{E}_k = \begin{bmatrix} 0 & 0 \\ E_{2,k} & 0 \end{bmatrix},\ \mathcal{F}_k = \begin{bmatrix} E_{1,k} & 0 \\ E_{1,k} & -K_k E_{3,k} \end{bmatrix}.$$

Our aim in this paper is to design an envelope-constrained $\mathcal{H}_\infty$ filter of the form (4) such that the following requirements are met simultaneously:

a) ($\mathcal{H}_\infty$ requirement) for any nonzero ζ_k, the output $\tilde{z}_k$ of the augmented system (6) satisfies

$$\sum_{k=0}^{N} \mathbb{E}\{||\tilde{z}_k||^2\} \le \gamma^2 \sum_{k=0}^{N} ||\zeta_k||^2 + \gamma^2 \sum_{i=-\ell}^{0} \mathbb{E}\{\eta_i^T \mathcal{W}_i \eta_i\} \tag{7}$$

where γ is a prescribed positive scalar and $\mathcal{W}_i > 0$ $(-\ell \le i \le 0)$ are some weighting matrices;

b) (envelope constraints) under the zero-initial condition, for the given input signal

$$\zeta_k^* = \begin{cases} \mathbf{1},\ k = 0 \\ 0,\ 1 \le k \le N \end{cases}$$

the corresponding output $\tilde{z}_k^*$ of the augmented system (6) satisfies

$$d_k^i - \varepsilon_k^i \le \mathbb{E}\{\mathbf{I}_i \tilde{z}_k^*\} \le d_k^i + \varepsilon_k^i,\ k \in [1,N], i \in [1,n_z] \tag{8}$$

where $\mathbf{I}_i := \big[\underbrace{0,\cdots,0}_{i-1},1,\underbrace{0,\cdots,0}_{n_z-i}\big]$, and $\{d_k^i\}_{k\in[0,\ N]}$ and $\{\varepsilon_k^i\}_{k\in[0,\ N]}$ are the sequences of the desired output and the tolerance band, respectively. Obviously, ε_k^i $(k \in [1,N],\ i \in [1,n_z])$ are positive scalars.

Remark 1 *(8) gives the envelope constraint on the individual estimation error in the mean square, which can be understood as the stochastic version of the envelope definition in [12] over a finite horizon. Similar to [12], an envelope-constrained $\mathcal{H}_\infty$ filtering system is shown in Fig. 1. The signal y_k is the measurement of the plant system with an energy bounded disturbance input v_k and the filter input signal $\tilde{y}_k$ is the output of transmission channel corrupted by an energy bounded noise w_k. $\hat{z}_k$ is the*

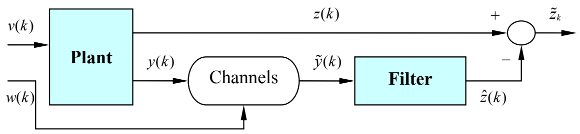

Fig. 1. The envelope-constrained $\mathcal{H}_\infty$ filtering system model.

estimate of z_k. The aim of the envelope-constrained filtering problem is twofold: 1) a filter gain is designed to reconstruct z_k by using the distorted signal $\tilde{y}_k$; and 2) for the given input signal w_k and v_k, $\tilde{z}_k$ is guaranteed to lie within the specified envelope at each time step. It should be pointed out that, the $\mathcal{H}_\infty$ criterion is concerned with the performance requirement as a whole over the finite-horizon $[0,\ N]$. In contrast, the envelope-constrained requirement (8) can be used to describe the transient dynamics of the filtering error process at each time step, that is, the outputs $\{\tilde{z}_k^*\}_{k\in[0,N]}$ stimulated by the given input $\{\zeta_k^*\}_{k\in[0,N]}$ are included in a desired envelope in the mean square sense at each step.

3 Main Results

In this section, by resorting to the stochastic analysis combined with the ellipsoid description on the estimation errors, some sufficient conditions are proposed to guarantee the $\mathcal{H}_\infty$ performance and achieve the desired envelope constraints for the $\mathcal{H}_\infty$ filter design over the given finite horizon.

3.1 $\mathcal{H}_\infty$ Performance Analysis

Denote $\eta_k^\ell = [\eta_{k-1}^T\ \cdots\ \eta_{k-\ell}^T]^T$, $\zeta_k^\ell = [\zeta_k^T\ \cdots\ \zeta_{k-\ell}^T]^T$ and $\tilde{\eta}_k = [\eta_k^T\ h_k^T\ (\eta_k^\ell)^T\ (\zeta_k^\ell)^T]^T$. The following lemma will be used in deriving our main results.

Lemma 1 *Let the external disturbances ζ_k and the initial values $\{\eta_k\}_{k\in[-\ell,\ 0]}$ be given. For the function*

$$V_k = \eta_k^T\mathcal{P}_k\eta_k + \sum_{j=1}^{\ell}\sum_{i=k-j}^{k-1}\eta_i^T\mathcal{R}_{i,j}\eta_i \tag{9}$$

where $\mathcal{P}_k$ and $\mathcal{R}_{i,j}$ are symmetric positive definite matrices with appropriate dimensions, the following relationship

$$\mathbb{E}\{\Delta V_k\} := \mathbb{E}\{V_{k+1} - V_k\} = \mathbb{E}\{\tilde{\eta}_k^T\Pi_1^k\tilde{\eta}_k\} \tag{10}$$

is true, where

$$\Pi_1^k = \begin{bmatrix} \Pi_{11}^k & \Pi_{12}^k & \Pi_{13}^k & \Pi_{14}^k \\ * & \Pi_{22}^k & \Pi_{23}^k & \Pi_{24}^k \\ * & * & \Pi_{33}^k & \Pi_{34}^k \\ * & * & * & \Pi_{44}^k \end{bmatrix},$$

$$\begin{aligned}
&\mathcal{I}_{0v} = diag\{\sqrt{\tilde{\vartheta}^0}I,\ \cdots,\ \sqrt{\tilde{\vartheta}^\ell}I\},\ \bar{\mathcal{S}}_{21} = diag_\ell\{\mathcal{S}_2\},\\
&\mathcal{I}_{1v} = diag\{\sqrt{\tilde{\vartheta}^1}I,\ \cdots,\ \sqrt{\tilde{\vartheta}^\ell}I\},\ \bar{\mathcal{S}}_{31} = diag_\ell\{\mathcal{S}_3\},\\
&\bar{\mathcal{C}}_{0k} = diag\{\mathcal{C}_k,\ \cdots,\ \mathcal{C}_{k-\ell}\},\ \mathcal{I}_D = [I\ 0\ \cdots\ 0],\\
&\bar{\mathcal{C}}_{1k} = diag\{\mathcal{C}_{k-1},\ \cdots,\ \mathcal{C}_{k-\ell}\},\ \tilde{\alpha} = \bar{\alpha}(1-\bar{\alpha}),\\
&\bar{\mathcal{E}}_{0k} = diag\{\mathcal{E}_k,\ \cdots,\ \mathcal{E}_{k-\ell}\},\ \Lambda_0 = [\bar{\vartheta}^0 I\ \cdots\ \bar{\vartheta}^\ell I],\\
&\bar{\mathcal{E}}_{1k} = diag\{\mathcal{E}_{k-1},\ \cdots,\ \mathcal{E}_{k-\ell}\},\ \Lambda_1 = [\bar{\vartheta}^1 I\ \cdots\ \bar{\vartheta}^\ell I],\\
&\Pi_{12}^k = \bar{\alpha}\mathcal{A}_k^T\mathcal{P}_{k+1},\quad \Pi_{13}^k = -\mathcal{A}_k^T\mathcal{P}_{k+1}\mathcal{K}_k\Lambda_1\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{21},
\end{aligned}$$

