A Test Problem for Visual Investigation of High-Dimensional Multi-Objective Search

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Abstract— **An inherent problem in multiobjective optimization is that visual observation of solution vectors with four or more objectives is infeasible, which brings big difficulties for algorithmic design, examination, and development. This paper presents a test problem, called the Rectangle problem, to aid the visual investigation of high-dimensional multiobjective search. Key features of the Rectangle problem are that the Pareto optimal solutions 1) lie in a rectangle in the two-variable decision space and 2) are similar (in the sense of Euclidean geometry) to their images in the four-dimensional objective space. In this case, it is easy to examine the behavior of objective vectors in terms of both convergence and diversity, by observing their proximity to the optimal rectangle and their distribution in the rectangle, respectively, in the decision space. Fifteen algorithms have been investigated. Underperformance of Pareto-based algorithms as well as most state-of-the-art many-objective algorithms indicates that the proposed problem not only is a good tool to help visually understand the behavior of multiobjective search in a high-dimensional objective space but also can be used as a challenging benchmark function to test algorithms' ability in balancing convergence and diversity of solutions.**

I. Introduction

COMMON existence of optimization problems with more than three objectives in industrial and engineering design leads to the emergence of a new research topic in the evolutionary multiobjective optimization (EMO) area, called many-objective optimization [1], [2]. However, problem's characteristics of the high dimension landscape, such as the curse of dimensionality and the ineffectiveness of the Pareto-based selection, bring great challenges for algorithmic designers and practitioners in the area. One of the challenges lies in the visualization of evolutionary search. In contrast to for two- or three-objective problems where it is straightforward to show the track of objective vectors during the evolutionary process, for problems with four or more objectives we cannot visually monitor how a set of objective vectors are evolved and visually understand how are their distribution in the space and their proximity to the Pareto front.

Effort has been made to ease this challenge. In general, there exist two classes of methods to visualize a set of vectors in the objective space. One, stemming from the multiple criterion decision-making (MCDM) community, is on the direct display by using a plane plot in the sense that objective vectors are displayed with no modifications, such as the parallel coordinate, bar chart, and star coordinate methods [3]. These methods, though, often come without information about the Pareto dominance relation between vectors. The other class is on the mapping of high-dimensional objective vectors to two- or three-dimensional ones for visualization. Key concerns under such mapping include the maintenance of the Pareto dominance relation between vectors and the reflection of their location information in the population. Many current studies originate from this motivation, presenting various interesting attempts [4], [5], [6], [7]. However, inevitable information loss associated with the dimension reduction will influence the observation and understanding of objective vectors. In addition, several methods of constructing or/and mapping some 'key' vectors from a objective vector set have been developed [8], [9], [10]. Despite failing to display all vectors in detail, these methods can provide an outline of the whole set, e.g., the range and the location of the set in the space.

On the other hand, some studies deal with the visualization challenge of evolutionary search from another prospective. Unlike the above methods which focus on objective vectors coming from any test problem, these studies propose (or introduce) a particular class of test problems to help the visual investigation of evolutionary search [11], [12]. Specifically, Köppen and Yoshida [13] presented a class of many-objective test problems whose Pareto optimal set is in a regular polygon in a two-dimensional decision space. This allows easy visualization and examination of the proximity of the obtained solutions to the optimal region and their distribution in the decision space. Later, Ishibuchi *et al.* [14], [15] extended and generalized this class of problems (called distance minimization problems), introducing multiple Pareto optimal polygons with same [14] or different shapes [16] as well as making decision variables' dimensionality scalable [15]. Overall, these problems provide a good alternative to help understand the behavior of multiobjective evolutionary search, and thus have been widely used to compare many-objective algorithms in recent studies [17], [18], [19].

However, one weakness of such a class of test problems is that from the decision space prospective it fails to exactly reflect the behavior and performance of objective vectors, i.e., their convergence and distribution with respect to the Pareto front. Even if a set of objective vectors are distributed perfectly over the optimal front, we cannot know this fact via the observation of the corresponding decision variables in the polygon(s).

In this paper, we construct a four-objective test problem

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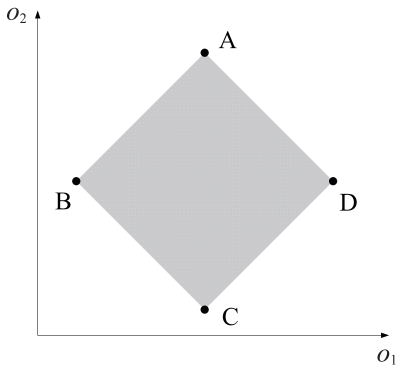


Fig. 1. An illustration of a four-objective distance minimization problem whose Pareto optimal region is determined by the four points.

