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INVARIANT MEASURES OF STOCHASTIC 2D NAVIER-STOKES EQUATIONS DRIVEN BY α-STABLE PROCESSES

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Abstract
In this note we prove the well-posedness for stochastic 2D Navier-Stokes equation driven by general Lévy processes (in particular, α-stable processes), and obtain the existence of invariant measures.

1 Introduction and Main Result

In this article we are concerned with the following stochastic 2D Navier-Stokes equation in torus $\mathbb{T}^2 = (0,1]^2$:

$$\mathrm{d}u_t = [\Delta u_t - (u_t \cdot \nabla)u_t + \nabla p_t]\mathrm{d}t + \mathrm{d}L_t, \;\; \mathrm{div}u_t = 0, \;\; u_0 = \varphi \in \mathbb{H}^0, \tag{1.1}$$

where $u_t(x) = (u_t^1(x), u_t^2(x))$ is the 2D-velocity field, p is the pressure, and $(L_t)_{t\geqslant 0}$ is an infinite dimensional cylindrical Lévy process given by

$$L_t = \sum_{j\in\mathbb{N}} \beta_j L_t^{(j)} e_j,$$

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where $\{(L_t^{(j)})_{t\geqslant 0}, j\in\mathbb{N}\}$ is a sequence of independent one dimensional purely discontinuous Lévy processes defined on some filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant 0};P)$ and with the same Lévy measure ν, $\{\beta_j, j\in\mathbb{N}\}$ is a sequence of positive numbers and $\{e_j, j\in\mathbb{N}\}$ is a sequence of orthonormal basis of Hilbert space $\mathbb{H}^0$, where for $\gamma\in\mathbb{R}$, $\mathbb{H}^\gamma$ with the norm $\|\cdot\|_\gamma$ and inner product $\langle\cdot,\cdot\rangle_\gamma$ denotes the usual Sobolev space of divergence free vector fields on $\mathbb{T}^2$ (see Section 2 for a definition).
As a continuous model, stochastic Navier-Stokes equation driven by Brownian motion has been extensively studied in the past decades (cf. [9, 3, 5, 8], etc.). Meanwhile, stochastic partial differential equation with jump has also been studied recently (cf. [11, 6]). However, in the well-known results, the assumption that the jump process has finite second order moments was required in order to obtain the usual energy estimate. This excludes the interest α-stable process. In this note, we establish the well posedness for stochastic 2D Navier-Stokes equation (1.1) driven by a general cylindrical Lévy process, and obtain the existence of invariant measures for this discontinuous model. More precisely, we shall prove that:

Theorem 1.1. *Suppose that for some $\theta\in(0,1]$,*

$$\mathbf{(H_\theta)}:\quad H_\theta := \int_{|x|>1}|x|^\theta\nu(\mathrm{d}x)+\int_{|x|\leqslant 1}|x|^{2\theta}\nu(\mathrm{d}x)+\sum_{j\in\mathbb{N}}|\beta_j|^\theta<+\infty.$$

Then for any $\varphi\in\mathbb{H}^0$, there exists a unique solution $(u_t)_{t\geqslant 0}=(u_t(\varphi))_{t\geqslant 0}$ to equation (1.1) satisfying that for P-almost all ω and for any $t>0$,

(i) $t\mapsto u_t(\omega)$ is right continuous and has left-hand limit in $\mathbb{H}^0$, and $\int_0^t\|\nabla u_s(\omega)\|_0^2\mathrm{d}s<+\infty$;

(ii) it holds that for any $\phi\in\mathbb{H}^1$,

$$\langle u_t(\omega),\phi\rangle_0=\langle\varphi,\phi\rangle_0+\int_0^t[\langle\Delta u_s(\omega),\phi\rangle_0+\langle u_s(\omega)\otimes u_s(\omega),\nabla\phi\rangle_0]\mathrm{d}s+\langle L_t(\omega),\phi\rangle_0.$$

Moreover, there exists a constant $C=C(H_\theta,\theta)>0$ such that for any $t>0$,

$$\mathbb{E}\left(\sup_{s\in[0,t]}\|u_s\|_0^\theta\right)+\mathbb{E}\left(\int_0^t\frac{\|\nabla u_s\|_0^2}{(\|u_s\|_0^2+1)^{1-\theta/2}}\mathrm{d}s\right)\leqslant C(1+\|\varphi\|_0^\theta+t). \tag{1.2}$$

In particular, there exists a probability measure μ on $(\mathbb{H}^0,\mathscr{B}(\mathbb{H}^0))$ called invariant measure of $(u_t(\varphi))_{t\geqslant 0}$ such that for any bounded measurable functional Φ on $\mathbb{H}^0$,

$$\int_{\mathbb{H}^0}\mathbb{E}\Phi(u_t(\varphi))\mu(\mathrm{d}\varphi)=\int_{\mathbb{H}^0}\Phi(\varphi)\mu(\mathrm{d}\varphi).$$

Remark 1.2. *Assumption $\mathbf{(H_\theta)}$ implies that cylindrical Lévy process $(L_t)_{t\geqslant 0}$ admits a càdlàg version in $\mathbb{H}^0$ and for any $t>0$ (cf. [12, p.159, Theorem 25.3]),*

$$\mathbb{E}\|L_t\|_0^\theta<+\infty.$$

In fact, for $\theta\in(0,1]$, by the elementary inequality $(a+b)^\theta\leqslant a^\theta+b^\theta$, we have

$$\mathbb{E}\|L_t\|_0^\theta\leqslant\mathbb{E}\left(\sum_{j\in\mathbb{N}}|\beta_j|\cdot|L_t^{(j)}|\right)^\theta\leqslant\sum_{j\in\mathbb{N}}|\beta_j|^\theta\cdot\mathbb{E}|L_t^{(j)}|^\theta=\mathbb{E}|L_t^{(1)}|^\theta\sum_{j\in\mathbb{N}}|\beta_j|^\theta<+\infty.$$

Moreover, $\mathbf{(H_\theta)}$ admits $\nu(\mathrm{d}x)=\mathrm{d}x/|x|^{1+\alpha}$ with $\alpha\in(\theta,2\theta)$.