$$\begin{aligned}
\Pi_{11}^k =\ & \mathcal{A}_k^T\mathcal{P}_{k+1}\mathcal{A}_k - \mathcal{P}_k + \mathcal{S}_1^T\mathcal{D}_k^T\mathcal{P}_{k+1}\mathcal{D}_k\mathcal{S}_1\\
& + \tilde{\vartheta}^0\mathcal{S}_3^T\mathcal{C}_k^T\mathcal{K}_k^T\mathcal{P}_{k+1}\mathcal{K}_k\mathcal{C}_k\mathcal{S}_3 + \sum\nolimits_{j=1}^{\ell}\mathcal{R}_{k,j},\\
\Pi_{14}^k =\ & -\mathcal{A}_k^T\mathcal{P}_{k+1}\mathcal{K}_k\Lambda_0\bar{\mathcal{E}}_{0k} + \mathcal{A}_k^T\mathcal{P}_{k+1}\mathcal{F}_k\mathcal{I}_D\\
& + \tilde{\vartheta}^0\big(\mathcal{K}_k\mathcal{C}_k\mathcal{S}_3\big)^T\mathcal{P}_{k+1}\mathcal{K}_k\mathcal{E}_k\mathcal{I}_D,\\
\Pi_{22}^k =\ & \bar{\alpha}^2\mathcal{P}_{k+1} + \tilde{\alpha}\mathcal{S}_1^T\mathcal{P}_{k+1}\mathcal{S}_1,\\
\Pi_{23}^k =\ & -\bar{\alpha}\mathcal{P}_{k+1}\mathcal{K}_k\Lambda_1\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{21},\\
\Pi_{24}^k =\ & -\bar{\alpha}\mathcal{P}_{k+1}\mathcal{K}_k\Lambda_0\bar{\mathcal{E}}_{0k} + \bar{\alpha}\mathcal{P}_{k+1}\mathcal{F}_k\mathcal{I}_D,\\
\Pi_{33}^k =\ & \bar{\mathcal{S}}_{21}^T\bar{\mathcal{C}}_{1k}^T\Lambda_1^T\mathcal{K}_k^T\mathcal{P}_{k+1}\mathcal{K}_k\Lambda_1\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{21}\\
& + (\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{31}\mathcal{I}_{1v})^T(I\otimes(\mathcal{K}_k^T\mathcal{P}_{k+1}\mathcal{K}_k))(\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{31}\mathcal{I}_{1v})\\
& - diag\{\mathcal{R}_{k-1,1},\mathcal{R}_{k-2,2},\cdots,\mathcal{R}_{k-\ell,\ell}\},\\
\Pi_{34}^k =\ & \bar{\mathcal{S}}_{21}^T\bar{\mathcal{C}}_{1k}^T\Lambda_1^T\mathcal{K}_k^T\mathcal{P}_{k+1}(\mathcal{K}_k\Lambda_0\bar{\mathcal{E}}_{0k} - \mathcal{F}_k\mathcal{I}_D)\\
& + [0\quad (\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{31}\mathcal{I}_{1v})^T(I\otimes(\mathcal{K}_k^T\mathcal{P}_{k+1}\mathcal{K}_k))(\bar{\mathcal{E}}_{1k}\mathcal{I}_{1v})],\\
\Pi_{44}^k =\ & (\mathcal{K}_k\Lambda_0\bar{\mathcal{E}}_{0k} - \mathcal{F}_k\mathcal{I}_D)^T\mathcal{P}_{k+1}(\mathcal{K}_k\Lambda_0\bar{\mathcal{E}}_{0k} - \mathcal{F}_k\mathcal{I}_D)\\
& + \mathcal{I}_{0v}^T\bar{\mathcal{E}}_{0k}^T(I\otimes(\mathcal{K}_k^T\mathcal{P}_{k+1}\mathcal{K}_k))\bar{\mathcal{E}}_{0k}\mathcal{I}_{0v}.
\end{aligned}$$

Proof: By calculating the difference of the first term in V_k along the trajectory of the system (6) and taking the mathematical expectation, one has

$$\begin{aligned}
&\mathbb{E}\{\eta_{k+1}^T\mathcal{P}_{k+1}\eta_{k+1} - \eta_k^T\mathcal{P}_k\eta_k\}\\
=\ & \mathbb{E}\Big\{\eta_k^T\mathcal{A}_k^T\mathcal{P}_{k+1}\mathcal{A}_k\eta_k - \eta_k^T\mathcal{P}_k\eta_k + 2\bar{\alpha}\eta_k^T\mathcal{A}_k^T\mathcal{P}_{k+1}h_k\\
& - 2\eta_k^T\mathcal{A}_k^T\mathcal{P}_{k+1}\mathcal{K}_k\Lambda_1\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{21}\eta_k^\ell - 2\eta_k^T\mathcal{A}_k^T\mathcal{P}_{k+1}\mathcal{K}_k\Lambda_0\\
& \times\bar{\mathcal{E}}_{0k}\zeta_k^\ell + 2\eta_k^T\mathcal{A}_k^T\mathcal{P}_{k+1}\mathcal{F}_k\mathcal{I}_D\zeta_k^\ell + \eta_k^T\mathcal{S}_1^T\mathcal{D}_k^T\mathcal{P}_{k+1}\mathcal{D}_k\\
& \times\mathcal{S}_1\eta_k + \bar{\alpha}^2 h_k^T\mathcal{P}_{k+1}h_k - 2\bar{\alpha}h_k^T\mathcal{P}_{k+1}\mathcal{K}_k\Lambda_1\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{21}\eta_k^\ell\\
& - 2\bar{\alpha}h_k^T\mathcal{P}_{k+1}\mathcal{K}_k\Lambda_0\bar{\mathcal{E}}_{0k}\zeta_k^\ell + 2\bar{\alpha}h_k^T\mathcal{P}_{k+1}\mathcal{F}_k\mathcal{I}_D\zeta_k^\ell\\
& + \bar{\alpha}(1-\bar{\alpha})h_k^T\mathcal{S}_1^T\mathcal{P}_{k+1}\mathcal{S}_1 h_k + (\eta_k^\ell)^T\bar{\mathcal{S}}_{21}^T\bar{\mathcal{C}}_{1k}^T\Lambda_1^T\mathcal{K}_k^T\\
& \times\mathcal{P}_{k+1}\mathcal{K}_k\Lambda_1\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{21}\eta_k^\ell + 2(\eta_k^\ell)^T\bar{\mathcal{S}}_{21}^T\bar{\mathcal{C}}_{1k}^T\Lambda_1^T\mathcal{K}_k^T\mathcal{P}_{k+1}\\
& \times\mathcal{K}_k\Lambda_0\bar{\mathcal{E}}_{0k}\zeta_k^\ell - 2(\eta_k^\ell)^T\bar{\mathcal{S}}_{21}^T\bar{\mathcal{C}}_{1k}^T\Lambda_1^T\mathcal{K}_k^T\mathcal{P}_{k+1}\mathcal{F}_k\mathcal{I}_D\zeta_k^\ell\\
& + \tilde{\vartheta}^0\eta_k^T\big(\mathcal{K}_k\mathcal{C}_k\mathcal{S}_3\big)^T\mathcal{P}_{k+1}\mathcal{K}_k\mathcal{C}_k\mathcal{S}_3\eta_k\\
& + (\eta_k^\ell)^T(\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{31}\mathcal{I}_{1v})^T(I\otimes(\mathcal{K}_k^T\mathcal{P}_{k+1}\mathcal{K}_k))\\
& \times(\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{31}\mathcal{I}_{1v})\eta_k^\ell + 2\tilde{\vartheta}^0\eta_k^T\mathcal{S}_3^T\mathcal{C}_k^T\mathcal{K}_k^T\mathcal{P}_{k+1}\mathcal{K}_k\mathcal{E}_k\mathcal{I}_D\zeta_k^\ell\\
& + 2\eta_k^{\ell\ T}[0\ (\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{31}\mathcal{I}_{1v})^T(I\otimes(\mathcal{K}_k^T\mathcal{P}_{k+1}\mathcal{K}_k))(\bar{\mathcal{E}}_{1k}\mathcal{I}_{1v})]\zeta_k^\ell\\
& + (\zeta_k^\ell)^T\bar{\mathcal{E}}_{0k}^T\Lambda_0^T\mathcal{K}_k^T\mathcal{P}_{k+1}\mathcal{K}_k\Lambda_0\bar{\mathcal{E}}_{0k}\zeta_k^\ell - 2(\zeta_k^\ell)^T\bar{\mathcal{E}}_{0k}^T\Lambda_0^T\mathcal{K}_k^T\\
& \times\mathcal{P}_{k+1}\mathcal{F}_k\mathcal{I}_D\zeta_k^\ell + (\zeta_k^\ell)^T(\bar{\mathcal{E}}_{0k}\mathcal{I}_{0v})^T(I\otimes(\mathcal{K}_k^T\mathcal{P}_{k+1}\mathcal{K}_k))\\
& \times(\bar{\mathcal{E}}_{0k}\mathcal{I}_{0v})\zeta_k^\ell + (\zeta_k^\ell)^T\mathcal{I}_D^T\mathcal{F}_k^T\mathcal{P}_{k+1}\mathcal{F}_k\mathcal{I}_D\zeta_k^\ell\Big\}.
\end{aligned} \tag{11}$$