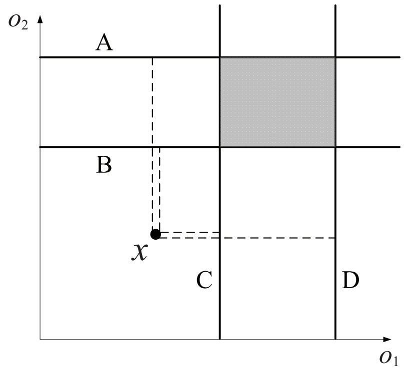


Fig. 2. An illustration of a Rectangle problem whose Pareto optimal region is determined by the four lines.

with its Pareto optimal region being a rectangle in a two-dimensional decision space, called the Rectangle problem. The key feature in this problem is that the Pareto optimal solutions in the decision space and their images in the objective space are similar (in the sense of Euclidean geometry). In other words, the ratio of the distance between any two Pareto optimal solutions to the distance between their corresponding objective vectors in the Pareto front is a constant. In this way, we can easily understand the behavior and performance of the objective vector set (e.g., its uniformity and coverage over the Pareto front) by observing the solution set in the two-variable decision space.

Using three instances of the proposed problem, we investigate the behavior of fifteen EMO algorithms, including well-known multiobjective algorithms and recently-developed many-objective ones. Interesting observations indicate that the Rectangle problem not only is a good tool to examine objective vectors' diversity that algorithms maintain but also provides a big challenge for algorithms to lead the solutions to search towards the optimal region.

The rest of this paper is organized as follows. Section II constructs and explains the proposed test problem. Section III briefly describes the considered EMO algorithms and their corresponding parameter settings. Section IV is devoted to visual investigation and understanding of the search behavior of the fifteen algorithms on three problem instances. Finally, conclusions are drawn in Section V.

II. Rectangle Problem

The distance minimization problems, proposed by Köppen and Yoshida [13] and generalized by Ishibuchi et al. [14], are a class of many-objective optimization problems that minimize the Euclidean distance from a solution to a given set of points in the two- or three- dimensional Euclidean space, where the distance to any of these points is treated as an independent objective. Figure 1 gives a four-objective example of the distance minimization problem with a set of points **A**, **B**, **C**, and **D**.

A significant feature of the distance minimization problems is that their Pareto optimal region is a convex polygon determined by the given point set [13]. This allows a clear observation of whether the considered solution set converges into the Pareto optimal region. However, a weakness of such problems is that they are unavailable for the distribution investigation of a solution set in the objective space. There is no explicit distribution relation between decision variables and their corresponding objective images in the problems.

Inspired by the above, this paper constructs a test problem whose Pareto optimal solutions lie in a rectangle in the decision space and more importantly are similar (in the sense of Euclidean geometry) to their images in the objective space. Unlike the distance minimization problems which consider the distance to a set of points, the proposed Rectangle problem takes into account the distance to a set of lines parallel to the coordinate axes. Figure 2 gives an example of the Rectangle problem where the Pareto optimal solutions are in the region enclosed by four lines **A**, **B**, **C**, and **D** (including the boundary).

Formally, the Rectangle problem minimizes the Euclidean distance from a solution, $\boldsymbol{x} = (x_1, x_2)$, to four lines parallel to the coordinate axes ($x_1 = a_1$, $x_1 = a_2$, $x_2 = b_1$, and $x_2 = b_1$):

$$\min \begin{array}{l} f_1(\boldsymbol{x}) = |x_1 - a_1| \\ f_2(\boldsymbol{x}) = |x_1 - a_2| \\ f_3(\boldsymbol{x}) = |x_2 - b_1| \\ f_4(\boldsymbol{x}) = |x_2 - b_2| \end{array} \tag{1}$$

Next, we explain the geometric similarity of the Rectangle problem between the Pareto optimal solutions and their images in the objective space. Let $\boldsymbol{x}^1 = (x_1^1, x_2^1)$ and $\boldsymbol{x}^2 = (x_1^2, x_2^2)$ be two Pareto optimal solutions for the problem given in Eq. (1) (here, without loss of generality, assuming $a_1 < a_2$ and $b_1 < b_2$). Then their Euclidean distance in the decision space is

$$D(\boldsymbol{x}^1, \boldsymbol{x}^2) = \left[(x_1^1 - x_1^2)^2 + (x_2^1 - x_2^2)^2\right]^{0.5} \tag{2}$$

Also, the distance of their images in the objective space is

$$\begin{aligned}
& D\big(\boldsymbol{f}(\boldsymbol{x}^1), \boldsymbol{f}(\boldsymbol{x}^2)\big) \\
= & \big[\big(f_1(\boldsymbol{x}^1) - f_1(\boldsymbol{x}^2)\big)^2 + \big(f_2(\boldsymbol{x}^1) - f_2(\boldsymbol{x}^2)\big)^2 + \\
& \big(f_3(\boldsymbol{x}^1) - f_3(\boldsymbol{x}^2)\big)^2 + \big(f_4(\boldsymbol{x}^1) - f_4(\boldsymbol{x}^2)\big)^2\big]^{0.5} \\
= & \big[(|x_1^1 - a_1| - |x_1^2 - a_1|)^2 + (|x_1^1 - a_2| - |x_1^2 - a_2|)^2 + \\
& (|x_2^1 - b_1| - |x_2^2 - b_1|)^2 + (|x_2^1 - b_2| - |x_2^2 - b_2|)^2\big]^{0.5}
\end{aligned} \tag{3}$$