Remark 1.3. *By estimate (1.2) and Poincàre's inequality, we have*

$$\begin{aligned}\mathbb{E}\left(\int_0^t \|\nabla u_s\|_0^\theta \mathrm{d}s\right) &\leqslant \mathbb{E}\left(\int_0^t \frac{\|\nabla u_s\|_0^\theta(\|u_s\|_0^{2-\theta}+1)}{(\|u_s\|_0^2+1)^{1-\theta/2}}\mathrm{d}s\right)\\ &\leqslant C\mathbb{E}\left(\int_0^t \frac{\|\nabla u_s\|_0^2+1}{(\|u_s\|_0^2+1)^{1-\theta/2}}\mathrm{d}s\right)\\ &\leqslant C(1+\|\varphi\|_0^\theta+t).\end{aligned}$$

This estimate in particular yields the existence of invariant measures by the classical Bogoliubov-Krylov's argument (cf. [4]).

Remark 1.4. *An obvious open question is about the uniqueness of invariant measures (i.e. ergodicity) for discontinuous system (1.1). The notion of asymptotic strong Feller property in [9] is perhaps helpful for solving this problem.*

This paper is organized as follows: In Section 2, we give some necessary materials. In Section 3, we prove the main result.

2 Preliminaries

In this section we prepare some materials for later use. Let $C_0^\infty(\mathbb{T}^2)^2$ be the space of all smooth $\mathbb{R}^2$-valued function on $\mathbb{T}^2$ with vanishing mean and divergence, i.e.,

$$\int_{\mathbb{T}^2} f(x)\mathrm{d}x = 0, \quad \mathrm{div} f(x) = 0.$$

For $\gamma \in \mathbb{R}$, let $\mathbb{H}^\gamma$ be the completion of $C_0^\infty(\mathbb{T}^2)^2$ with respect to the norm

$$\|f\|_\gamma = \left(\int_{\mathbb{T}^2} |(-\Delta)^{\gamma/2} f(x)|^2 \mathrm{d}x\right)^{1/2},$$

where $(-\Delta)^{\gamma/2}$ is defined through Fourier's transform. Thus, $(\mathbb{H}^\gamma, \|\cdot\|_\gamma)$ is a separable Hilbert space with the obvious inner product

$$\langle f, g\rangle_\gamma := \int_{\mathbb{T}^2} (-\Delta)^{\gamma/2} f(x) \cdot (-\Delta)^{\gamma/2} g(x)\mathrm{d}x.$$

Below, we shall fix an orthonormal basis $\{e_j, j \in \mathbb{N}\} \subset C_0^\infty(\mathbb{T}^2)^2$ of $\mathbb{H}^0$ consisting of the eigenvectors of Δ, i.e.,

$$\Delta e_j = -\lambda_j e_j, \quad \langle e_j, e_j\rangle_0 = 1, \quad j = 1, 2, \cdots, \tag{2.1}$$

where $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_j \uparrow \infty$.

Let $\{(L_t^{(j)})_{t\geqslant 0}, j \in \mathbb{N}\}$ be a sequence of independent one dimensional purely discontinuous Lévy processes with the same characteristic function, i.e.,

$$\mathbb{E}e^{\mathrm{i}\xi L_t^{(j)}} = e^{-t\psi(\xi)}, \ \forall t \geqslant 0, j = 1, 2, \cdots,$$

where $\psi(\xi)$ is a complex valued function called Lévy symbol given by

$$\psi(\xi) = \int_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}\xi y} - 1 - \mathrm{i}\xi y 1_{|y|\leqslant 1})\nu(\mathrm{d}y),$$

where ν is the Lévy measure and satisfies that

$$\int_{\mathbb{R}\setminus\{0\}} 1 \wedge |y|^2 \nu(\mathrm{d}y) < +\infty.$$

For $t > 0$ and $\Gamma \in \mathscr{B}(\mathbb{R}\setminus\{0\})$, the Poisson random measure associated with $L_t^{(j)}$ is defined by

$$N^{(j)}(t,\Gamma) := \sum_{s\in(0,t]} 1_\Gamma(L_s^{(j)} - L_{s-}^{(j)}).$$

The compensated Poisson random measure is given by

$$\tilde{N}^{(j)}(t,\Gamma) = N^{(j)}(t,\Gamma) - t\nu(\Gamma).$$

By Lévy-Itô's decomposition (cf. [2, p.108, Theorem 2.4.16]), one has

$$L_t^{(j)} = \int_{|x|\leqslant 1} x\tilde{N}^{(j)}(t,\mathrm{d}x) + \int_{|x|>1} xN^{(j)}(t,\mathrm{d}x).$$