On the other hand, it is not difficult to show that

$$\begin{aligned}
&\mathbb{E}\Big\{\sum_{j=1}^{\ell}\Big(\sum_{i=k-j+1}^{k}\eta_i^T\mathcal{R}_{i,j}\eta_i - \sum_{i=k-j}^{k-1}\eta_i^T\mathcal{R}_{i,j}\eta_i\Big)\Big\}\\
=\ & \mathbb{E}\Big\{\sum\nolimits_{j=1}^{\ell}\eta_k^T\mathcal{R}_{k,j}\eta_k\\
& - \eta_k^{\ell\ T}\mathrm{diag}\{\mathcal{R}_{k-1,1},\mathcal{R}_{k-2,2},\cdots,\mathcal{R}_{k-\ell,\ell}\}\eta_k^\ell\Big\}.
\end{aligned} \tag{12}$$

Therefore, it follows from (11) and (12) that the equality

(10) holds, which completes the proof.

Theorem 1 *Let the positive scalar $\gamma > 0$, the positive definite matrices $\mathcal{W}_i > 0$ ($-\ell \le i \le 0$) and the filter gain matrices $\{K_k\}_{k\in[0,\ N]}$ be given. For the augmented system (6), the $\mathcal{H}_\infty$ performance requirement defined in (7) is guaranteed for all nonzero ζ_k if there exist families of positive scalars $\{\lambda_k\}_{k\in[0,\ N]}$, positive definite matrices $\{\mathcal{P}_k\}_{k\in[0,\ N+1]}$ and $\{\mathcal{R}_{i,j}\}_{i\in[-\ell,\ N],\ j\in[1,\ \ell]}$ satisfying the following recursive matrix inequalities*

$$\Pi_k = \begin{bmatrix} \tilde{\Pi}_{11}^k & \Pi_{12}^k + \lambda_k \mathcal{U}_{1k}^T & \Pi_{13}^k & \Pi_{14}^k \\ * & \Pi_{22}^k - \lambda_k I & \Pi_{23}^k & \Pi_{24}^k \\ * & * & \Pi_{33}^k & \Pi_{34}^k \\ * & * & * & \Pi_{44}^k - \frac{\gamma^2}{\ell+1} I \end{bmatrix} < 0 \quad (13)$$

and

$$\mathcal{P}_0 \le \gamma^2 \mathcal{W}_0,\ \sum_{j=i}^{\ell} \mathcal{R}_{-i,j} \le \gamma^2 \mathcal{W}_{-i},\ i = 1, 2, \cdots, \ell \quad (14)$$

where $\tilde{\Pi}_{11}^k = \Pi_{11}^k - \lambda_k \mathcal{U}_{2k} + \mathcal{L}_k^T \mathcal{L}_k$, $\mathcal{U}_{1k} = I \otimes (\Phi_k + \Psi_k)/2$, $\mathcal{U}_{2k} = I \otimes (\Phi_k^T \Psi_k + \Psi_k^T \Phi_k)/2$ *and the other corresponding matrices are defined in Lemma 1.*

Proof: In order to analyze the $\mathcal{H}_\infty$ performance of the augmented system (6), we introduce the following function

$$\begin{aligned} \mathcal{J}_k =& \eta_{k+1}^T \mathcal{P}_{k+1} \eta_{k+1} - \eta_k^T \mathcal{P}_k \eta_k \\ &+ \sum_{j=1}^{\ell} \Big(\sum_{i=k-j+1}^{k} \eta_i^T \mathcal{R}_{i,j} \eta_i - \sum_{i=k-j}^{k-1} \eta_i^T \mathcal{R}_{i,j} \eta_i \Big). \end{aligned} \quad (15)$$

It is easy to see from (2) that

$$\big[h_k - (I \otimes \Phi_k)\eta_k\big]^T \big[h_k - (I \otimes \Psi_k)\eta_k\big] \le 0. \quad (16)$$

Then, substituting (10) and (16) into (15) results in

$$\begin{aligned} \mathbb{E}\{\mathcal{J}_k\} \le \mathbb{E}\Big\{& \tilde{\eta}_k^T \Pi_1^k \tilde{\eta}_k - \lambda_k \big[h_k \\ &- (I \otimes \Phi_k)\eta_k\big]^T \big[h_k - (I \otimes \Psi_k)\eta_k\big]\Big\}. \end{aligned} \quad (17)$$

Due to $\{\zeta_k\}_{k\in[-\ell,\ -1]} = 0$, adding the zero term $\tilde{z}_k^T \tilde{z}_k - \gamma^2 \zeta_k^T \zeta_k - (\tilde{z}_k^T \tilde{z}_k - \gamma^2 \zeta_k^T \zeta_k)$ to $\mathbb{E}\{\mathcal{J}_k\}$ yields

$$\begin{aligned} \mathbb{E}\{\mathcal{J}_k\} \le& \mathbb{E}\Big\{ \tilde{\eta}_k^T \Pi_1 \tilde{\eta}_k + ||\tilde{z}_k||^2 \\ &- \lambda_k \big[h_k - (I \otimes \Phi_k)\eta_k\big]^T \big[h_k - (I \otimes \Psi_k)\eta_k\big] \\ &- \frac{\gamma^2}{\ell+1} \sum_{s=0}^{\ell} ||\zeta_{k-s}||^2 + \frac{\gamma^2}{\ell+1} \sum_{s=0}^{\ell} ||\zeta_{k-s}||^2 \\ &- \gamma^2 ||\zeta_k||^2 \Big\} - \mathbb{E}\Big\{ \|\tilde{z}_k\|^2 - \gamma^2 \|\zeta_k\|^2 \Big\} \\ \le& \mathbb{E}\Big\{ \tilde{\eta}_k^T \Pi_k \tilde{\eta}_k \Big\} - \mathbb{E}\Big\{ \|\tilde{z}_k\|^2 - \gamma^2 \|\zeta_k\|^2 \Big\} \\ &+ \mathbb{E}\Big\{ \frac{\gamma^2}{\ell+1} \sum_{s=0}^{\ell} ||\zeta_{k-s}||^2 - \gamma^2 ||\zeta_k||^2 \Big\}. \end{aligned} \quad (18)$$