Since $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are two Pareto optimal solutions of the problem, it holds that $a_1 \leq x_1^1, x_1^2 \leq a_2$ and $b_1 \leq x_2^1, x_2^2 \leq b_2$. So Eq. (3) can be further expressed as

$$\begin{aligned}
& = \big[(x_1^1 - x_1^2)^2 + (x_1^1 - x_1^2)^2 + (x_2^1 - x_2^2)^2 + (x_2^1 - x_2^2)^2\big]^{0.5} \\
& = \sqrt{2}\big[(x_1^1 - x_1^2)^2 + (x_2^1 - x_2^2)^2\big]^{0.5} \\
& = \sqrt{2} D(\boldsymbol{x}^1, \boldsymbol{x}^2)
\end{aligned} \tag{4}$$

The above equation indicates that the ratio of the distance between any two Pareto optimal solutions to the distance between their corresponding objective vectors is a constant. As such, it is easy to understand the distribution of the objective vectors in a Pareto front approximation by observing their position and crowding degree in the rectangle in the two-dimensional decision space.

Note that this two-dimensional problem with respect to decision variables can be extended to the three-dimensional scenario. In this way, the proposed problem will minimize the Euclidean distance from a solution (i.e., $\boldsymbol{x} = (x_1, x_2, x_3)$) to six lines parallel to the three coordinate axes, and the Pareto optimal region will become a cuboid enclosed by these six lines. In addition, it is necessary to point out that unlike in the distance minimization problems where objective dimensionality can be set freely, in the Rectangle problem the number of objectives (i.e., the considered lines) is determined by the number of decision variables (two lines corresponding to one coordinate axis). It is not easy (or even impossible) to add new lines while keeping the dimensionality of decision space unchanged, because the geometric similarity between the Pareto optimal solutions and their objective images will be violated when one coordinate axis corresponds to more than two lines.

III. Fifteen Algorithms Investigated

In this section, we briefly describe the considered algorithms and their corresponding parameter settings in the experimental studies. In all, fifteen EMO algorithms are investigated, including well-known multiobjective algorithms and recently-developed many-objective ones. Readers seeking more details on these algorithms should refer to their original literature.

- **Nondominated Sorting Genetic Algorithm II (NSGA-II)** [20]. As a representative Pareto-based algorithm, NSGA-II is well-known by its nondominated sorting and crowding distance-based density estimation strategies in fitness assignment.
- **Strength Pareto Evolutionary Algorithm 2 (SPEA2)** [21]. SPEA2 is also a prevalent Pareto-based algorithm, which adopts a so-called fitness strength value and the k-th nearest neighbor to rank individuals in the population.
- **Multiple Single Objective Pareto Sampling (MSOPS)** [22]. MSOPS uses the idea of single-objective aggregated optimization to search in parallel. Specifying an individual with a number of weight vectors, MSOPS is popular to deal with many-objective optimization problems.
- **Indicator-Based Evolutionary Algorithm (IBEA)** [23]. IBEA aims to integrate the preference information of the decision maker into multiobjective search. The main idea is to define the optimization goal in terms of a binary performance measure and then to directly use this measure to guide search.
- **ϵ-dominance Multiobjective Evolutionary Algorithm (ϵ-MOEA)** [24]. ϵ-MOEA is a steady-state algorithm using ϵ-dominance to strengthen selection pressure. Dividing the objective space into many hyperboxes, ϵ-MOEA allows each hyperbox at most one solution based on ϵ-dominance and the distance from solutions to the utopia point in the hyperbox.
- **S Metric Selection EMO Algorithm (SMS-EMOA)** [25]. SMS-EMOA, like IBEA, is also an indicator-based algorithm, which maximizes the hypervolume contribution of a population during the evolutionary process. Combined with the concept of nondominated sorting, SMS-EMOA can produce a well-converged and well-distributed solution set.
- **Multiobjective Evolutionary Algorithm based on Decomposition (MOEA/D)** [26]. Using a predefined set of weight vectors to maintain a diverse set of solutions, MOEA/D converts a multiobjective problem into many single-objective problems and tackles them simultaneously. As one of the most popular algorithms developed recently, MOEA/D can work well with different aggregation functions, such as Tchebycheff and penalty-based boundary intersection (here denoted as MOEA/D-TCH and MOEA/D-PBI, respectively).
- **Average Ranking (AR)** [27]. AR, ranking solutions in each objective and then summing all the rank values to evaluate them, has been found to be successful in terms of convergence for many-objective problems [33], despite the risk of leading the solutions to gather into a sub-area of the Pareto front due to the lack of a diversity maintenance scheme [34], [28].
- **Average Ranking combined with Grid (AR+Grid)** [28]. AR+Grid is a hybrid method which uses grid to enhance diversity for AR in many-objective optimization. In AR+Grid, the AR strategy is employed to provide the selection pressure towards the Pareto front, and the grid device is introduced to prevent solutions from being crowded in the objective space.
- **Hypervolume Estimation Algorithm (HypE)** [29].