For a Polish space $(\mathbb{G},\rho)$, let $\mathbb{D}(\mathbb{R}_+;\mathbb{G})$ be the space of all right continuous functions with left-hand limits from $\mathbb{R}_+$ to $\mathbb{G}$, which is endowed with the Skorohod topology:

$$d_{\mathbb{G}}(u,v) := \inf_{\lambda\in\Lambda}\left[\sup_{s\neq t}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right| \vee \int_0^\infty \sup_{t\geqslant 0}(\rho(u_{t\wedge r}, v_{\lambda(t)\wedge r})\wedge 1)e^{-r}\mathrm{d}r\right], \tag{2.2}$$

where Λ is the space of all continuous and strictly increasing function from $\mathbb{R}_+ \to \mathbb{R}_+$ with $\lambda(0) = 0$ and $\lambda(\infty) = \infty$. Thus, $(\mathbb{D}(\mathbb{R}_+;\mathbb{G}); d_{\mathbb{G}})$ is again a Polish space (cf. [7, p.121, Theorem 5.6]).
We need the following tightness criterion, which is a direct combination of [10, Corollary 5.2] and Aldous's criterion [1].

Theorem 2.1. *Let $\{(X_t^n)_{t\geqslant 0}, n\in\mathbb{N}\}$ be a sequence of $\mathbb{H}^{-1}$-valued stochastic processes with càdlàg path. Assume that*

(i) for each $\phi\in C_0^\infty(\mathbb{T}^2)^2$ and $t>0$, $\lim_{K\to\infty}\sup_{n\in\mathbb{N}} P\left\{\sup_{s\in[0,t]}|\langle X_s^n,\phi\rangle_{-1}|\geqslant K\right\} = 0$;

(ii) for each $\phi\in C_0^\infty(\mathbb{T}^2)^2$ and $t,a>0$,

$$\lim_{\varepsilon\to 0+}\sup_{n\in\mathbb{N}}\sup_{\tau\in\mathscr{S}_t} P\left\{|\langle X_\tau^n - X_{\tau+\varepsilon}^n,\phi\rangle_{-1}|\geqslant a\right\} = 0,$$

where $\mathscr{S}_t$ denotes the class of all $(\mathscr{F}_t)$-stopping times with bound t;

(iii) for every $\varepsilon>0$ and $t>0$,

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}} P\left(\sup_{s\in[0,t]}\sum_{j=m}^\infty \langle X_s^n, e_j\rangle_{-1}^2 \geqslant \varepsilon\right) = 0.$$

Then the laws of $(X^n_t)_{t\geqslant 0}$ *in* $\mathbb{D}(\mathbb{R}_+;\mathbb{H}^{-1})$ *are tight.*

The following result comes from [7, p.131 Theorem 7.8].

Theorem 2.2. *Suppose that stochastic processes sequence* $\{(X^n_t)_{t\geqslant 0}, n\in\mathbb{N}\}$ *weakly converges to* $(X_t)_{t\geqslant 0}$ *in* $\mathbb{D}(\mathbb{R}_+;\mathbb{H}^{-1})$*. Then, for any* $t>0$ *and* $\phi\in\mathbb{H}^1$*, there exists a sequence* $t_n\downarrow t$ *such that for any bounded continuous function* f*,*

$$\lim_{n\to\infty}\mathbb{E}f(\langle X^n_{t_n},\phi\rangle_{-1})=\mathbb{E}f(\langle X_t,\phi\rangle_{-1}).$$

We also need the following technical result.

Lemma 2.3. *Suppose that sequence* $\{u^n, n\in\mathbb{N}\}$ *converges to* u *in* $\mathbb{D}(\mathbb{R}_+;\mathbb{H}^{-1})$*. Then for any* $T>0$ *and* $m\in\mathbb{N}$*,*

$$\sup_{t\in[0,T]}\|u_t\|_0\leqslant\varliminf_{n\to\infty}\sup_{t\in[0,T+\frac{1}{m}]}\|u^n_t\|_0. \tag{2.3}$$

If in addition, for Lebesgue almost all t*,* u^n_t *converges to* u_t *in* $\mathbb{H}^0$*, then for any* $\beta>0$*,*

$$\int_0^T\frac{\|\nabla u_t\|_0^2}{(1+\|u_t\|_0^2)^\beta}\mathrm{d}t\leqslant\varliminf_{n\to\infty}\int_0^T\frac{\|\nabla u^n_t\|_0^2}{(1+\|u^n_t\|_0^2)^\beta}\mathrm{d}t. \tag{2.4}$$

Proof. Without loss of generality, we assume that the right hand side of (2.3) is finite. For any $\phi\in\mathbb{H}^1$, it is clear that $t\mapsto\langle u_t,\phi\rangle_0$ is a càdlàg real valued function, and by definition (2.2) of Skorohod metric, we have

$$d_{\mathbb{R}}(\langle u^n,\phi\rangle_0,\langle u,\phi\rangle_0)\leqslant(2+\|\phi\|_1)d_{\mathbb{H}^{-1}}(u^n,u),$$

and so $\langle u^n,\phi\rangle_0$ converges to $\langle u,\phi\rangle_0$ in $\mathbb{D}(\mathbb{R}_+;\mathbb{R})$ as $n\to\infty$. Since the discontinuous points of $\langle u_\cdot,\phi\rangle_0$ are at most countable, for any $T>0$ and $m\in\mathbb{N}$, there exists a time $T_m\in(T,T+1/m)$ such that $\langle u_\cdot,\phi\rangle_0$ is continuous at T_m. Thus, we have (cf. [7, p.119, Proposition 5.3])

$$\lim_{n\to\infty}\sup_{t\in[0,T_m]}|\langle u^n_t,\phi\rangle_0|=\sup_{t\in[0,T_m]}|\langle u_t,\phi\rangle_0|.$$