Summing up (18) on both sides from 0 to N with respect to k, one has

$$\begin{aligned} &\sum_{k=0}^{N} \mathbb{E}\{\mathcal{J}_k\} = \mathbb{E}\{V_{N+1}\} - \mathbb{E}\{V_0\} \\ &\le \mathbb{E}\Big\{ \sum_{k=0}^{N} \tilde{\eta}_k^T \Pi_2^k \tilde{\eta}_k \Big\} - \mathbb{E}\Big\{ \sum_{k=0}^{N} (\|\tilde{z}_k\|^2 - \gamma^2 \|\zeta_k\|^2) \Big\} \\ &\quad + \mathbb{E}\Big\{ \frac{\gamma^2}{\ell+1} \sum_{s=0}^{\ell} \sum_{k=0}^{N} \big(||\zeta_{k-s}||^2 - ||\zeta_k||^2 \big) \Big\}. \end{aligned} \quad (19)$$

Finally, it can be easily concluded from (13), (14) and (19) that

$$\begin{aligned} &0 \le \mathbb{E}\{V_{N+1}\} + \mathbb{E}\Big\{ \gamma^2 \sum_{i=-\ell}^{0} \eta_i^T \mathcal{W}_i \eta_i - V_0 \Big\} \\ &\le \mathbb{E}\Big\{ \sum_{k=0}^{N} (\gamma^2 \|\zeta_k\|^2 - \|\tilde{z}_k\|^2) + \gamma^2 \sum_{i=-\ell}^{0} \eta_i^T \mathcal{W}_i \eta_i \Big\} \end{aligned} \quad (20)$$

which means that the $\mathcal{H}_\infty$ performance index (7) holds, and the proof is now complete.

3.2 Envelope Constraint Analysis

Let us now deal with the analysis issue on the envelope constraints for the addressed discrete time-varying stochastic systems by employing the idea of ellipsoid description borrowed from the set-membership filtering method.

Theorem 2 *Let the filter gain matrices $\{K_k\}_{k\in[0,N]}$ as well as the sequence $\{d_k^i, \varepsilon_k^i\}_{k\in[1,N]}$ of desired outputs and tolerance bands be given. For the augmented system (6) with $\eta_i = 0$ ($-\ell \le i \le 0$), the envelope constraints defined in (8) are guaranteed for the given input $\{\zeta_k^*\}_{k\in[0,N]}$ if there exist families of positive scalars $\{\mu_{k+1}, \pi_k, \tau_0^k, \tau_1^k, \cdots, \tau_\ell^k\}_{k\in[0,\ N]}$ and positive definite matrices $\{\mathcal{Q}_k\}_{k\in[1,\ N+1]}$ satisfying the following recursive matrix inequalities*

$$\Xi_k = \begin{bmatrix} -\mu_{k+1}^{\varepsilon,i} & 0 & \bar{\mathcal{M}}_k^T \\ * & -\mu_{k+1} \mathcal{Q}_{k+1}^{-1} & \mathcal{Q}_{k+1}^{-1} \mathcal{L}_{k+1}^T \mathbf{I}_i^T \\ * & * & -I \end{bmatrix} < 0 \quad (21a)$$

$$\Omega_k = \begin{bmatrix} \Xi_{0k} & \Xi_{1k}^T & \Xi_{2k}^T & \Xi_{3k}^T & \Xi_{4k}^T \\ * & -\mathcal{Q}_{k+1}^{-1} & 0 & 0 & 0 \\ * & * & -\Delta_k^Q & 0 & 0 \\ * & * & * & -\mathcal{Q}_{k+1}^{-1} & 0 \\ * & * & * & * & -\Delta_k^Q \end{bmatrix} < 0 \quad (21b)$$

where

$$\mu_{k+1}^{\varepsilon,i} = (\varepsilon_{k+1}^i)^2 - \mu_{k+1},\ \ i \in [1,\ n_z],$$

$$\Delta_k^1 = -\pi_k \eta_k^{*T} \mathcal{U}_{2k} \eta_k^* + \sum_{i=0}^{\ell} \tau_i^k - 1,\ \ \Delta_k^Q = I \otimes \mathcal{Q}_{k+1}^{-1},$$

$$\Delta_k^2 = (I \otimes \mathcal{K}_k)(\bar{\mathcal{C}}_{1k} \bar{\mathcal{S}}_{31} \mathcal{I}_{1v} \bar{\eta}_k^{\ell *} + \bar{\mathcal{E}}_{1k} \mathcal{I}_{1v} \zeta_k^{\ell *}),$$

$$\Theta_k = I\ (k = -\ell, \cdots, 0),\ \ \Theta_k \Theta_k^T = \mathcal{Q}_k^{-1}\ (k > 0),$$

$$\Gamma_k = diag\{\tau_1^k, \cdots, \tau_\ell^k\},\ \ \bar{\mathbf{\Theta}}_k = diag\{\Theta_{k-1}, \cdots, \Theta_{k-\ell}\},$$

$$\Xi_{0k} = \begin{bmatrix} \Delta_k^1 & -\pi_k \eta_k^{*T} \mathcal{U}_{2k} \Theta_k & 0 & \pi_k \eta_k^{*T} \mathcal{U}_{1k}^T \\ * & -\pi_k \Theta_k^T \mathcal{U}_{2k} \Theta_k - \tau_0^k I & 0 & \pi_k \Theta_k^T \mathcal{U}_{1k}^T \\ * & * & -I \otimes \Gamma_k & 0 \\ * & * & * & -\pi_k I \end{bmatrix},$$

$$\Xi_{1k} = \left[\, -\bar{\alpha}\bar{h}_k \quad \mathcal{A}_k \Theta_k \quad -\mathcal{K}_k \Lambda_1 \bar{\mathcal{C}}_{1k} \bar{\mathcal{S}}_{21} \bar{\mathbf{\Theta}}_k \quad \bar{\alpha} I \, \right],$$

$$\Xi_{2k} = \begin{bmatrix} \mathcal{D}_k \mathcal{S}_1 \bar{\eta}_k^* & \mathcal{D}_k \mathcal{S}_1 \Theta_k & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\tilde{\alpha}} I \end{bmatrix},$$

$$\Xi_{3k} = \left[\sqrt{\tilde{\vartheta}^0} \mathcal{K}_k (\mathcal{C}_k \mathcal{S}_3 \bar{\eta}_k^* + \mathcal{E}_k \zeta_k^*) \quad \sqrt{\tilde{\vartheta}^0} \mathcal{K}_k \mathcal{C}_k \mathcal{S}_3 \Theta_k \ 0 \ \ 0 \right],$$

$$\Xi_{4k} = \left[\, \Delta_k^2 \quad 0 \quad (I \otimes \mathcal{K}_k) \bar{\mathcal{C}}_{1k} \bar{\mathcal{S}}_{31} \bar{\mathbf{\Theta}}_k \mathcal{I}_{1v} \quad 0 \, \right],$$

$$\bar{\mathcal{M}}_k = \mathbf{I}_i \mathcal{L}_{k+1} \mathcal{M}_k - d_{k+1}^i, \ \bar{\eta}_0^* = 0, \ \bar{\eta}_0^{\ell *} = 0, \ \zeta_0^{\ell *} = 0,$$

$$\begin{aligned} \mathcal{M}_k = {} & \mathcal{A}_k \bar{\eta}_k^* + \bar{\alpha} \bar{h}_k^* - \mathcal{K}_k \Lambda_1 \bar{\mathcal{C}}_{1k} \bar{\mathcal{S}}_{21} \bar{\eta}_k^{\ell *} \\ & - \mathcal{K}_k \Lambda_1 \bar{\mathcal{E}}_{1k} \zeta_k^{\ell *} - \bar{\vartheta}^0 \mathcal{K}_k \mathcal{E}_k \zeta_k^* + \mathcal{F}_k \zeta_k^*, \end{aligned}$$

$$\bar{\eta}_k^{\ell *} = [\, \bar{\eta}_{k-1}^{*T}, \ \cdots, \ \bar{\eta}_{k-\ell}^{*T} \,]^T, \ \zeta_k^{\ell *} = [\, \zeta_{k-1}^{*T}, \ \cdots, \ \zeta_{k-\ell}^{*T} \,]^T.$$

Here, $\bar{\eta}_k^*$ satisfies the following recursive dynamics

$$\begin{aligned} \bar{\eta}_{k+1}^* = {} & \mathcal{A}_k \bar{\eta}_k^* + \bar{\alpha} \bar{h}_k^* - \sum_{s=1}^{\ell} \bar{\vartheta}^s \mathcal{K}_k \mathcal{C}_{k-s} \mathcal{S}_2 \bar{\eta}_{k-s}^* \\ & - \sum_{s=0}^{\ell} \bar{\vartheta}^s \mathcal{K}_k \mathcal{E}_{k-s} \zeta_{k-s}^* + \mathcal{F}_k \zeta_k^* \end{aligned} \tag{22}$$

with $\bar{h}_k^* = \left[h^T\big(k, [I \ 0] \bar{\eta}_k^*\big) \quad h^T\big(k, [I \ 0] \bar{\eta}_k^*\big) - h^T\big(k, [I \ 0] \bar{\eta}_k^* - [0 \ I] \bar{\eta}_k^*\big) \right]^T$.