TABLE I

THE PARAMETER SETTING AND THE SOURCE OF THE TESTED ALGORITHMS

Algorithm	Parameter(s)	Source
NSGA-II [20]		*http://www.iitk.ac.in/kangal*
SPEA2 [21]		*http://www.tik.ee.ethz.ch/pisa*
MSOPS [22]	weight vectors 200	*http://code.evanhughes.org/*
IBEA [23]	$\kappa = 0.05$	*http://www.tik.ee.ethz.ch/pisa*
ϵ-MOEA [24]	$\epsilon = 0.85$	*http://www.iitk.ac.in/kangal*
SMS-EMOA [25]		*http://jmetal.sourceforge.net/index.html*
MOEA/D-TCH [26]	neighborhood size 10%	*http://dces.essex.ac.uk/staff/qzhang/*
MOEA/D-PBI [26]	neighborhood size 10%, penalty parameter 2.0	*http://dces.essex.ac.uk/staff/qzhang/*
AR [27]		written by ourselves
AR+Grid [28]	grid division 30	*http://www.brunel.ac.uk/ cspgmml1/*
HypE [29]	sampling point $10,000$	*http://www.tik.ee.ethz.ch/pisa*
DMO [30]		written by ourselves
GrEA [31]	grid division 25	*http://www.brunel.ac.uk/ cspgmml1/*
FD-NSGA-II [32]	$\sigma = 0.5$	provided by its authors
SPEA2+SDE [18]		*http://www.brunel.ac.uk/ cspgmml1/*

Adopting Monte Carlo simulation to approximate the exact hypervolume value, HypE significantly reduces the time cost of the HV calculation and enables hypervolume-based search to be easily applied to many-objective optimization problems [29].

- **Diversity Management Operator (DMO)** [30]. Based on the basic framework of NSGA-II, DMO improves the diversity maintenance mechanism by adaptively tuning it according to the search range of the current evolutionary population.
- **Grid-based Evolutionary Algorithm (GrEA)** [31]. GrEA introduces three grid-based criteria to distinguish between individuals in mating and environmental selection. Together with a fitness adjustment strategy, these criteria are used to avoid partial overcrowding as well as guiding the search towards different promising directions in the space.
- **Fuzzy Dominance-based NSGA-II (FD-NSGA-II)** [32]. To deal with the failure of Pareto dominance in many-objective optimization, FD-NSGA-II develops a fuzzy dominance-based fitness evaluation mechanism to continuously differentiate individuals into different degrees of optimality. The concept of fuzzy logic is adopted in the algorithm to quantify the difference between individuals in a population.
- **SPEA2 with Shift-based Density Estimation (SPEA2+SDE)** [18]. Shifting individuals' position before estimating their density, SDE attempts to make Pareto-based algorithms suitable for many-objective optimization. In contrast to traditional density estimation which only involves individuals' distribution, SDE covers both the distribution and convergence information of individuals. The Pareto-based algorithm SPEA2 has been demonstrated to be very competitive in a high-dimensional space when cooperating with SDE.

A crossover probability $p_c = 1.0$ and a mutation probability $p_m = 1/L$ (where L denotes the number of decision variables) were used. The operators for crossover and mutation are simulated binary crossover (SBX) and polynomial mutation with both distribution indexes 20. The population size was set to 120 (also the archive set maintained with the same if required) and the termination criterion of a run was 30,000 evaluations (i.e., 250 generations) for all the algorithms. In ϵ-MOEA, the size of the archive set is determined by the ϵ value. For a fair comparison, we set ϵ so that the archive set is approximately of the same size as that of the other algorithms. Table I summarizes parameter settings as well as the source of all the algorithms. The setting of these parameters in our experimental studies either follows the suggestion in their original papers or has been found to make the algorithm perform well on the tested problem.