Hence,

$$\begin{aligned}\sup_{t\in[0,T]}\|u_t\|_0&=\sup_{t\in[0,T]}\sup_{\phi\in\mathbb{H}^1;\|\phi\|_0\leqslant 1}|\langle u_t,\phi\rangle_0|\\&\leqslant\sup_{\phi\in\mathbb{H}^1;\|\phi\|_0\leqslant 1}\sup_{t\in[0,T_m]}|\langle u_t,\phi\rangle_0|\\&=\sup_{\phi\in\mathbb{H}^1;\|\phi\|_0\leqslant 1}\lim_{n\to\infty}\sup_{t\in[0,T_m]}|\langle u^n_t,\phi\rangle_0|\\&\leqslant\varliminf_{n\to\infty}\sup_{\phi\in\mathbb{H}^1;\|\phi\|_0\leqslant 1}\sup_{t\in[0,T_m]}|\langle u^n_t,\phi\rangle_0|\\&=\varliminf_{n\to\infty}\sup_{t\in[0,T_m]}\|u^n_t\|_0.\end{aligned}$$

Thus, (2.3) is proven.

For proving (2.4), let $\mathscr{N}$ be the Lebesgue null set such that for all $t\notin\mathscr{N}$, u^n_t converges to u_t in $\mathbb{H}^0$. Fixing a $t\notin\mathscr{N}$, then as above, we have

$$\frac{\|\nabla u_t\|_0^2}{(1+\|u_t\|_0^2)^\beta}\leqslant\frac{\varliminf_{n\to\infty}\|\nabla u^n_t\|_0^2}{(1+\lim_{n\to\infty}\|u^n_t\|_0^2)^\beta}\leqslant\varliminf_{n\to\infty}\frac{\|\nabla u^n_t\|_0^2}{(1+\|u^n_t\|_0^2)^\beta}.$$

Estimate (2.4) now follows by Fatou's lemma. □

3 Proof of Theorem 1.1

We first give the following definition about the weak solutions to equation (1.1).

Definition 3.1. *A probability measure P on $\mathbb{D}(\mathbb{R}_+;\mathbb{H}^{-1})$ is called a weak solution of equation (1.1) if*

(i) for any $t>0$, $P\left(u\in\mathbb{D}(\mathbb{R}_+;\mathbb{H}^{-1}):\sup_{s\in[0,t]}\|u_s\|_0+\int_0^t\|\nabla u_s\|_0^2\mathrm{d}s<+\infty\right)=1$;

(ii) for any $j\in\mathbb{N}$,

$$M_t^{(j)}(u):=\langle u_t,e_j\rangle_0-\langle u_0,e_j\rangle_0-\int_0^t[\langle u_s,\Delta e_j\rangle_0+\langle u_s\otimes u_s,\nabla e_j\rangle_0]\mathrm{d}s \tag{3.1}$$

is a Lévy process with the characteristic function

$$\mathbb{E}e^{\mathrm{i}\xi M_t^{(j)}}=\exp\left\{t\int_{\mathbb{R}\setminus\{0\}}(e^{\mathrm{i}\xi y\beta_j}-1-\mathrm{i}\xi y\beta_j 1_{|y|\leqslant 1})\nu(\mathrm{d}y)\right\},$$

and $\{(M_t^{(j)})_{t\geqslant 0},j\in\mathbb{N}\}$ is a sequence of independent Lévy processes.

Proof of Existence of Weak Solutions: We use Galerkin's approximation to prove the existence of weak solutions and divide the proof into three steps.
(Step 1): For $n\in\mathbb{N}$, set

$$\mathbb{H}_n^0:=\mathrm{span}\{e_1,e_2,\cdots,e_n\},$$

and let Π_n be the projection from $\mathbb{H}^0$ to $\mathbb{H}_n^0$ and define

$$L_t^n:=\sum_{j=1}^n\beta_j L_t^{(j)}e_j=\sum_{j=1}^n\int_{|y|\leqslant 1}y\beta_j e_j\tilde{N}^{(j)}(t,\mathrm{d}y)+\sum_{j=1}^n\int_{|y|>1}y\beta_j e_j N^{(j)}(t,\mathrm{d}y).$$

Consider the following finite dimensional SDE driven by finite dimensional Lévy process L_t^n:

$$\mathrm{d}u_t^n=[\Delta u_t^n-\Pi_n((u_t^n\cdot\nabla)u_t^n)]\mathrm{d}t+\mathrm{d}L_t^n,\ \ u_0^n=\Pi_n\varphi. \tag{3.2}$$