Proof: To start with, let us first propose an ellipsoid description on the errors between the states of the augmented system (6) (with the given input ζ_k^*) and the recursive dynamics (22). Then, based on the obtained ellipsoid description, the envelope constraints given in (8) can be transformed into a set of recursive matrix inequalities that are easy to handle. For this purpose, define an ellipsoid $\Omega(\mathcal{Q}, \eta_s, 1)$ in the mean-square sense as follows

$$\begin{aligned} \Omega(\mathcal{Q}, \eta_s, 1) = \big\{ & \eta \in \mathbb{R}^{2n_x} : \eta_s \in \mathbb{R}^{2n_x}, \\ & \mathbb{E}\{(\eta - \eta_s)^T \mathcal{Q} (\eta - \eta_s)\} \leq 1 \big\} \end{aligned} \tag{23}$$

where $\mathcal{Q} \in \mathbb{R}^{2n_x \times 2n_x}$ is a positive definite matrix and η_s is the centre of the ellipsoid $\Omega(\mathcal{Q}, \eta_s, 1)$.

For $-\ell \leq i \leq 0$, because of $\bar{\eta}_i^* = \eta_i = 0$, one has that η_i belongs to the ellipsoid $\Omega(I, \bar{\eta}_i^*, 1)$. Similar to [6], for $-\ell \leq s \leq 0$, there exists a set of random vectors ϖ_s ($-\ell \leq s \leq 0$) with $\mathbb{E}\{\varpi_s^T \varpi_s\} \leq 1$ satisfying $\eta_s = \bar{\eta}_s^* + \Theta_s \varpi_s$ where $\Theta_s = I$.

In what follows, by using the *mathematical induction* method, we will first prove the following assertion.

Assertion: The solution $\mathcal{Q}_k$ of (21b) satisfies

$$\mathbb{E}\{(\eta_k - \bar{\eta}_k^*)^T \mathcal{Q}_k (\eta_k - \bar{\eta}_k^*)\} \leq 1, \quad 1 \leq k \leq N \tag{24}$$

or, $\eta_k \in \Omega(\mathcal{Q}_k, \bar{\eta}_k^*, 1)$, where $\bar{\eta}_k^*$ is determined by the dynamics (22).

The proof of the above assertion is divided into two steps, namely, *the initial step* and *the inductive step.*

Initial step. For $i = 1$, denote $\varpi_0^\ell = [\varpi_{-1}^T, \cdots, \varpi_{-\ell}^T]^T$ and $\varrho_0 = [1, \varpi_0^T, \varpi_0^{\ell T}, h_0^T]^T$. Considering $\mathbb{E}\{\varpi_s^T \varpi_s\} < 1$ ($s = -\ell, \cdots, 0$) and $\aleph^T(\bar{\eta}_0^*, \varpi_0) \aleph(\bar{\eta}_0^*, \varpi_0) \leq 0$ (with $\aleph(\bar{\eta}_0^*, \varpi_0) := h_0 - (I \otimes \Psi(0))(\bar{\eta}_0^* + \Theta_0 \varpi_0)$), one has

$$\begin{aligned} & \mathbb{E}\big\{(\eta_1 - \bar{\eta}_1^*)^T \mathcal{Q}_1 (\eta_1 - \bar{\eta}_1^*)\big\} - 1 \\ \leq & \mathbb{E}\Big\{ \varrho_0^T \Big(\Xi_{10}^T \mathcal{Q}_1 \Xi_{10} + \Xi_{20}^T (I \otimes \mathcal{Q}_1) \Xi_{20} \\ & + \Xi_{30}^T \mathcal{Q}_1 \Xi_{30} + \Xi_{40}^T (I \otimes \mathcal{Q}_1) \Xi_{40} \Big) \varrho_0 - 1 \\ & - \sum_{s=-\ell}^{0} \tau_s^0 (\varpi_s^T \varpi_s - 1) - \pi_0 \big[h_0 - (I \otimes \Phi(0)) \\ & \times (\bar{\eta}_0^* + \Theta_0 \varpi_0) \big]^T \big[h_0 - (I \otimes \Psi(0))(\bar{\eta}_0^* + \Theta_0 \varpi_0) \big] \Big\} \\ = & \mathbb{E}\Big\{ \varrho_0^T \Big(\Xi_{00}^T + \Xi_{10}^T \mathcal{Q}_1 \Xi_{10} + \Xi_{20}^T (I \otimes \mathcal{Q}_1) \Xi_{20} \\ & + \Xi_{30}^T \mathcal{Q}_1 \Xi_{30} + \Xi_{40}^T (I \otimes \mathcal{Q}_1) \Xi_{40} \Big) \varrho_0 \Big\}. \end{aligned} \tag{25}$$

Therefore, by using the Schur Complement Lemma, it can be verified from (25) that the solution $\mathcal{Q}_1$ of (21b) satisfies

$$\mathbb{E}\{(\eta_1 - \bar{\eta}_1^*)^T \mathcal{Q}_1 (\eta_1 - \bar{\eta}_1^*)\} \leq 1,$$

which means that η_1 belongs to the ellipsoid $\Omega(\mathcal{Q}_1, \bar{\eta}_1^*, 1)$.

Inductive step. So far, we have proved that the assertion is true of $i = 1$. Next, given that the assertion is true for $i = k$, we aim to show that the same assertion is true for $i = k + 1$.

Since the assertion is true for $i = k$, it follows again from [6] that there exists a set of random vectors ϖ_i (with $\mathbb{E}\{\varpi_i^T \varpi_i\} \leq 1$) satisfying $\eta_i = \bar{\eta}_i^* + \Theta_i \varpi_i$ where Θ_i is a factorization of $\mathcal{Q}_i^{-1} = \Theta_i \Theta_i^T$. It remains to show that, for $i = k + 1$, the solution $\mathcal{Q}_{k+1}$ of the recursive matrix inequalities (21b) guarantees $\eta_{k+1} \in \Omega(\mathcal{Q}_{k+1}, \bar{\eta}_{k+1}^*, 1)$.