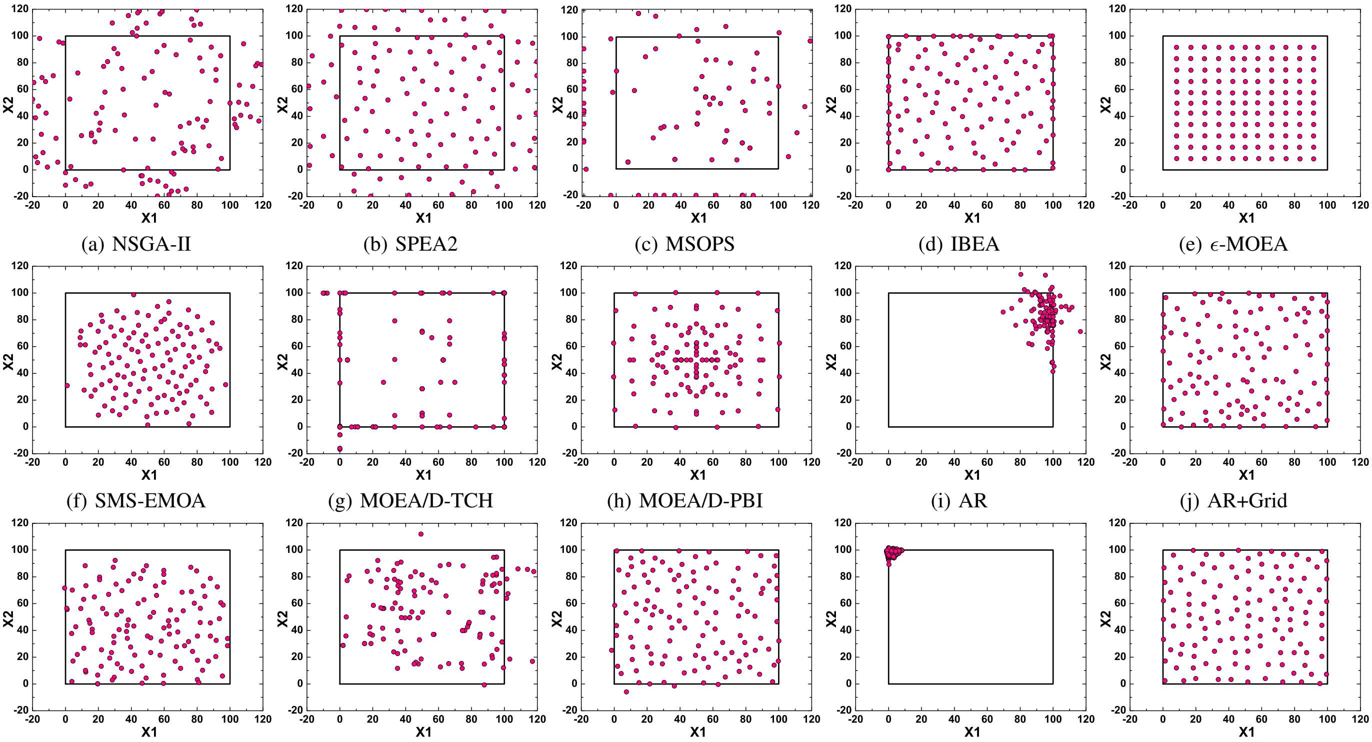
IV. RESULTS AND DISCUSSION

In this section, we investigate the behavior of the fifteen EMO algorithms in terms of convergence and diversity on the Rectangle problem by demonstrating their solutions in the two-dimensional decision space. Three problem instances with the same objective lines ($x_1 = 0$, $x_1 = 100$, $x_2 = 0$, and $x_2 = 100$) are introduced, where only difference lies in the search (decision) space's range, thus providing different challenges for an algorithm to balance convergence and diversity.

A. *Instance I*

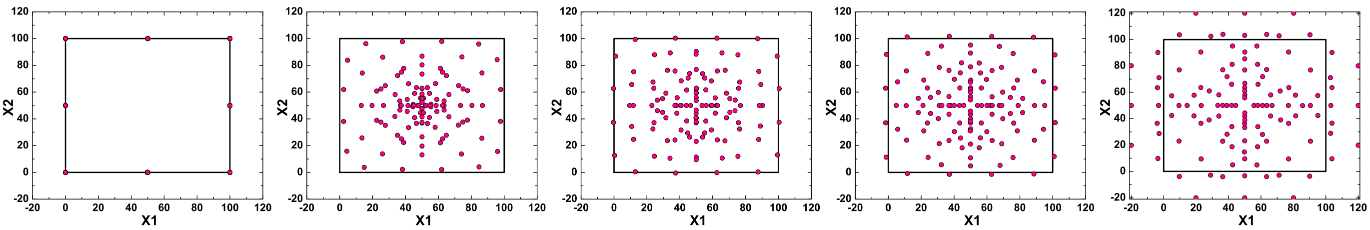
Figure 3 shows the final solution sets obtained by one typical run of the 15 algorithms on a problem instance where the search range of both x_1 and x_2 is $[-20, 120]$. From different behaviors of their solutions in the figure, these algorithms can be divided into four groups.

The first group corresponds to the algorithms which fail to converge, including NSGA-II, SPEA2, and MSOPS. Among them, SPEA2 performs better than the other two algorithms in terms of diversity, although its solutions seem to cover the whole search space rather than the optimal region. Two algorithms, AR and FD-NSGA-II, belong to the second group where the obtained solutions concentrate in a small part of the Pareto optimal region. The algorithms in the third group struggle to maintain uniformity although most of their



(a) NSGA-II (b) SPEA2 (c) MSOPS (d) IBEA (e) ϵ-MOEA
(f) SMS-EMOA (g) MOEA/D-TCH (h) MOEA/D-PBI (i) AR (j) AR+Grid
(k) HypE (l) DMO (m) GrEA (n) FD-NSGA-II (o) SPEA2+SDE

Fig. 3. The final solution set of the fifteen algorithms on the Rectangle problem where $x_1, x_2 \in [-20, 120]$.