Since for any $R>0$ and $u,v\in\mathbb{H}_n^0$ with $\|u\|_0,\|v\|_0\leqslant R$,

$$\|\Pi_n((u\cdot\nabla)u-(v\cdot\nabla)v)\|_0\leqslant C_{R,n}\|u-v\|_0$$

and

$$\langle u,\Delta u-\Pi_n((u\cdot\nabla)u)\rangle_0=-\|\nabla u\|_0,\ \ \forall u\in\mathbb{H}_n^0, \tag{3.3}$$

finite dimensional SDE (3.2) is thus well-posed.
Define a smooth function f_n on $\mathbb{H}_n^0$ by

$$f_n(u):=(\|u\|_0^2+1)^{\theta/2},\ \ u\in\mathbb{H}_n^0.$$

By simple calculations, we have

$$\nabla f_n(u)=\frac{\theta u}{(\|u\|_0^2+1)^{1-\theta/2}},\ \ \nabla^2 f_n(u)=\frac{\theta\sum_{i=1}^n e_i\otimes e_i}{(\|u\|_0^2+1)^{1-\theta/2}}-\frac{\theta(2-\theta)u\otimes u}{(\|u\|_0^2+1)^{2-\theta/2}}, \tag{3.4}$$

and for all $u, v \in \mathbb{H}_n^0$,

$$|f_n(u) - f_n(v)| \leqslant |(\|u\|_0^2 + 1)^{1/2} - (\|v\|_0^2 + 1)^{1/2}|^\theta \leqslant \|u - v\|_0^\theta. \tag{3.5}$$

By (3.2), (3.3), (3.4) and Itô's formula (cf. [2, p.226, Theorem 4.4.7]), we have

$$\begin{aligned}
f_n(u_t^n) &= f_n(u_0^n) - \int_0^t \frac{\theta \|\nabla u_s^n\|_0^2}{(\|u_s^n\|_0^2 + 1)^{1-\theta/2}} \mathrm{d}s + \sum_{j=1}^n \int_0^t \int_{|y|\leqslant 1} [f_n(u_s^n + y\beta_j e_j) - f_n(u_s^n)] \tilde{N}^{(j)}(\mathrm{d}s, \mathrm{d}y) \\
&\quad + \sum_{j=1}^n \int_0^t \int_{|y|\leqslant 1} \left[f_n(u_s^n + y\beta_j e_j) - f_n(u_s^n) - \frac{\theta \langle u_s^n, y\beta_j e_j \rangle_0}{(|u_s^n|^2 + 1)^{1-\theta/2}} \right] \nu(\mathrm{d}y)\mathrm{d}s \\
&\quad + \sum_{j=1}^n \int_0^t \int_{|y|>1} \left[f_n(u_s^n + y\beta_j e_j) - f_n(u_s^n) \right] N^{(j)}(\mathrm{d}s, \mathrm{d}y) \\
&=: f_n(u_0^n) - I_1^n(t) + I_2^n(t) + I_3^n(t) + I_4^n(t).
\end{aligned}$$

For $I_2^n(t)$, by Burkholder's inequality and (3.5), we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\in[0,T]} I_2^n(t) \right) &\leqslant C \sum_{j=1}^n \mathbb{E}\left(\int_0^T \int_{|y|\leqslant 1} |f_n(u_s^n + y\beta_j e_j) - f_n(u_s^n)|^2 N^{(j)}(\mathrm{d}s, \mathrm{d}y) \right)^{1/2} \\
&\leqslant C \sum_{j=1}^n \left(\mathbb{E} \int_0^T \int_{|y|\leqslant 1} |f_n(u_s^n + y\beta_j e_j) - f_n(u_s^n)|^2 \nu(\mathrm{d}y)\mathrm{d}s \right)^{1/2} \\
&\leqslant C T^{1/2} \sum_{j=1}^n |\beta_j|^\theta \left(\int_{|y|\leqslant 1} |y|^{2\theta} \nu(\mathrm{d}y) \right)^{1/2} \leqslant C T^{1/2} \sum_{j=1}^\infty |\beta_j|^\theta.
\end{aligned}$$

where we have used condition $(\mathbf{H}_\theta)$. Here and after, the constant C is independent of n, T. For $I_3^n(t)$, by Taylor's expansion and (3.4), we have

$$\mathbb{E}\left(\sup_{t\in[0,T]} I_3^n(t) \right) \leqslant C \sum_{j=1}^n \beta_j^2 \int_0^T \int_{|y|\leqslant 1} |y|^2 \nu(\mathrm{d}y)\mathrm{d}s \leqslant C T \sum_{j=1}^\infty |\beta_j|^\theta \int_{|y|\leqslant 1} |y|^2 \nu(\mathrm{d}y).$$

For $I_4^n(t)$, by (3.5), we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\in[0,T]} I_4^n(t) \right) &\leqslant \sum_{j=1}^n \mathbb{E}\left(\int_0^T \int_{|y|>1} |f_n(u_s^n + y\beta_j e_j) - f_n(u_s^n)| N^{(j)}(\mathrm{d}s, \mathrm{d}y) \right) \\
&= \sum_{j=1}^n \mathbb{E}\left(\int_0^T \int_{|y|>1} |f_n(u_s^n + y\beta_j e_j) - f_n(u_s^n)| \nu(\mathrm{d}y)\mathrm{d}s \right) \\
&\leqslant C T \sum_{j=1}^\infty |\beta_j|^\theta \int_{|y|>1} |y|^\theta \nu(\mathrm{d}y).
\end{aligned}$$

Combining the above calculations, we obtain that

$$\mathbb{E}\left(\sup_{t\in[0,T]} (\|u_t^n\|_0^2 + 1)^{\theta/2} \right) + \mathbb{E} \int_0^T \frac{\theta \|\nabla u_s^n\|_0^2}{(\|u_s^n\|_0^2 + 1)^{1-\theta/2}} \mathrm{d}s \leqslant (\|\varphi\|_0^2 + 1)^{\theta/2} + C T + C T^{1/2}. \tag{3.6}$$