For notational simplicity, denote $\varpi_k^\ell = [\, \varpi_{k-1}^T, \cdots, \varpi_{k-\ell}^T \,]^T$ and $\varrho_k = [\, 1, \ \varpi_k^T, \ (\varpi_k^\ell)^T, \ h_k^T \,]^T$. Similar to the initial step for $i = 1$, it can be derived that

$$\begin{aligned} & \mathbb{E}\big\{(\eta_{k+1} - \bar{\eta}_{k+1}^*)^T \mathcal{Q}_{k+1} (\eta_{k+1} - \bar{\eta}_{k+1}^*)\big\} - 1 \\ \leq & \mathbb{E}\Big\{ \varrho_k^T \Big(\Xi_{1k}^T \mathcal{Q}_{k+1} \Xi_{1k} + \Xi_{2k}^T (I \otimes \mathcal{Q}_{k+1}) \Xi_{2k} \\ & + \Xi_{3k}^T \mathcal{Q}_{k+1} \Xi_{3k} + \Xi_{4k}^T (I \otimes \mathcal{Q}_{k+1}) \Xi_{4k} \Big) \varrho_k - 1 \\ & - \sum_{s=k-\ell}^{k} \tau_s^k (\varpi_s^T \varpi_s - 1) - \pi_k \big[h_k - (I \otimes \Phi(k)) \\ & \times (\bar{\eta}_k^* + \Theta_k \varpi_k) \big]^T \big[h_k - (I \otimes \Psi(k))(\bar{\eta}_k^* + \Theta_k \varpi_k) \big] \Big\} \\ = & \mathbb{E}\Big\{ \varrho_k^T \Big(\Xi_{0k}^T + \Xi_{1k}^T \mathcal{Q}_{k+1} \Xi_{1k} + \Xi_{2k}^T (I \otimes \mathcal{Q}_{k+1}) \Xi_{2k} \\ & + \Xi_{3k}^T \mathcal{Q}_{k+1} \Xi_{3k} + \Xi_{4k}^T (I \otimes \mathcal{Q}_{k+1}) \Xi_{4k} \Big) \varrho_k \Big\}, \end{aligned} \tag{26}$$

which implies that η_{k+1} belongs to $\Omega(\mathcal{Q}_{k+1}, \bar{\eta}_{k+1}^*, 1)$ if (21b) holds. Therefore, by the induction, it can be concluded that the solution $\mathcal{Q}_k$ of (21b) satisfies (24).

Having proved the assertion, let us now consider the envelope constraints (8) which are equivalent to the following inequalities

$$\big[\mathbb{E}\{(\mathbf{I}_i \tilde{z}_{k+1} - d_{k+1}^i)\} \big]^2 \leq (\varepsilon_{k+1}^i)^2, \quad i \in [1, n_z]. \tag{27}$$

Noticing that there exists a random vector ϖ_{k+1} satisfying

$$\eta_{k+1} = \bar{\eta}^*_{k+1} + \Theta_{k+1}\varpi_{k+1}, \quad \mathbb{E}\{\varpi^T_{k+1}\varpi_{k+1}\} \leq 1,$$

one has

$$\begin{aligned}
&\big[\mathbb{E}\{(\mathbf{I}_i\tilde{z}_{k+1} - d^i_{k+1})\}\big]^2 - (\varepsilon^i_{k+1})^2 \\
&= \big[(\mathbf{I}_i\mathcal{L}_{k+1}\mathcal{M}_k - d^i_{k+1}) \\
&\quad + \mathbf{I}_i\mathcal{L}_{k+1}\Theta_{k+1}\mathbb{E}\{\varpi_{k+1}\}\big]^2 - (\varepsilon^i_{k+1})^2 \\
&= \chi^T_k \left\{ \begin{bmatrix} -(\varepsilon^i_{k+1})^2 & 0 \\ 0 & 0 \end{bmatrix} \right. \\
&\quad \left. + \begin{bmatrix} \bar{\mathcal{M}}^T_k \\ \Theta^T_{k+1}\mathcal{L}^T_{k+1}\mathbf{I}^T_i \end{bmatrix} \begin{bmatrix} \bar{\mathcal{M}}^T_k \\ \Theta^T_{k+1}\mathcal{L}^T_{k+1}\mathbf{I}^T_i \end{bmatrix}^T \right\} \chi_k
\end{aligned} \tag{28}$$

where $\chi_k := [\,1 \quad \mathbb{E}\{\varpi^T_{k+1}\}]^T$.

In light of $\mathbb{E}\{\varpi^T_{k+1}\}\mathbb{E}\{\varpi_{k+1}\} \leq \mathbb{E}\{\varpi^T_{k+1}\varpi_{k+1}\} < 1$, it follows from (28) that

$$\begin{aligned}
&\big[\mathbb{E}\{(\mathbf{I}_i\tilde{z}_{k+1} - d^i_{k+1})\}\big]^2 - (\varepsilon^i_{k+1})^2 \\
&< \big[\mathbb{E}\{(\mathbf{I}_i\tilde{z}_{k+1} - d^i_{k+1})\}\big]^2 - (\varepsilon^i_{k+1})^2 + \mu_{k+1} \\
&\quad - \mu_{k+1}\mathbb{E}\{\varpi^T_{k+1}\}\mathbb{E}\{\varpi_{k+1}\} \\
&= \chi^T_k \left\{ \begin{bmatrix} -(\varepsilon^i_{k+1})^2 + \mu_{k+1} & 0 \\ 0 & -\mu_{k+1}I \end{bmatrix} \right. \\
&\quad \left. + \begin{bmatrix} \bar{\mathcal{M}}^T_k \\ \Theta^T_{k+1}\mathcal{L}^T_{k+1}\mathbf{I}^T_i \end{bmatrix} \begin{bmatrix} \bar{\mathcal{M}}^T_k \\ \Theta^T_{k+1}\mathcal{L}^T_{k+1}\mathbf{I}^T_i \end{bmatrix}^T \right\} \chi_k. \\
&= \chi^T_k\tilde{\Upsilon}_k\chi_k
\end{aligned} \tag{29}$$

Obviously, it follow from (29) that (27) (or (8)) holds if $\tilde{\Upsilon}_k < 0$ which is, according to the Schur Complement Lemma, equivalent to

$$\begin{bmatrix} -(\varepsilon^i_{k+1})^2 + \mu_{k+1} & 0 & \bar{\mathcal{M}}^T_k \\ * & -\mu_{k+1}I & \Theta^T_{k+1}\mathcal{L}^T_{k+1}\mathbf{I}^T_i \\ * & * & -I \end{bmatrix} < 0. \tag{30}$$

By performing the congruence transformation diag$\{I, \Theta_{k+1}, I\}$ to (30), it is not difficult to obtain the inequality (21a) and therefore (30) is true. It can now be concluded that the envelope constraints (8) are achieved, which completes the proof.

Remark 2 *It is worth mentioning that the idea of the set-membership filtering is to construct an ellipsoidal state estimation set of all system states consistent with the measured outputs and the given disturbance information (i.e. a specified ellipsoid description) [9, 16]. Different from the traditional point estimation approaches (e.g. the $\mathcal{H}_\infty$ state estimation, the Bayes' estimation and the method of moments), the set-membership filtering approach can be utilized to obtain a certain region encompassing the system states rather than the estimation vector. In this paper, borrowed from the set-membership filtering method, the idea of employing the ellipsoid description on the estimation errors is used to convert the envelope constraints (8) into a set of matrix inequalities (21a) which can be easily handled via standard software package.*

3.3 Envelope-Constrained $\mathcal{H}_\infty$ Filter Design

Having established the analysis results, we are in a position to deal with the filter design problem. For this purpose, denote

$$\bar{\Pi}^k_{11} = \sum_{j=1}^{\ell}\mathcal{R}_{k,j} - \mathcal{P}_k - \lambda_k\mathcal{U}_{2k} + \mathcal{L}^T_k\mathcal{L}_k,$$

$$\bar{\Pi}^k_{33} = \mathrm{diag}\{\mathcal{R}_{k-1,1}, \mathcal{R}_{k-2,2}, \cdots, \mathcal{R}_{k-\ell,\ell}\},$$

$$\Upsilon_{0k} = \begin{bmatrix} \bar{\Pi}^k_{11} & \lambda_k\mathcal{U}_{1k} & 0 & 0 \\ * & -\lambda_k I & 0 & 0 \\ * & * & -\bar{\Pi}^k_{33} & 0 \\ * & * & * & -\frac{\gamma^2}{\ell+1}I \end{bmatrix}, \quad \tilde{\mathcal{P}}_{k+1} = \mathcal{P}^{-1}_{k+1},$$