(a) MOEA/D-PBI(0.1) (b) MOEA/D-PBI(1.5) (c) MOEA/D-PBI(2.0) (d) MOEA/D-PBI(2.5) (e) MOEA/D-PBI(5.0)

Fig. 4. The final solution set of the five implementations of MOEA/D-PBI with different penalty parameter values on the Rectangle problem where $x_1, x_2 \in [-20, 120]$. The number in the bracket denotes the penalty parameter value of the algorithm.

solutions can converge into the optimal region. MOEA/D-TCH, MOEA/D-PBI, HypE, and DMO fall into this group, their solutions overcrowded in some regions of the rectangle, thus leading to vacancy in other ones.

The last group involves the remaining algorithms (i.e., IBEA, ϵ-MOEA, SMS-EMOA, AR+Grid, GrEA, and SPEA2+SDE) which perform well in terms of convergence and diversity. More specifically, the solutions obtained by ϵ-MOEA tend to be perfectly uniform, but fail to cover the boundary of the optimal region. SMS-EMOA has the same problem, with most of its solutions on the middle part of the rectangle. This observation is interesting in view of that SMS-EMOA has been reported to perform well in maintaining solutions' extensity [25], [35]. Although the solutions of IBEA, AR+Grid, and GrEA can reach the boundary the optimal region, they are not so uniform as those of ϵ-MOEA and SPEA2+SDE. SPEA2+SDE appears to be the only algorithm with excellent performance in terms of both extensity and uniformity, and its solutions are distributed uniformly over the whole Pareto optimal region.

In addition, it is worth mentioning that among different implementations of MOEA/D regarding aggregation functions, MOEA/D-PBI with the penalty parameter value 2 (i.e., the setting considered here) performs best on the rectangle problem; this is not the case for the distance minimization problem where MOEA/D-TCH and MOEA/D-PBI with the penalty parameter value 0.1 (denoted as MOEA/D-PBI(0.1)) have been found to work well [36], [18]. Figure 4 shows the result of five implementations of MOEA/D-PBI with different penalty parameter values on the tested instance. Clearly, MOEA/D-PBI(0.1) fails to maintain diversity, while MOEA/D-PBI(5.0) to make all of its solutions converge into the optimal region. Despite having similar distribution with MOEA/D-PBI(1.5) and MOEA/D-PBI(2.5), MOEA/D-

PBI(2.0) seems to achieve a better balance between convergence and diversity, some of its solutions located exactly in the boundary of the rectangle.

B. Instance II

The Rectangle problem instance considered in this section greatly enlarges the search space of instance I, with $x_1, x_2 \in [-10000, 10000]$. Figure 3 shows the final solution sets obtained by one typical run of the 15 algorithms on the instance.

It is clear from the figure that most of the algorithms face big challenges in balancing convergence and diversity on this problem instance. The solution sets obtained by NSGA-II, SPEA2, AR, AR+Grid, DMO, and FD-NSGA-II fail to approach the Pareto optimal region. The first five sets are distributed in the form of a cross and the last one is located in a rhombic region. All solutions of MSOPS overlap in six points near the rectangle. MOEA/D-TCH and GrEA perform similarly—most of their solutions can converge into the Pareto optimal region, but there still exist several solutions far away from the optimal rectangle. Although all the solutions obtained by IBEA and HypE are the Pareto optimal solutions, the two algorithms struggle to maintain uniformity, leading their solutions to concentrate (or even coincide) in some areas of the rectangle.

The remaining four algorithms, ϵ-MOEA, SMS-EMOA, MOEA/D-PBI, and SPEA2+SDE, perform significantly better than the previous ones. The solutions of ϵ-MOEA have almost perfect uniformity. Despite some shortcomings in terms of extensity or uniformity, the solution set obtained by SMS-EMOA and MOEA/D-PBI largely covers the whole optimal region. SPEA2+SDE, like the case on the problem instance I, achieves the best performance in balancing solutions' uniformity and extensity.

Contrast the results on instance II with those on instance I: only the four algorithms (i.e., ϵ-MOEA, SMS-EMOA, MOEA/D-PBI, and SPEA2+SDE) perform similarly; some algorithms' solution set, like NSGA-II's and SPEA2's, is far away from the optimal region and distributed crosswise. This is not the case for the distance minimization problem where the solution set obtained by Pareto-based algorithms can easily approach the Pareto optimal region even when the number of objectives reaches ten [19]. Figure 6 gives an illustration to explain why this happen. Let $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ be two solutions for the Rectangle problem in the figure. $\boldsymbol{x}^1$ is located in the middle of the two objective lines parallel to coordinate axis o_2, and $\boldsymbol{x}^2$ in the right upper area to the four objective lines. The region that Pareto-dominates $\boldsymbol{x}^1$ is a line segment, far smaller than that dominating $\boldsymbol{x}^2$, although $\boldsymbol{x}^2$ is closer to the optimal region than $\boldsymbol{x}^1$. In fact, any solution located between two parallel objective lines (except the Pareto optimal solutions) is dominated only by a line segment parallel to the two objective lines, given that any improvement of the solution's distance to the one objective line will lead to the degradation to the other. This characteristic of the Rectangle problem (i.e., some non-Pareto optimal solutions dominated by only a linear