(Step 2): In this step, we use Theorem 2.1 to show that $\{(u_t^n)_{t\geqslant 0}, n\in\mathbb{N}\}$ is tight in $\mathbb{D}(\mathbb{R}_+;\mathbb{H}^{-1})$. For any $\phi\in C_0^\infty(\mathbb{T}^2)^2$, by equation (3.2), we have

$$\begin{aligned}\langle u_t^n,\phi\rangle_{-1} &= \langle u_0^n,\phi\rangle_{-1}+\int_0^t[\langle\Delta u_s^n,\phi\rangle_{-1}-\langle(u_s^n\cdot\nabla)u_s^n,\phi\rangle_{-1}]\mathrm{d}s+\langle L_t^n,\phi\rangle_{-1}\\ &= \langle u_0^n,\phi\rangle_{-1}+\int_0^t[\langle u_s^n,\Delta\phi\rangle_{-1}+\langle u_s^n\otimes u_s^n,\nabla\phi\rangle_{-1}]\mathrm{d}s+\langle L_t^n,\phi\rangle_{-1}.\end{aligned}$$

Thus, for $\varepsilon>0$ and any stopping time τ bounded by t, we have

$$\begin{aligned}\langle u_{\tau+\varepsilon}^n-u_\tau^n,\phi\rangle_{-1} &= \int_\tau^{\tau+\varepsilon}[\langle u_s^n,\Delta\phi\rangle_{-1}+\langle u_s^n\otimes u_s^n,\nabla\phi\rangle_{-1}]\mathrm{d}s+\langle L_{\tau+\varepsilon}^n-L_\tau^n,\phi\rangle_{-1}\\ &\leqslant \varepsilon\sup_{s\in[0,t]}\left(\|u_s^n\|_0\cdot\|\phi\|_0+\|u_s^n\|_0^2\cdot\|\nabla(-\Delta)^{-1}\phi\|_\infty\right)\\ &\quad+\sum_{j=1}^n|\beta_j|\cdot|L_{\tau+\varepsilon}^{(j)}-L_\tau^{(j)}|\cdot\|(-\Delta)^{-1}\phi\|_0.\end{aligned}$$

Using $(a+b)^\theta\leqslant a^\theta+b^\theta$ provided that $\theta\in(0,1]$, we get

$$\mathbb{E}|\langle u_{\tau+\varepsilon}^n-u_\tau^n,\phi\rangle_{-1}|^{\theta/2}\leqslant C_\phi\mathbb{E}\left(\sup_{s\in[0,t]}\|u_s^n\|_0^\theta+1\right)\varepsilon^{\theta/2}+C_\phi\left(\mathbb{E}\sum_{j=1}^n|\beta_j|^\theta\cdot|L_{\tau+\varepsilon}^{(j)}-L_\tau^{(j)}|^\theta\right)^{\frac{1}{2}}.$$

By the strong Markov property of Lévy process (cf. [12, p.278, Theorem 40.10]), we have

$$\mathbb{E}|L_{\tau+\varepsilon}^{(j)}-L_\tau^{(j)}|^\theta=\mathbb{E}|L_\varepsilon^{(j)}|^\theta=\mathbb{E}|L_\varepsilon^{(1)}|^\theta,\ \forall j\in\mathbb{N}.$$

Thus, by (3.6) and $(\mathbf{H}_\theta)$,

$$\mathbb{E}|\langle u_{\tau+\varepsilon}^n-u_\tau^n,\phi\rangle_{-1}|^{\theta/2}\leqslant C\left[\varepsilon^{\theta/2}+(\mathbb{E}|L_\varepsilon^{(1)}|^\theta)^{1/2}\right],\tag{3.7}$$

where the constant C is independent of n,τ and ε. On the other hand, by (2.1), we have

$$\mathbb{E}\left(\sup_{s\in[0,t]}\sum_{j=m}^\infty\langle u_s^n,e_j\rangle_{-1}^2\right)^{\theta/2}=\mathbb{E}\left(\sup_{s\in[0,t]}\sum_{j=m}^\infty\frac{\langle u_s^n,e_j\rangle_0^2}{\lambda_j^2}\right)^{\theta/2}\leqslant\frac{1}{\lambda_m^\theta}\mathbb{E}\left(\sup_{s\in[0,t]}\|u_s^n\|_0^\theta\right).\tag{3.8}$$

By Theorem 2.1 and (3.6)-(3.8), one knows that the law of $(u_t^n)_{t\geqslant 0}$ in $\mathbb{D}(\mathbb{R}_+;\mathbb{H}^{-1})$ denoted by P_n is tight.

(Step 3): Let P be any accumulation point of $\{P_n,n\in\mathbb{N}\}$. In this step, we show that P is a weak solution of equation (1.1) in the sense of Definition 3.1. First of all, by Skorohod's embedding theorem, there exists a probability space $(\tilde\Omega,\tilde{\mathscr{F}},\tilde P)$ and $\mathbb{D}(\mathbb{R}_+;\mathbb{H}^{-1})$-valued random variables X^n and X such that

(i) Law of X^n under $\tilde P$ is P_n and law of X under $\tilde P$ is P.

(ii) X^n converges to X in $\mathbb{D}(\mathbb{R}_+;\mathbb{H}^{-1})$ a.s. as $n\to\infty$.