$$\Upsilon_{2k} = [\, \sqrt{\tilde{\vartheta}^0}\mathcal{K}_k\mathcal{C}_k\mathcal{S}_3 \quad 0 \quad 0 \quad \sqrt{\tilde{\vartheta}^0}\mathcal{K}_k\mathcal{E}_k\mathcal{I}_D \,],$$

$$\Upsilon_{3k} = [\, 0 \;\; 0 \;\; (I\otimes\mathcal{K}_k)\bar{\mathcal{C}}_{1k}\bar{\mathcal{S}}_{31}\mathcal{I}_{1v} \quad [\, 0 \;\; (I\otimes\mathcal{K}_k)\bar{\mathcal{E}}_{1k}\mathcal{I}_{1v}]],$$

$$\Upsilon_{4k} = \begin{bmatrix} \mathcal{D}_k\mathcal{S}_1 & 0 & 0 & 0 \\ 0 & \sqrt{\bar{\alpha}}\mathcal{S}_1 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathcal{Q}}_{k+1} = \mathcal{Q}^{-1}_{k+1},$$

$$\bar{\tilde{\mathcal{P}}}_{k+1} = I\otimes\tilde{\mathcal{P}}_{k+1}, \quad \bar{\tilde{\mathcal{Q}}}_{k+1} = I\otimes\tilde{\mathcal{Q}}_{k+1}.$$

It is not difficult to see that the inequalities (13), (21a) and (21b) in Theorem 1 and 2 are, respectively, equivalent to

$$\Pi_k = \begin{bmatrix} \Upsilon_{0k} & \Upsilon^T_{1k} & \Upsilon^T_{2k} & \Upsilon^T_{3k} & \Upsilon^T_{4k} \\ * & -\tilde{\mathcal{P}}_{k+1} & 0 & 0 & 0 \\ * & * & -\tilde{\mathcal{P}}_{k+1} & 0 & 0 \\ * & * & * & -\bar{\tilde{\mathcal{P}}}_{k+1} & 0 \\ * & * & * & * & -\bar{\tilde{\mathcal{P}}}_{k+1} \end{bmatrix} < 0 \tag{31a}$$

$$\begin{aligned}
\tilde{\Xi}_k &= \begin{bmatrix} -\mu^{\varepsilon,i}_{k+1} & 0 & \bar{\mathcal{M}}^T_k \\ * & -\mu_{k+1}I & \tilde{\mathcal{Q}}_{k+1}\mathcal{L}^T_{k+1}\mathbf{I}^T_i \\ * & * & -I \end{bmatrix} \\
&\quad + \mathrm{diag}\Big\{0, \mu_{k+1}(I - \tilde{\mathcal{Q}}_{k+1}), 0\Big\} < 0, i\in[1, n_z]
\end{aligned} \tag{31b}$$

$$\tilde{\Omega}_k = \begin{bmatrix} \Xi_{0k} & \Xi^T_{1k} & \Xi^T_{2k} & \Xi^T_{3k} & \Xi^T_{4k} \\ * & -\tilde{\mathcal{Q}}_{k+1} & 0 & 0 & 0 \\ * & * & -\bar{\tilde{\mathcal{Q}}}_{k+1} & 0 & 0 \\ * & * & * & -\tilde{\mathcal{Q}}_{k+1} & 0 \\ * & * & * & * & -\bar{\tilde{\mathcal{Q}}}_{k+1} \end{bmatrix} < 0 \tag{31c}$$

Furthermore, it is apparent that $\tilde{\Xi}_k < 0$ if both $\tilde{\mathcal{Q}}_{k+1} \geq I$ and

$$\begin{bmatrix} -\mu^{\varepsilon,i}_{k+1} & 0 & \bar{\mathcal{M}}^T_k \\ * & -\mu_{k+1}I & \tilde{\mathcal{Q}}_{k+1}\mathcal{L}^T_{k+1}\mathbf{I}^T_i \\ * & * & -I \end{bmatrix} < 0, \; i\in[1, \; n_z] \tag{32}$$

hold with $\mu^{\varepsilon,i}_{k+1} = (\varepsilon^i_{k+1})^2 - \mu_{k+1}$. Note that the introduction of $\tilde{\mathcal{Q}}_{k+1} \geq I$ and (32) is for computational convenience.

According to Theorem 1 and Theorem 2, we have the following filter design scheme.

Theorem 3 *Let the disturbance attenuation level $\gamma > 0$, the positive definite matrices $\mathcal{W}_i$ $(i = -\ell, \cdots, 0)$, and the sequence of desired output and tolerance band $\{d_k^i, \ \varepsilon_k^i\}_{k\in[1,\ N]}$ be given. For the discrete time-varying stochastic system (1) with the envelope-constrained $\mathcal{H}_\infty$ filter (4), the output estimation errors $\{\tilde{z}_k\}_{k\in[0,N]}$ satisfy both the desired $\mathcal{H}_\infty$ performance (7) for any nonzero inputs and the envelope constraint requirement (8) for the certain inputs $\{\zeta_k^*\}_{k\in[0,N]}$ if there exist families of positive scalars $\{\lambda_k, \mu_{k+1}, \pi_k, \tau_0^k, \tau_1^k, \cdots, \tau_\ell^k\}_{k\in[0,\ N]}$, positive definite matrices $\mathcal{P}_0$, $\{\tilde{\mathcal{P}}_k,\ \tilde{\mathcal{Q}}_k\}_{k\in[1,\ N+1]}$, $\{\mathcal{R}_{i,j}\}_{i\in[-\ell,\ N],\ j\in[1,\ \ell]}$, and real-valued matrices $\{K_k\}_{k\in[0,\ N]}$ satisfying $\tilde{\mathcal{Q}}_{k+1} \geq I$, matrix inequalities(14), the recursive matrix inequalities (31a), (31c) and (32) with the parameters updated by $\mathcal{P}_{k+1} = \tilde{\mathcal{P}}_{k+1}^{-1}$ and $\mathcal{Q}_{k+1} = \tilde{\mathcal{Q}}_{k+1}^{-1}$.*

Remark 3 *In this paper, the envelope-constrained $\mathcal{H}_\infty$ filtering problem is investigated for a class of discrete time-varying stochastic systems with fading measurements, RONs and mixed noises. The main result established in Theorem 3 contains all the information about the $\mathcal{H}_\infty$ index, the envelope constraints, the occurring probability of RONs and the statistical information of channel coefficients. The main novelty is twofold: 1) a new envelope-constrained performance criterion is proposed to describe the transient dynamics of the filtering error process; and 2) by employing the ellipsoid description on the estimation errors, the envelope constraints (8) are transformed into a set of matrix inequalities and the filter gain matrix is obtained by solving these matrix inequalities.*

4 Numerical Example

In this section, a numerical example is presented to illustrate the effectiveness of the proposed envelope-constrained $\mathcal{H}_\infty$ filter design scheme for discrete time-varying stochastic systems (1) with fading measurements (3). The corresponding parameters are given as follows

$$A_k = \begin{bmatrix} 0.73 + 0.2\sin(1.5k) & 0.30 \\ -0.35 & 0.44 \end{bmatrix},\ D_k = \begin{bmatrix} 0.10 & -0.2 \\ 0 & 0.08 \end{bmatrix},$$

$$E_{1k} = \begin{bmatrix} 0.2 \\ -0.16 \end{bmatrix}, \quad C_k = [-0.6 \ \ 0.75],$$

$$E_{2k} = 0.06, \quad E_{3k} = 0.105, \quad L_k = [-0.072 \ 0.064].$$

The probability of RONs is taken as $\bar{\alpha} = 0.45$ and the nonlinear vector-valued function $h(k, x_k)$ is chosen as

$$h(k, x_k) = \begin{cases} \begin{bmatrix} h_1(k, x_k) \\ 0.06x_{2k} - \tanh(0.02x_{2k}) \end{bmatrix}, & 0 \leq k < 6 \\ \begin{bmatrix} 0.04x_{1k} - \tanh(0.02x_{1k}) \\ 0.05x_{2k} \end{bmatrix}, & 6 \leq k \leq 17 \end{cases}$$

where $h_1(k, x_k) = -0.06x_{1k} + 0.03x_{2k} + \tanh(0.03x_{1k})$ and $x_i(k)$ $(i = 1, 2)$ denotes the i-th element of the system state $x(k)$. It is easy to see that the constraint (2) is met with