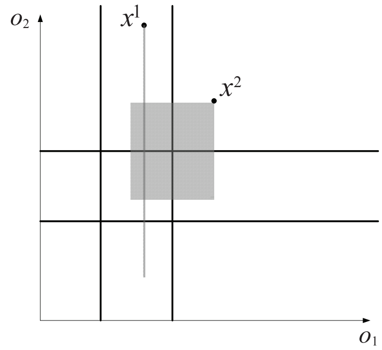


Fig. 6. An illustration of the difficulty for algorithms to converge on the Rectangle problem. The shadows are the regions that dominate $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$, respectively.

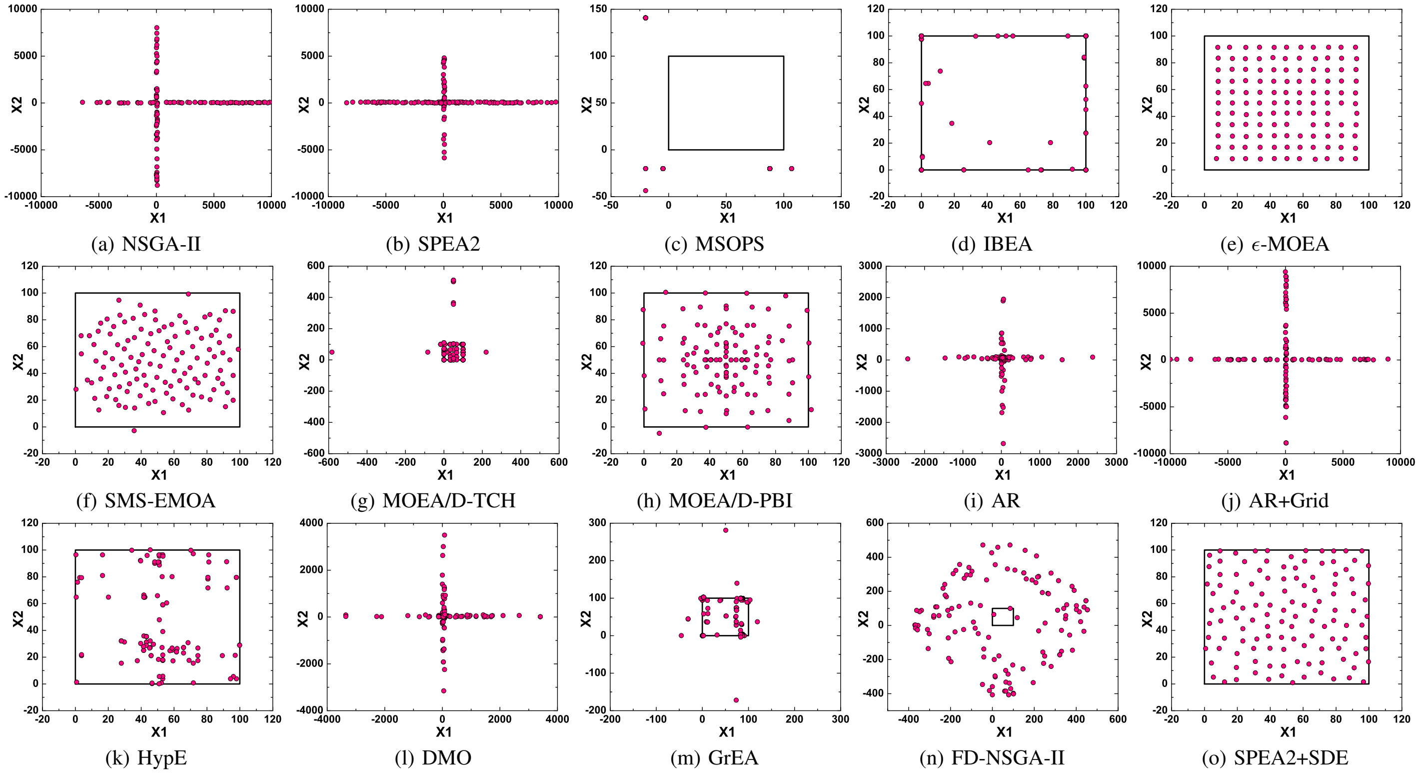
region) will bring a great challenge for algorithms which use Pareto dominance as the sole selection criterion in terms of convergence, usually leading their solutions to be distributed crisscross in the space.

In addition, it is necessary to mention that on instance II the optimal setting of the algorithms' parameter(s) (if existing) is likely to be different from that on instance I. The characteristic of the Rectangle problem explained above can make the optimal setting of the parameter(s) vary for the search space with different ranges. For instance, a significantly large grid division of GrEA (say 500) can make the algorithm's solutions converge into the optimal rectangle as well as having good diversity on instance II. Similar cases lie in the algorithms IBEA, AR+Grid, and FD-NSGA-II.