Thus, by (3.6), we have

$$\tilde{\mathbb{E}}\left(\sup_{t\in[0,T]}\|X_t^n\|_0^\theta\right)+\tilde{\mathbb{E}}\left(\int_0^T\frac{\theta\|\nabla X_s^n\|_0^2}{(\|X_s^n\|_0^2+1)^{1-\theta/2}}\mathrm{d}s\right)\leqslant C(1+\|\varphi\|_0^\theta+T).\tag{3.9}$$

By Lemma 2.3 and Fatou's lemma, for any $m \in \mathbb{N}$, we have

$$\mathbb{E}^P\left(\sup_{t\in[0,T]}\|u_t\|_0^\theta\right)=\tilde{\mathbb{E}}\left(\sup_{t\in[0,T]}\|X_t\|_0^\theta\right)\leqslant\varliminf_{n\to\infty}\tilde{\mathbb{E}}\left(\sup_{t\in[0,T+1/m]}\|X_t^n\|_0^\theta\right)$$
$$\leqslant(\|\varphi\|_0^2+1)^{\theta/2}+C(T+1/m)+C(T+1/m)^{1/2}. \tag{3.10}$$

On the other hand, for any $\delta\in(0,\theta/4)$, by Hölder's inequality and (3.9), we have

$$\tilde{\mathbb{E}}\left(\int_0^T\|X_s^n-X_s\|_0^\delta\mathrm{d}s\right)\leqslant\tilde{\mathbb{E}}\left(\int_0^T\|X_s^n-X_s\|_{-1}^{\delta/2}\|X_s^n-X_s\|_1^{\delta/2}\mathrm{d}s\right)$$
$$\leqslant\left(\tilde{\mathbb{E}}\int_0^T\|X_s^n-X_s\|_{-1}^\delta\mathrm{d}s\right)^{1/2}\left(\tilde{\mathbb{E}}\int_0^T\|X_s^n-X_s\|_1^\delta\mathrm{d}s\right)^{1/2}\to0.$$

So, there exists a subsequence still denoted by n such that for $\tilde{P}\times\mathrm{d}t$-almost all (ω,s), $X_s^n(\omega)$ converges to $X_s(\omega)$ in $\mathbb{H}^0$. By Lemma 2.3 and (3.9), we then obtain

$$\mathbb{E}^P\left(\int_0^T\frac{\theta\|\nabla u_s\|_0^2}{(\|u_s\|_0^2+1)^{1-\theta/2}}\mathrm{d}s\right)=\tilde{\mathbb{E}}\left(\int_0^T\frac{\theta\|\nabla X_s\|_0^2}{(\|X_s\|_0^2+1)^{1-\theta/2}}\mathrm{d}s\right)$$
$$\leqslant\varliminf_{n\to\infty}\tilde{\mathbb{E}}\left(\int_0^T\frac{\theta\|\nabla X_s^n\|_0^2}{(\|X_s^n\|_0^2+1)^{1-\theta/2}}\mathrm{d}s\right)$$
$$\leqslant C(1+\|\varphi\|_0^\theta+T). \tag{3.11}$$

Combining (3.10) and (3.11) gives (1.2). In particular, $\sup_{t\in[0,T]}\|u_t\|_0$ and $\int_0^T\frac{\theta\|\nabla u_s\|_0^2}{(\|u_s\|_0^2+1)^{1-\theta/2}}\mathrm{d}s$ are finite P-almost surely, which produces (i) of Definition 3.1.

Fixing $j\in\mathbb{N}$, in order to show that $M_t^{(j)}$ defined by (3.1) is a Lévy process, we only need to prove that for any $0\leqslant s<t$,

$$\mathbb{E}^Pe^{\mathrm{i}\xi(M_t^{(j)}-M_s^{(j)})}=\tilde{\mathbb{E}}e^{\mathrm{i}\xi(\tilde{M}_t^{(j)}-\tilde{M}_s^{(j)})}=\exp\left\{(t-s)\int_{\mathbb{R}\setminus\{0\}}(e^{\mathrm{i}\xi y\beta_j}-1-1_{|y|\leqslant1}\mathrm{i}\xi y\beta_j)\nu(\mathrm{d}y)\right\}, \tag{3.12}$$

where

$$\tilde{M}_t^{(j)}:=\langle X_t,e_j\rangle_0-\langle X_0,e_j\rangle_0-\int_0^t[\langle X_r,\Delta e_j\rangle_0+\langle X_r\otimes X_r,\nabla e_j\rangle_0]\mathrm{d}r.$$

Fix $0\leqslant s<t$ below. By Theorem 2.2, there exists $(s_n,t_n)\downarrow(s,t)$ such that

$$\lim_{n\to\infty}\tilde{\mathbb{E}}e^{\mathrm{i}\xi\langle X_{t_n}^n,e_j\rangle_0}=\tilde{\mathbb{E}}e^{\mathrm{i}\xi\langle X_t,e_j\rangle_0},\quad\lim_{n\to\infty}\tilde{\mathbb{E}}e^{\mathrm{i}\xi\langle X_{s_n}^n,e_j\rangle_0}=\tilde{\mathbb{E}}e^{\mathrm{i}\xi\langle X_s,e_j\rangle_0}.$$