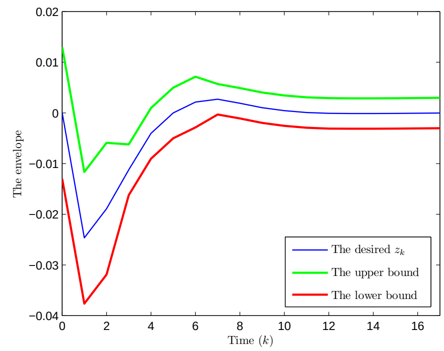

Fig. 2. The desired envelope.

$$\Phi_k = \begin{cases} \begin{bmatrix} -0.03 & 0.03 \\ 0 & 0.06 \end{bmatrix}, & 0 \leq k < 6, \\ \mathrm{diag}\{0.02, 0.05\}, & 6 \leq k \leq 17, \end{cases}$$

$$\Psi_k = \begin{cases} \begin{bmatrix} -0.06 & 0.03 \\ 0 & 0.04 \end{bmatrix}, & 0 \leq k < 6, \\ \mathrm{diag}\{0.04, 0.05\}, & 6 \leq k \leq 17. \end{cases}$$

The order of the fading model is $\ell = 1$ and the probability density functions of channel coefficients are as follows

$$\begin{cases} f(\vartheta^0) = 0.0005(e^{9.89\vartheta^0} - 1), & 0 \leq \vartheta^0 \leq 1, \\ f(\vartheta^1) = 8.5017e^{-8.5\vartheta^1}, & 0 \leq \vartheta^2 \leq 1. \end{cases} \tag{33}$$

It can be obtained that the mathematical expectation $\bar{\vartheta}_s$ $(s = 0, 1)$ are 0.8991 and 0.1174, and the variance $\tilde{\vartheta}_s$ $(s = 0, 1)$ are 0.0133 and 0.01364. Furthermore, by utilizing 2000 times simulation to system (1) with the given input ζ_k^*, the desired output d_k and the tolerance band ε_k are, respectively, selected as

$$d_k = 0, \quad 0 \leq k \leq 20; \quad \varepsilon_k = \begin{cases} 0.013, & 0 \leq k < 4, \\ 0.005, & 4 \leq k < 8, \\ 0.003, & 8 \leq k \leq 17. \end{cases}$$

The desired envelope is shown in Fig. 2. Let the positive scalar γ and the corresponding matrices be taken as $\gamma = 0.75$, $\mathcal{W}_0 = 21I$ and $\mathcal{W}_i = I$ $(i = -\ell, \cdots, -1)$. By applying Theorem 3, the desired filter parameters are obtained and shown in Tab. 1. Other matrices are omitted for space saving.

Table 1
The filter parameter K_k

k	0	1	2	3	$\cdots$
K_k	$\begin{bmatrix} -0.2890 \\ 0.2583 \end{bmatrix}$	$\begin{bmatrix} 0.3166 \\ 0.1592 \end{bmatrix}$	$\begin{bmatrix} 0.4553 \\ 0.0189 \end{bmatrix}$	$\begin{bmatrix} 0.4589 \\ 0.0117 \end{bmatrix}$	$\cdots$

In the simulation, the exogenous disturbance inputs are selected as $w_k = 0.5e^{-0.2k}\sin(k)$ and $v_k = 4\cos(k)/(k+1)$. The initial values $x(k)$ are randomly generated that obey uniform distribution over $[-0.5,\ 0.5]$. By utilizing

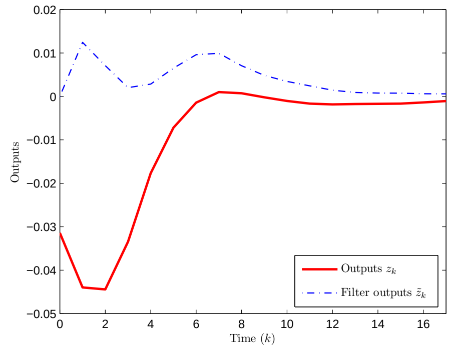

Fig. 3. The outputs.

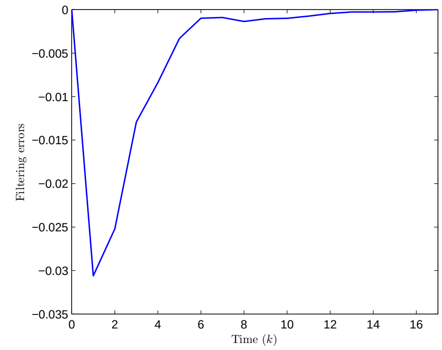

Fig. 4. The filtering errors.

500-times independent simulation trials, the average trajectories of the filter outputs and the filtering errors are shown in Fig. 3 and Fig. 4, respectively. The simulation results have confirmed that the designed filter performs very well.

In what follows, it would be interesting to see the relationship between the disturbance attenuation level γ and the probability $\bar{\alpha}$. With the same parameters with the previous trials, by applying Theorem 3, we can obtain the permitted γ with respect to the different probability $\bar{\alpha}$ as shown in Table 2, from which we can see that the disturbance attenuation performance (in terms of γ) deteriorates with increased $\bar{\alpha}$. Furthermore, in order to see the relationship between γ and the channel number ℓ, we assume that the probability $\bar{\alpha}$ is 0.45, and the parameters $\bar{\vartheta}_s$ and $\tilde{\vartheta}_s$ for all $s \in (0, 1, \cdots \ell)$ are 0.35 and 0.0134, respectively. In this case, the permitted γ is shown in Table 3, from which we can observe that a smaller ℓ leads to a better filtering performance. Finally, let us show the relationship between γ and the tolerance band ε_k. For this purpose, we assume that the probability density function $f(\vartheta^s)$ $(s = 1, 2)$ is the same as (33) and the probability $\bar{\alpha}$ is 0.45. By utilizing Theorem 3 again, we can obtain the permitted γ for different tolerance bands ε_k as listed in Table 4. We can find from Table 4 that a larger sequence value of the tolerance bands results in a smaller γ, indicating that a larger sequence value improves the disturbance attenuation performance, which deserves further theoretical investigation.

Table 2
The permitted γ with different $\bar{\alpha}$

$\bar{\alpha}$	0.40	0.41	0.42	0.43	0.44
γ	0.423674	0.427785	0.431567	0.436177	0.440454

Table 3
The permitted γ with different ℓ

ℓ	1	2	3	4	5
γ	0.4654	0.4844	0.4940	0.5046	0.5177

Table 4
The permitted γ with different tolerance bands

The tolerance band	$1.02\varepsilon_k$	$1.01\varepsilon_k$	ε_k	$0.99\varepsilon_k$
γ	0.44413	0.44441	0.44497	0.44525

5 Conclusions

In this paper, we have addressed the envelope-constrained $\mathcal{H}_\infty$ filtering problem for a class of discrete time-varying stochastic systems with fading measurements, randomly occurring nonlinearities (RONs) and mixed noises. Some uncorrelated random variables have been introduced, respectively, to govern the phenomena of RONs and fading measurements. A novel envelope-constrained performance criterion has been proposed to describe the transient dynamics of the filtering error process. By employing the stochastic analysis approach combined with the ellipsoid description on the estimation errors, some sufficient conditions have been established in the form of recursive matrix inequalities and the desired filter gain matrices have been obtained in terms of the solution to these matrix inequalities. Further research topics include the investigation on the blind equalization problems with fading measurements and RONs.

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