C. Instance III

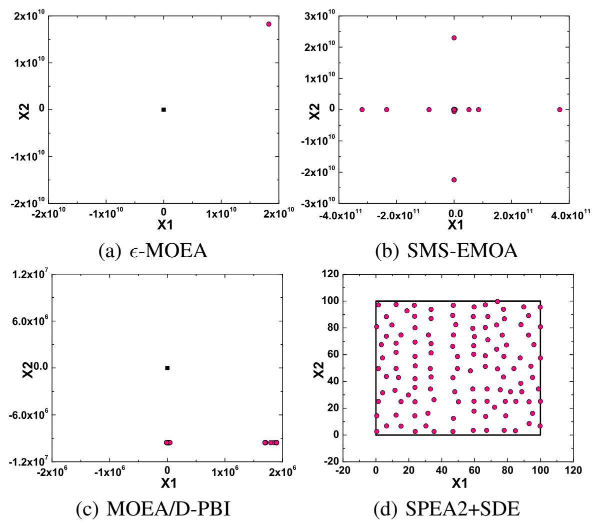
From the result comparison on instance I and instance II, the four algorithms, ϵ-MOEA, SMS-EMOA, MOEA/D-PBI, and SPEA2+SDE, have been found to perform steadily with the change of search space. In this section, we tremendously enlarges the search range of solutions in order to further test the algorithms' ability of leading solutions to converge towards the Pareto optimal region when working in a huge space. Figure 7 shows the final solution sets obtained by one typical run of the four algorithms on the problem instance with $x_1, x_2 \in [-10^{12}, 10^{12}]$.

Clearly, only SPEA2+SDE works well on this instance, with its solutions located in the rectangle as well as having good coverage. The archive set of ϵ-MOEA has only one individual far from the optimal region. In fact, however setting the ϵ value of the algorithm, there is only one solution left in the final archive set when the problem's search space becomes huge. Likewise, SMS-EMOA and MOEA/D-PBI fail to lead their solution set to approach the optimal region. The solutions of the former have a crisscross distribution and the solutions of the latter gather into two clusters parallel to the horizontal axis.



(a) NSGA-II (b) SPEA2 (c) MSOPS (d) IBEA (e) ϵ-MOEA
(f) SMS-EMOA (g) MOEA/D-TCH (h) MOEA/D-PBI (i) AR (j) AR+Grid
(k) HypE (l) DMO (m) GrEA (n) FD-NSGA-II (o) SPEA2+SDE

Fig. 5. The final solution set of the fifteen algorithms on the Rectangle problem where $x_1, x_2 \in [-10000, 10000]$.



(a) ϵ-MOEA (b) SMS-EMOA
(c) MOEA/D-PBI (d) SPEA2+SDE

Fig. 7. The final solution set of the four algorithms on the Rectangle problem where $x_1, x_2 \in [-10^{12}, 10^{12}]$.

V. Conclusions

Visual investigation of evolutionary search in a high-dimensional space is an important issue in the EMO area, which can help understand the behavior of the existing EMO algorithms, facilitate their modification, and further develop new algorithms for many-objective optimization problems. Unlike the existing studies which mainly focus on the mapping of high-dimensional objective vectors to two- or three-dimensional ones for visualization, this paper develops a test function, called the Rectangle problem, where the Pareto optimal solutions in the two-variable decision space have similar distribution to their images in the four-dimensional objective space. In this case, it is feasible to visually investigate high-dimensional objective vectors of the problem by observing their behavior in the decision space.

Fifteen EMO algorithms have been investigated on three instances of the proposed problem with varying range of search space which present different challenges for an algorithm to converge. Different behaviors of the test algorithms have been demonstrated. The Pareto-based algorithms (i.e., NSGA-II and SPEA2) fail to guide their solutions evolving towards the Pareto optimal region even if the optimal region accounts for a large proportion of the whole search space. IBEA, AR+Grid, and GrEA can achieve a good balance between convergence and diversity on instance I, but struggle when the search space become larger. Although ϵ-MOEA, SMS-EMOA, and MOEA/D-PBI work well on instances I and II, their solutions fail to approach the optimal rectangle when a huge problem's search space is introduced. SPEA2+SDE is the only algorithm with good performance on all the three test instances, its solutions distributed uniformly over the whole Pareto optimal region all along.

Despite involving only four objectives, the Rectangle problem has posed big challenges to most of the many-objective algorithms. This is different from the case that on most four-objective problems in the literature (e.g., the DTLZ problem suite [37]) the algorithms developed specifically for many-objective optimization (or even only based on Pareto selection criterion) perform fairly well [35]. One major future work is on the changeability of objectives' number of the Rectangle problem. More research is desired to improve the proposed problem or develop new problems whose objectives' number is independent of the dimensionality of decision space.

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