By equation (3.2), it is well-known that for any $n\geqslant j$,

$$\tilde{\mathbb{E}}\exp\left\{\mathrm{i}\xi\left[\langle X_{t_n}^n-X_{s_n}^n,e_j\rangle_0-\int_{s_n}^{t_n}[\langle X_r^n,\Delta e_j\rangle_0+\langle X_r^n\otimes X_r^n,\nabla e_j\rangle_0]\mathrm{d}r\right]\right\}$$
$$=\mathbb{E}^{P_n}\exp\left\{\mathrm{i}\xi\left[\langle u_{t_n}^n-u_{s_n}^n,e_j\rangle_0-\int_{s_n}^{t_n}[\langle u_r^n,\Delta e_j\rangle_0+\langle u_r^n\otimes u_r^n,\nabla e_j\rangle_0]\mathrm{d}r\right]\right\}$$

$$= \exp\left\{(t_n - s_n)\int_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}\xi y\beta_j} - 1 - 1_{|y|\leqslant 1}\mathrm{i}\xi y\beta_j)\nu(\mathrm{d}y)\right\}.$$

Thus, for proving (3.12), it suffices to prove the following limits:

$$\lim_{n\to\infty}\tilde{\mathbb{E}}\left|\exp\left\{\mathrm{i}\xi\int_s^t \langle X_r^n\otimes X_r^n, \nabla e_j\rangle_0 \mathrm{d}r\right\} - \exp\left\{\mathrm{i}\xi\int_s^t \langle X_r\otimes X_r, \nabla e_j\rangle_0 \mathrm{d}r\right\}\right| = 0,$$

$$\lim_{n\to\infty}\tilde{\mathbb{E}}\left|\exp\left\{\mathrm{i}\xi\int_s^t \langle X_r^n, \Delta e_j\rangle_0 \mathrm{d}r\right\} - \exp\left\{\mathrm{i}\xi\int_s^t \langle X_r, \Delta e_j\rangle_0 \mathrm{d}r\right\}\right| = 0,$$

$$\lim_{n\to\infty}\tilde{\mathbb{E}}\left|\exp\left\{\mathrm{i}\xi\int_{s_n}^{t_n} \langle X_r^n\otimes X_r^n, \nabla e_j\rangle_0 \mathrm{d}r\right\} - \exp\left\{\mathrm{i}\xi\int_s^t \langle X_r^n\otimes X_r^n, \nabla e_j\rangle_0 \mathrm{d}r\right\}\right| = 0,$$

$$\lim_{n\to\infty}\tilde{\mathbb{E}}\left|\exp\left\{\mathrm{i}\xi\int_{s_n}^{t_n} \langle X_r^n, \Delta e_j\rangle_0 \mathrm{d}r\right\} - \exp\left\{\mathrm{i}\xi\int_s^t \langle X_r^n, \Delta e_j\rangle_0 \mathrm{d}r\right\}\right| = 0.$$

Let us only prove the first limit, the others are similar. Noticing that for any $\delta\in(0,1)$ and $a, b\in\mathbb{R}$,

$$|e^{\mathrm{i}a} - e^{\mathrm{i}b}| \leqslant 2(|a-b|\wedge 1) \leqslant 2|a-b|^\delta,$$

by Hölder's inequality and $\|u\|_0 \leqslant \|u\|_{-1}^{1/2}\|u\|_1^{1/2}$, we have for $\delta < \theta/4$,

$$\begin{aligned}
&\tilde{\mathbb{E}}\left|\exp\left\{\mathrm{i}\xi\int_s^t \langle X_r^n\otimes X_r^n, \nabla e_j\rangle_0 \mathrm{d}r\right\} - \exp\left\{\mathrm{i}\xi\int_s^t \langle X_r\otimes X_r, \nabla e_j\rangle_0 \mathrm{d}r\right\}\right| \\
&\leqslant 2|\xi|^\delta\tilde{\mathbb{E}}\left|\int_s^t \langle X_r^n\otimes X_r^n - X_r\otimes X_r, \nabla e_j\rangle_0 \mathrm{d}r\right|^\delta \\
&\leqslant C\tilde{\mathbb{E}}\left(\int_s^t \|X_r^n - X_r\|_0(\|X_r^n\|_0 + \|X_r\|_0)\mathrm{d}r\right)^\delta \\
&\leqslant C\tilde{\mathbb{E}}\left(\sup_{r\in[s,t]}(\|X_r^n\|_0 + \|X_r\|_0)\int_s^t \|X_r^n - X_r\|_{-1}^{1/2}\|X_r^n - X_r\|_1^{1/2}\mathrm{d}r\right)^\delta \\
&\leqslant C\tilde{\mathbb{E}}\Bigg(\sup_{r\in[s,t]}(\|X_r^n\|_0 + \|X_r\|_0 + 1)^{2\delta-(\theta\delta/2)}\left(\int_s^t \|X_r^n - X_r\|_{-1}\mathrm{d}r\right)^{\delta/2} \\
&\quad\times\left(\int_s^t \frac{(\|X_r^n\|_1 + \|X_r\|_1)}{(\|X_r^n\|_0^2 + \|X_r\|_0^2 + 1)^{1-\theta/2}}\mathrm{d}r\right)^{\delta/2}\Bigg) \\
&\leqslant C\left[\tilde{\mathbb{E}}\left(\int_s^t \|X_r^n - X_r\|_{-1}\mathrm{d}r\right)^{2\delta}\right]^{1/4} \to 0,
\end{aligned}$$

as $n\to\infty$, where in the last inequality, we have used (3.9) and Hölder's inequality. As for the independence of $M^{(j)}$ for different $j\in\mathbb{N}$, it can be proved in a similar way. □

Proof of Theorem 1.1: The pathwise uniqueness follows by the classical result for 2D deterministic Navier-Stokes equation. As for the existence of invariant measures, basing on (1.2) (see Remark 1.3), it follows by the classical Bogoliubov-Krylov's argument. □

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